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SEMANTICS FOR FIRST AND HIGHER ORDER REALIZABILITY

Abstract. First order Kleene realizability is given a semantic interpretation, including arithmetic and other types. These types extend at a stroke to full higher order intuitionistic logic. They are also useful themselves, e.g., as models for lambda calculi, for which see *Asperti and Longo 1991* and papers on PERs and polymorphism (*IEEE 1990*).

This semantics is simpler and more explicit than in *Hyland 1982*, giving the logical content of Freyd, Carboni, and Scedrov's (*1988*) assemblies. Category theory here is only an organizing device and can be skipped, except in Lemmas 11–12 verifying the higher order logic. We verify some higher order constructive recursive analysis, and prove two metatheorems by generalizing the construction to other toposes.

1. ASSEMBLIES

We enumerate the partial recursive functions and write $n.m$ for the value of the n-th function applied to m. Of course $n.m$ may be undefined. We use a surjective recursive pairing function $\langle\ ,\ \rangle$ with recursive projections, l and r, read "left" and "right". So we have $l\langle n,m\rangle = n$ and $r\langle n,m\rangle = m$, and $n = \langle ln, rn\rangle$.

An *assembly* (A) consists of a set A called the *carrier* and an infinite sequence of its subsets $(A_1, A_2, \cdots)$ whose union is all of A. Each A_n is called the n-th *caucus*. An *arrow* between assemblies, written $f : (A) \to (B)$, is a function between the carriers $f : A \to B$ with at least one *modulus*—that is a number e such that whenever $a \in A_n$ then $e.n$ is defined and $fa \in B_{e.n}$. Arrows compose, with the composite of moduli as a modulus for the composite. So we have a well defined category.

Assemblies appear in *Freyd, Carboni, and Scedrov 1988, Freyd and Scedrov 1990,* and *McLarty 1992,* and as ω-sets in *Asperti and Longo 1991.*

Let the assembly (1) have a singleton set 1 as carrier, and the same 1 for every caucus. There is an assembly $(\mathbb{N})$ with $\mathbb{N}$ the set of natural numbers and each caucus $\mathbb{N}_n$ the singleton $\{n\}$. An arrow $f : (\mathbb{N}) \to (\mathbb{N})$ must be recursive, with its own codes as moduli. Categorists note (1) is terminal and $(\mathbb{N})$ a natural number object.

Assemblies (A) and (B) have a product $(A \times B)$ whose n-th caucus is $A_{ln} \times B_{rn}$. The projections $(A \times B) \to (A)$ and $(A \times B) \to (B)$ are well defined. The exponential (B^A) has the set of arrows from (A) to (B) as carrier, and the n-th caucus contains those arrows with n as modulus. E.g.,

C. Anthony Anderson and M. Zelëny (eds.), Logic, Meaning and Computation, 353-363.

the carrier of $(\mathbb{N}^{\mathbb{N}})$ is the set of recursive functions, each one in those caucuses which code it. The category is cartesian closed.

A *subassembly* of (A) is a monic arrow $f : (P) \rightarrowtail (A)$, i.e., any arrow one-to-one on the carriers. This subassembly is contained in $g : (Q) \rightarrowtail (A)$ if there is an arrow $h : (P) \to (Q)$ with $g \circ h = f$. Subassemblies are equivalent if each contains the other. The restriction of (A) to a subset S of the carrier is the subassembly $(S) \rightarrowtail (A)$ with carrier S and with each caucus S_n the intersection of S with A_n. Given arrows f and g both from (A) to (B), the restriction $(E) \rightarrowtail (A)$ of (A) to those x such that $fx = gx$ is an equalizer for f and g. The category has all finite limits.

2. ASSEMBLY REALIZABILITY

The first order logic of the category of assemblies is multi-sorted, with a sort for each assembly (B).

Each variable of sort (B) is a term of sort (B). For each $b : (1) \to (B)$ we have a term b of sort (B), a constant. For any arrow $f : (A) \to (B)$ and term t of sort (A) there is a term ft of sort (B). Since a closed term of sort (B) is a constant or suitably sorted arrows applied to constants, it determines an element of the carrier B. Our notation confuses closed terms with elements when the context makes it clear.

For each subassembly $(P) \rightarrowtail (A)$ there is a predicate symbol P applicable to terms of sort (A). For each subassembly of a product, $(R) \rightarrowtail (A \times B)$, we have a two place relation symbol R, applying to pairs of terms of sort (A) and (B). Similarly for n-ary relations for any n. For any (A) the diagonal $(A) \rightarrowtail (A \times A)$ is the *equality* relation, the restriction of $(A \times A)$ to pairs $\langle a, a\rangle$. We write $=$ for its relation symbol.

We define realizability only for sentences. For any $i : (R)(A \times B \times C)$ and closed terms t_A, t_B, t_C of sorts (A), (B), and (C) respectively, n realizes the atomic sentence $Rt_At_Bt_C$ iff some r in R_n has $i(r) = \langle tA, tB, tC\rangle$. Generalization to n-ary relations is obvious.

Let S and S' be sentences and Sy any formula whose sole free variable is y of sort (B). Then n realizes:

$S \,\&\, S'$ iff ln realizes S and rn realizes S'.

$S \vee S'$ iff either: $ln = 0$ and rn realizes S or $ln = 1$ and rn realizes S'.

$S \Rightarrow S'$ iff, for any m that realizes S, $n.m$ is defined and realizes S'.

$\sim S$ iff no m realizes S.

$(\forall y)Sy$ iff, for every b in B_m, $n.m$ is defined and realizes Sb.

$(\exists y)Sy$ iff for some b in B_{ln}, rn realizes Sb.

REMARK 2.1. Just asking whether a sentence is realized by some number gives standard truth conditions, except for $\forall$. I.e., $S \Rightarrow S'$ is realized iff: if S is realized so is S'. $(\exists y.(B))Sy$ is realized iff some Sb is. Similarly for &, $\vee$, and $\sim$. Notice $(\forall y.(B)) \sim\sim Sy$ is realized iff Sb is realized for all b in B. (Constructively, throughout the "if" clauses for $\Rightarrow$ and $\forall$ read "not-not-realized".)

Let $Pxyz$ be a formula with no free variables but x, y, and z, of sorts (A), (B), and (C) respectively. We write $[xyz \mid Pxyz]$ for the *extension* of P over that list of variables. The n-th caucus $[xyz \mid Pxyz]_n$ contains those triples $\langle a, b, c\rangle$ such that $\langle a, b, c\rangle$ is in $(A \times B \times C)_{ln}$ and rn realizes $Pabc$. The inclusion makes this a subassembly of $(A \times B \times C)$.

Even if y, for example, does not occur in $Pxyz$, so b does not appear in $Pabc$, we define $[xyz \mid Pxyz]$ as a subassembly of $(A \times B \times C)$ as above. We define an extension $[x^- \mid P]$ for any list x^- of distinct variables including all those free in P. A subassembly $(P) \rightarrowtail (A)$ need not be $[x \mid Px]$ but is equivalent to it

We write "$x.(A)$" or "$x^-.(A \times B \times C)$" to show the sorts of variables.

3. THE LOGIC OF ASSEMBLIES

These extensions admit an intuitionist sequent calculus, agreeing with the first order part of topos logic (see *Bell 1988* or *McLarty 1992*). So this interpretation agrees with the categorical one.

A *sequent* is an expression $\Gamma : P$ with Γ a finite set of formulas and P a formula. We use a new formula "fa", read as "false". The extension of fa over any list of variables is the empty subassembly of the corresponding product of assemblies.

Let x^- list the variables free in $\Gamma : P$. Let $[x^- \mid \Gamma]$ be the intersection of the extensions over x^- of all formulas in Γ, or equivalently the extension of the conjunction of formulas in Γ. The sequent $\Gamma : P$ is true if $[x^- : \Gamma]$ is contained in $[x^- \mid P]$. In that case, we write

$$\Gamma \vdash P.$$

So we have $\vdash P$ iff the extension of P is the entire product it is defined over.

Calculation shows the sequents above any double line here are true iff the one below is. Here Γ is any finite set of formulas and P, P', Q, and Q' any formulas, and x any variable not free in Γ or Q

$$\frac{\Gamma : P \text{ and } \Gamma : P'}{\Gamma : P \,\&\, P'} \qquad \frac{\Gamma, Q : P \text{ and } \Gamma, Q' : P}{\Gamma, Q \vee Q' : P}$$

$$\frac{\Gamma, Q : P}{\Gamma : Q \Rightarrow P} \qquad \frac{\Gamma, Q : fa}{\Gamma : \sim Q}$$

$$\frac{\Gamma : P}{\Gamma : (\forall x)P} \qquad \frac{\Gamma, P : Q}{\Gamma, (\exists x)P : Q}$$

We use a cut rule: If $\Gamma, Q : P$ and $\Gamma : Q$ are true so is $\Gamma : P$, unless there is a free variable over the empty assembly in Q but none in $\Gamma : P$. Every sequent with a free variable over the empty assembly is true.

This logic does not prove the law of excluded middle, "$P \vee \sim P$". Any formula "P" implies "$\sim\sim P$" but the converse need not hold.

4. ON NEGATION, AND EQUIVALENCE RELATIONS

A subassembly $(C) \rightarrowtail (A)$ is *complemented*, or *decidable*, if the formula $(\forall x.(A))(Cx \vee \sim Cx)$ is true. The terminology is natural here since a subassembly of $(\mathbb{N})$ is complemented iff it is equivalent to the restriction of $(\mathbb{N})$ to some recursive set of numbers. The diagonal $(\mathbb{N}) >> (\mathbb{N} \times \mathbb{N})$ is complemented—i.e., equality is decidable for natural numbers. It is not for $(\mathbb{N}^{\mathbb{N}})$, because equality of recursive functions is not recursively decidable.

A subassembly $(C) \rightarrowtail (A)$ is *double negation closed*, or *d.n. closed* for short, iff

$$\vdash \sim\sim Cx \Rightarrow Cx. \qquad x.(A)$$

So it is d.n. closed iff it is equivalent to a restriction of (A). Every decidable subassembly is d.n. closed, as is equality on any assembly.

Call a formula in the language of assemblies *almost negative* if all its predicates and relations are d.n. closed and its only connectives are &, $\Rightarrow$, $\sim$, and $\forall$.

THEOREM 4.1. *Let P be any almost negative formula with free variables $x^-.(A \times \cdots \times K)$. Construe the predicates and relations in P as subsets and relations of the carriers $A \cdots K$. Then $[x^- \mid P]$ is equivalent to the restriction of $(A \times \cdots \times K)$ to the set theoretic extension of P over x^-.*

Proof. By the logic of assemblies, almost negative formulas are d.n. closed. And every n realizes $\sim\sim S$ if some number realizes S. Then use Remark 2.1. ⊣

An equivalence relation is a relation $(R) \rightarrowtail (A \times A)$ satisfying the usual sequents for reflexivity, symmetry, and transitivity. A quotient for (R) is defined categorically as a coequalizer for the two projections $(R) \to (A)$ with those projections as kernel pair. In terms of logic it is an arrow $q : (A) \to (Q)$ such that:

$$\vdash (Rxy \Rightarrow qx = qy) \,\&\, (qx = qy \Rightarrow Rxy), \qquad x, y.(A)$$
$$\vdash (\exists x.(A))qx = z. \qquad z.(Q)$$

THEOREM 4.2. *An equivalence relation (R) on (A) has a quotient assembly iff it is d.n. closed.*

Proof. Equality is always d.n. closed, so given the first sequent above, (R) is also. Conversely, if (R) is d.n. closed, it gives an equivalence relation on the set A with a projection $q : A \rightarrow Q$ to the set of equivalence classes. Define (Q) by making each Q_n the set of equivalence classes of members of A_n. Then $q : (A) \rightarrow (Q)$ is a quotient of (R). ⊣

REMARK 4.3. For any function $f : \mathbb{N} \rightarrow \mathbb{N}$, define an assembly (T) where each natural number m occurs only in the caucus $T\langle m, fm\rangle$. The inclusion defines a subassembly $(T) \rightarrowtail (\mathbb{N})$ whose double negation is all of $(\mathbb{N})$. And f is a well defined arrow $f : (T) \rightarrow (\mathbb{N})$. I.e., every function from $\mathbb{N}$ to $\mathbb{N}$ is an arrow from some *double negation dense* subassembly of $(\mathbb{N})$ to $(\mathbb{N})$. But $(T) \rightarrowtail (\mathbb{N})$ is equivalent to all of $(\mathbb{N})$ iff f is recursive.

5. ARITHMETIC IN ASSEMBLIES

The assembly $(\mathbb{N})$ with successor arrow s verifies the Peano axioms. By trivial realizability:

$$\vdash \sim(0 = sx), \qquad x, y. (\mathbb{N})$$
$$\vdash sx = sy \Rightarrow x = y, \qquad z. (A)$$

$$P(z, 0), (\exists x)(P(z, x) \Rightarrow P(z, sx)) \vdash (\exists x)P(z, x)$$

where $(P) \rightarrowtail (A \times \mathbb{N})$ is any relation to $(\mathbb{N})$.

All the classical theory of primitive recursive functions applies in assemblies, since all the proofs in *Kleene 1952,* chapter IX, work in the logic of assemblies. Furthermore, primitive recursive definitions are almost negative. So the extension of any n-ary primitive recursive predicate in assemblies is the restriction of $(\mathbb{N} \times \cdots \times \mathbb{N})$ to that predicate's set theoretic extension. This includes all graphs of primitive recursive functions.

The classical theory of general recursive functions in *Kleene 1952,* chapter XI, works in the logic of assemblies except for the diagonal arguments. Those use excluded middle, which fails for the claim "function f is defined for argument n" in assemblies due to the undecidability of the halting problem. The classical proof that there are non-recursive functions from the natural numbers to themselves fails here, and in fact for assemblies, all such functions are recursive.

To show this, define Kleene's T-predicate $T(e, x, y)$ and output function $U(y)$ in the logic of assemblies. The usual theorems follow in that logic. So just as in the classical case, we read $T(e, x, y)$ as "e is Gödel number of a definition (or, e is a code for a partial recursive function) which when applied to argument x gives a calculation with Gödel number y", and read $U(y)$ as "the final output from calculation y".

THEOREM 5.1. *Church's Thesis is true for assemblies*:

$$\vdash (\forall f. (\mathbb{N}^{\mathbb{N}}))(\exists e)(\forall x)(\exists y)(T(e, x, y) \,\&\, U(y) = fx).$$

Proof. Since the T-predicate and U are primitive recursive they agree in extension with the classical ones. Let $c(n, m)$ be the Gödel number for the completed calculation (if there is one) obtained from definition number n applied to argument m. Church's Thesis is realized by any code for the function taking n to $\langle n, h(\mathbb{N})\rangle$ where $h(\mathbb{N})$ codes the function taking any k to $\langle c(n,k), \langle\langle n, k, c(n,k)\rangle, \langle n.k, n.k\rangle\rangle\rangle$.

This is easier to check than to read. ⊣

THEOREM 5.2. *First order Markov principle. For any relation* $(P) \rightarrowtail (\mathbb{N} \times A)$:

$$(\forall y.(\mathbb{N}))(Pyx \vee \sim Pyx), \sim\sim(\exists y)Pyx \vdash (\exists y)Pyx. \qquad x.(A)$$

Proof. It suffices to find a partial recursive y such that: For all a in A, if k realizes $(\forall y)(Pya \vee \sim Pya)$ and some Pra is realized, then ψk realizes $(\exists y)Pya$. Define ϕk to be the smallest i such that $l(k.i) = 0$ (so $r(k.i)$ realizes Pia). For ψk take $\langle \phi k, r(k.\phi k)\rangle$. This uses Markov's principle in the metalanguage, since constructively a realizer for $\sim\sim(\exists y)Pyx$ only implies $(\exists y)Pyx$ is not-not realized. ⊣

Since $(\mathbb{N})$ is isomorphic to $(\mathbb{N} \times \mathbb{N})$, Markov's principle extends to two or more quantifiers $\exists$ over $(\mathbb{N})$ before a decidable formula. We count such formulas as almost negative, and Theorem 4.1 still holds.

The axiom of choice is not true in assemblies, but the axiom of choice from $(\mathbb{N})$ to any assembly is.

THEOREM 5.3. *For any assemblies* (A) *and* (B) *and relation* $(P) \rightarrowtail (\mathbb{N} \times B \times A)$ *we have*:

$$(\forall y)(\exists x.(B))Pyxz \vdash (\exists f.(B^{\mathbb{N}}))(\forall y)P(y, fy, z). \qquad z.(A)$$

Proof. Given a in A, let n realize $(\forall y)(\exists x.(B))Pyxa$. I.e., for all m there exists b in $B_{l(n.m)}$ such that $r(n.m)$ realizes $Rmba$. So (assuming countable choice in our metalanguage) there is an arrow $f_a : (\mathbb{N}) \to (B)$, with the function taking m to $l(n.m)$ as modulus, such that $r(n.m)$ realizes $Rm(f_a m)a$. This easily gives a realizer for $(\exists f.(B^{\mathbb{N}}))(\forall y)P(y, fy, z)$ recursive in n. ⊣

An assembly is *modest* iff each caucus A_n has at most one member. These amount to the "strictly effective objects" of Hyland (*1982*). An arrow between them is determined by any of its moduli. They are the quotients of sub-objects of $(\mathbb{N})$ by closed equivalence relations and appear in other notations as PERs (see *IEEE 1990, Asperti and Longo 1991*).

Notice that for modest (B), the proof of choice from $(\mathbb{N})$ needs no choice principle in the metalanguage.

Ordinary arithmetized analysis defines the integers $(\mathbb{I})$ and rationals $(\mathbb{Q})$ as quotients of products of $(\mathbb{N})$ by decidable equivalence relations. The assembly

$(\mathbb{Q}^{\mathbb{N}})$ of sequences of rationals, all recursive by Church's Thesis, has a sub-assembly (CS) of Cauchy sequences. Hyland (*1982*) shows that if we require a certain rate of convergence for Cauchy sequences, then their equivalence is closed. Thus, the Cauchy reals form the assembly of recursive reals. In fact, all these are modest. Hyland (*1982*) says more on this. His "canonically separated objects" amount to assemblies.

6. THE EFFECTIVE TOPOS

This construction of $\mathcal{E}\!f\!f$ was first sketched categorically in *Freyd, Carboni, and Scedrov 1988*. Further generalities are in *Freyd and Scedrov 1990* and McLarty 1992.

The objects of $\mathcal{E}\!f\!f$ are equivalence relations of assemblies. When we take $(R) \rightarrowtail (A \times A)$ as an object in $\mathcal{E}\!f\!f$ we may call it (A/R) to suggest the quotient of (A) by (R). An arrow in $\mathcal{E}\!f\!f$ from (A/R) to (B/R') is given by a relation $(F) \rightarrowtail (A \times B)$ which is equivariant for (R) and (R') in this sense:

$$Rxy, Fyz \vdash Fxz; \qquad x, y.\,(A)$$
$$Fxz, R'zw \vdash Fxw; \qquad z, w.\,(B)$$

and functional in that:

$$\begin{aligned} Fxz, Fxw &\vdash R'zw, \\ &\vdash (\exists z)Fxz. \end{aligned}$$

Relations $(F) \rightarrowtail (A \times B)$ and $(F') \rightarrowtail (A \times B)$ give the same arrow from (A/R) to (B/R') if they are equivalent sub-objects of $(A \times B)$.

Given such an $\mathcal{E}\!f\!f$ arrow (F) and another (G) from (B/R') to some (C/R''), the relation $(\exists y.\,(B))(Fxy \& Gyz)$ defines an arrow $(G \circ F)$. This is associative, and (R) is the identity from (A/R) to (A/R). So $\mathcal{E}\!f\!f$ is a category.

THEOREM 6.1. *The category $\mathcal{E}\!f\!f$ has all finite limits:*

(i) *The object $(1/=)$ is terminal.*

(ii) *Any (A/R) and (B/R') have a product $(A \times B/R \times R')$ with $R \times R'\langle x, z\rangle \langle y, w\rangle$ defined by $(Rxy \,\&\, R'zw)$.*

(iii) *Any arrows (F) and (G) both from (A/R) to (B/R') have an equalizer, $[x.\,(A) \mid (\exists y)(Fxy \,\&\, Gxy)]$ modulo the equivalence relation on it induced by R.*

Proof. The natural proofs of these for quotient sets are sound for assemblies. ⊣

Some results fit assemblies into $\mathcal{E}\!f\!f$.

LEMMA 6.2. *A relation* $(F) \rightarrowtail (A \times B)$ *satisfies:*

$$Fxy \,\&\, Fxz \vdash y = z, \qquad x.(A)$$
$$\vdash (\exists y)Fxy \qquad y,z.(B)$$

in the logic of assemblies iff there is a unique arrow of assemblies $f : (A) \to (B)$ *such that*

$$\vdash Fxy \longleftrightarrow fx = y.$$

In other words, F is equivalent to the graph of f.

Proof. By realizability, if the sequents are true, (F) is isomorphic to (A). Composing with the projection of (F) to (B) gives f. ⊣

So there is a full and faithful functor to $\mathcal{E}\!f\!f$ taking each assembly (A) to $(A/=)$ and each arrow to its graph. We treat assemblies and their arrows as objects and arrows of $\mathcal{E}\!f\!f$ via this functor.

LEMMA 6.3.

(i) *Each* $(R) : (A) \to (A/R)$ *is a quotient of the two projections from* (R) *to* (A)*. It is their coequalizer and they are its kernel pair.*

(ii) *If* $(F) : (A/R) \rightarrowtail (B)$ *is monic in* $\mathcal{E}\!f\!f$*, then* (A/R) *is isomorphic to an assembly.*

Proof. For (i), use the logic of assemblies. For (ii), (F) induces a relation between (A) and $[y.(B) \mid (\exists x.(A))Fxy]$ equivariant for (R) and functional in both directions, thus an isomorphism in $\mathcal{E}\!f\!f$. ⊣

For any equivalence relation $(R) \rightarrowtail (A \times A)$ it is easy to see that R-equivariance as defined by

$$Rxy, Hyz \vdash Fxz \qquad x,y.(A) \quad z.(B)$$

is equivalent to saying a relation $(H) \rightarrowtail (A \times B)$ has the same pullback along the two projections $(R \times B) \to (A \times B)$.

LEMMA 6.4. *Relations in* $\mathcal{E}\!f\!f$ *from* (A/R) *to an assembly* (B) *correspond to* R*-equivariant relations* $(H) \rightarrowtail (A \times B)$*.*

Proof. By Lemma 6.3(i) any relation from (A/R) to (B) pulls back to such an (H). Conversely, let (H) be R-equivariant and (R') the equivalence relation on the assembly (H) induced by the relation (R) on (A). Then (H/R') is the unique (up to equivalence) relation from (A/R) to (B) with pullback (H). ⊣

There is a kind of truth value assembly (W) with each caucus W_n the powerset of the natural numbers. For a subassembly $(M) \rightarrowtail (W)$ counting as *true*, let each M_n contain the sets of natural numbers which include n. Given any $(C) \rightarrowtail (B)$, define $f : (B) \to (W)$ with each fb the set of realizers for Cb. Then $(C) \rightarrowtail (B)$ is equivalent to the pullback of (M) along f, but not only f. We say f *weakly classifies* $(C) \rightarrowtail (B)$.

There is an equivalence relation $E(E) \rightarrowtail (W \times W)$ such that: Any $\langle f, g\rangle : (B) \rightarrow (W \times W)$ factors through (E) iff f and g weakly classify the same subassembly of (B). Each E_n contains those pairs $\langle S, S'\rangle$ of sets of numbers such that for every m in S the value $ln.m$ is defined and is in S', and for every m in S' the value $rn.m$ is defined and in S. Intuitively, n realizes $(S \Rightarrow S')$ & $(S' \Rightarrow S)$.

LEMMA 6.5. *Every $\mathcal{E}\!f\!f$ object (B/R') has a power object.*

Proof. Arrows $(A) \rightarrow (W^B/E^B)$ uniquely classify relations from (A) to (B). A relation from any (A/R) to (B) amounts to an arrow $(A) \rightarrow (W^B/E^B)$ coequalizing the projections $(R) \rightarrow (A)$. This induces an arrow from (A/R) to (W^B/E^B) classifying the original relation. So (B) has power object (W^B/E^B).

The equalizer $(S) \rightarrowtail (W^B)$ of the two induced arrows from (W^B) to $(W^{R'})$ has an equivalence relation (E') induced by the relation (E^B) on (W^B). And (S/E') is a power object for (B/R'). ⊣

So $\mathcal{E}\!f\!f$ is a topos and has its own higher order logic, related to the logic of assemblies by:

LEMMA 6.6. *A formula with all assembly sorts is true in the logic of $\mathcal{E}\!f\!f$ iff it is true in the logic of assemblies. An $\mathcal{E}\!f\!f$ formula Qx, with x a variable over the power object of an assembly (B), is true iff every $Q(P)$ obtained by replacing x with a relation of assemblies $(P) \rightarrowtail (A \times B)$ is true.*

Proof. Both claims follow from Lemma 6.3(ii). For the second, every $\mathcal{E}\!f\!f$ object is surjective image of an assembly, so we need only know Qs for every generalized element $s : (A) \rightarrow P(B)$ defined over an assembly, thus every subassembly $(P) \rightarrowtail (A \times B)$. ⊣

REMARK 6.7. Any (A/R) is d.n. separated in $\mathcal{E}\!f\!f$ iff it has d.n. closed equality, so iff (R) is d.n. closed in $\mathcal{E}\!f\!f$. By Lemma 6.6 that is iff (R) is d.n. closed in assemblies. And so iff (A/R) is isomorphic to an assembly. The category of sets, *Set*, is isomorphic to the full subcategory of assemblies containing those (A) with each A_n equal to the carrier A. Call these assemblies *codiscrete.* They are d.n. sheaves in the category of assemblies. Each (B) has a d.n. dense monic to the codiscrete based on the set B, and is a sheaf iff this is an isomorphism. So the d.n. sheaves in $\mathcal{E}\!f\!f$ are (up to isomorphism) the codiscretes, i.e., they are (up to functor equivalence) *Set* embedded in $\mathcal{E}\!f\!f$.

7. ARITHMETIC IN EFF

The assembly $(\mathbb{N})$ is a natural number object in $\mathcal{E}\!f\!f$. To see this, use the Peano axioms where the induction axiom has P, a variable over the power object of $(\mathbb{N})$:

$$\vdash \sim(0 = sx),$$

$$\vdash sx = sy \Rightarrow x = y,$$

$$P(0) \,\&\, (\forall x)(P(x) \Rightarrow P(sx)) \vdash (\forall x)P(x).$$

The first two deal only with an assembly and so remain true in $\mathcal{E}\!f\!f$. Lemma 6.6 proves the induction axiom from the induction scheme for assemblies. Church's Thesis also deals only with assemblies. Lemma 6.6 gives the Markov principle with P a free variable over the power object of $(\mathbb{N})$:

$$(\forall n)(Pn \vee \sim Pn), \qquad \sim\sim(\exists n)Pn \vdash (\exists n)Pn.$$

Consider choice from $(\mathbb{N})$ to any (B/R), with P a variable over the power object $P(N \times (B/R))$:

$$(\forall y.\, (\mathbb{N}))(\exists x.\, (B/R))Pyx \vdash (\exists f.\, ((B/R)\mathbb{N}))(\forall y)P(y, fy).$$

Certainly this is true for (B/R) if it is true for (B). Then use Lemma 6.6. And again, if (B) is modest, we have choice from $(\mathbb{N})$ to any (B/R) without assuming choice in the metalanguage.

In $\mathcal{E}\!f\!f$ we can define the Dedekind real numbers by Dedekind cuts. As Hyland (*1982*) remarks, the Cauchy and Dedekind reals agree (up to isomorphism) in $\mathcal{E}\!f\!f$ since choice from $(\mathbb{N})$ to (Q) shows every Dedekind cut gives a Cauchy sequence. We have good conditions for recursive analysis in full higher order intuitionistic logic.

8. METATHEOREMS

The above reasoning is constructive, except for particulars of arithmetic, so it works in any topos. This quickly gives two standard results in our context:

THEOREM 8.1. *Realizable realizability is realizability. I.e., realizability in $\mathcal{E}\!f\!f$ of sentences with all terms of sort* $(\mathbb{N})$ *agrees with classical realizability.*

Proof. Via Gödel numbering of those sentences we regard realizability as a relation between natural numbers. Its definitions are almost negative, except for $S \vee S'$. And equality of natural numbers is decidable, so that clause is equivalent to the almost negative: If $ln = 0$ then rn realizes S; and if not $ln = 0$, then $ln = 1$ and rn realizes S'. ⊣

THEOREM 8.2. *More arithmetic is realized classically than intuitionistically; or than follows intuitionistically from the Peano axioms plus Church's Thesis.*

Proof. The second claim follows from the first as the Peano axioms and Church's thesis are intuitionistically realized. For the first we give a topos where Markov's principle fails in ordinary arithmetic. By the proof of Theorem 5.2, Markov's principle is also not realized there.

Take presheaves (Kripke models) on this poset: The set of pairs $(n, 0)$ and $(n, 1)$ for all natural numbers n, where each $(n, 0)$ precedes $(n, 1)$ and each $(m, 0)$ and $(m, 1)$ with $n < m$, while each $(n, 1)$ is a dead-end preceding nothing. The natural numbers here are the presheaf assigning $\mathbb{N}$ at every node. Markov's principle fails for the sub-presheaf S which is empty at each $(n, 0)$ and at each $(n, 1)$ is the singleton $\{n\}$. ⊣

Van Oosten (*1991*) gives several other higher order realizabilities, all of which can be presented in this way, using a first order logic for variants of assemblies and then getting a topos and higher order logic by adjoining quotients. They realize different parts of arithmetic, which is much of the point. And Remark 6.7 fails for some.

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