``Emmy Noether's `Set Theoretic' Topology: From Dedekind to the rise
of functors'',

In Jeremy Gray and José Ferreirós ed.s The Architecture of Modern Mathematics: Essays in history and philosophy, Oxford, 2006, 211--35.

Emmy Noether's 'Set Theoretic' Topology: From Dedekind to the Rise of Functors

Colin McLarty

> 'In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain.' (Weyl 1939, 500)

If Hermann Weyl ever put faces on these spirits they were his good friends, the angel Luitzen Brouwer, and the devil Emmy Noether. Weyl well describes the scope of their ambition. But topology and algebra were not fighting each other. They would come to share the soul of most of mathematics. And the angel and devil had worked together. In 1926 and 1927 Emmy Noether induced young topologists gathered around Brouwer to use her algebra to organize the kind of work on topological maps that Brouwer taught. This created the still current basis for 'algebraic topology.' It was a huge advance for the structuralist conception of mathematics. And it turned Noether from a great algebraist to a decisive figure for twentieth-century mathematics.[1]

The algebra was not her first great work. A recent check of 300 references to Noether in *Mathematical Reviews* found 80% of those with identifiable topic concerned her earlier conservation theorems in mathematical physics. Among many surveys of this work, see (Byers 1996). Yet the algebra is central to her reputation. Alexandroff notes

> Emmy Noether herself was partly responsible for her early work being remembered less frequently than would be natural. For with all the fervor of her nature she was herself ready to forget what had been done in the first years of her mathematical activity, considering these results as standing apart from her true mathematical path—the creation of a general abstract algebra. (Alexandroff 1981, 101).

This algebra would be so general as to apply over all mathematics.

---

[1] (Alexandroff 1932) is a beautiful 50-page introduction to just the topology studied here. Originally commissioned as an appendix to (Hilbert & Cohn-Vossen 1932), this gem highlights Brouwer's and Noether's ideas.

It is widely said that Noether created 'algebraic topology' by bringing groups into *homology theory*. The fullest historical studies up to now are by Jean Dieudonné and Saunders Mac Lane.[2] Mac Lane concludes that Poincaré already knew of homology groups. Dieudonné rejects this, saying Poincaré wrote a great deal about groups and never mentioned homology groups. Mac Lane says no one used these groups until Noether in Göttingen and Vietoris independently in Vienna, about 1926. Dieudonné accepts that. We will see Mac Lane is entirely right about Poincaré. But Noether was not in Göttingen at the crucial moment, nor was Vietoris in Vienna. They were both in tiny Laren, Holland, visiting Brouwer.

The place is important because the event combined Brouwer's ideas in topology (not foundations) with Noether's algebra. The young topologists used some of his technical tools. More to the point, they followed him in giving continuous maps at least as much attention as point sets in topology. His most famous result, his *fixed point theorem*, is explicitly about maps: Every continuous map $f : B^n \to B^n$, from an n-dimensional ball to itself, has a fixed point, a point $x \in B^n$ such that $f(x) = x$. His greatest single paper is titled 'On mappings of manifolds' (Brouwer 1911). His theorems do not all refer to mappings, but his proofs do.[3]

The time is important because Noether was just then developing what she called her set-theoretic foundations for algebra. This was not what we now call set theory. It was not the idea of using sets in basic definitions and reasoning. She took that more or less for granted, as did other Göttingers by the 1920s. Rather, her project was to get abstract algebra away from thinking about operations on elements, such as addition or multiplication of elements in groups or rings. Her algebra would describe structures in terms of selected subsets (such as normal subgroups of groups) and homomorphisms. Alexandroff applied her tools to Brouwer's use of continuous maps—though Vietoris was the first to make it work.

Noether brought something much deeper and more comprehensive to topology than just the use of homology groups. The next section (and a more technical appendix) will show that groups were familiar in homology before her. She brought an entire programme of looking at groups, and other structures in algebra, and other structures outside of algebra like topological spaces, in terms of the homomorphisms between them. She called this 'set-theoretic foundations.' Section 6 describes the programme in general, and Section 6 quotes Noether on her chief tools for making this work: her homomorphism and isomorphism theorems. She credits these to Dedekind and they are probably the most important single case for her famous slogan 'Es steht alles schon bei Dedekind (this is all already in Dedekind).' Section 6 describes what she got from him on this point

---

[2] (Dieudonné 1984), (Mac Lane 1986), and Dieudonné's review of Mac Lane in *Math. Reviews* 87e:01027.

[3] Brouwer's key concept, the *degree* of a map, remains central to the topology of manifolds. It is called the *Brouwer degree* in (Vick 1994, 25).

and how she developed it. It is far from a comprehensive account of Dedekind's influence on Noether and Noether's way of finding things in Dedekind that no one could see before her.[4] Section 6 is a close look at how the young topologists in Laren took these ideas into topology. Then section 6 looks more broadly at the role of Noether's 'set-theoretic foundations' especially in category theory and the current use of 'structure' in mathematics.

1 Homology before Noether

For a rough idea of Poincaré on homology we can consider the torus.[5] Poincaré said a closed curve C on the torus is *homologous to zero*, and wrote $C \sim 0$, if it cut out a piece of the torus. A circle around a little piece of the surface cuts out that piece, but a curve drawn around the small circumference of the torus, like the solid and dotted line in the figure, cuts the torus open to become a cylinder without cutting out any piece. It is not homologous to zero.

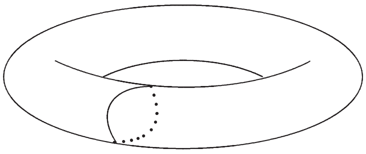

Fig. 1.

Let C_α be that curve and C'_α another curve like it, to one side of it, so that C_α and C'_α cut a cylindrical piece out of the torus. For this, Poincaré would write $C_\alpha - C'_\alpha \sim 0$ and say that the difference $C_\alpha - C'_\alpha$ is *homologous to zero*. The minus sign shows the two curves lie on opposite sides of the cylindrical piece. Let C_β be a curve going the whole way around the large circumference of the torus (like the elliptical outline of the torus in the figure). The curves C_α and C_β cut the torus into a cylinder and then a flat sheet, but do not cut out any piece.

The key fact about the homology of the torus is that given any list of three different curves some combination of them will cut out a piece of the torus. Maybe one of them alone will do it, or some two of them, but anyway some combination of them will. At most, two curves can be *independent* on the torus. The two C_α, C_β are independent. So we say the 1-dimensional *Betti number* of the torus is 2.

In a higher-dimensional space, Poincaré said a sum of curves $C_1 + C_2, + \cdots + C_n$ is homologous to zero if they jointly form the boundary of some surface in that

---

[4] On Dedekind, see Jeremy Avigad's contribution to this volume. We do not address Dedekind's work on foundations, including his correspondence with Cantor, which Noether coedited (Noether & Cavaillès 1937), see also (Ferreirós 2005).

[5] See (Herremann 2000), (Pont 1974), (Sarkaria 1999). Poincaré was often vague in his definitions and would change them from one page to the next without comment. We ignore all such issues.

space. He used formal sums with integer coefficients such as

$$3C_1 - 2C_2 - C_3 + 5C_4$$

to say that curve C_3 is used three times, and the opposite to C_2 is used twice and so on. He said two formal sums were homologous:

$$C_1 + 2C_2 \sim C_3 + C_4$$

if and only if their formal difference was homologous to zero:

$$C_1 + 2C_2 - C_3 - C_4 \sim 0.$$

He remarked that 'in homologies the terms compose according to the rules of ordinary addition' (Poincaré 1899–1904, 449–50). He means the associative, commutative, and negation rules:

$$C_1 + (C_2 + C_3) \sim (C_1 + C_2) + C_3$$
$$C_1 + C_2 \sim C_2 + C_1 \qquad C - C \sim 0$$

Poincaré's homology was 'algebraic' as that term was understood at the time, and also in our sense of using group theory. Alain Herreman has nicely laid out the earlier meaning. He quotes Öystein Ore from 1931: 'it might be said that algebra deals with the formal combinations of symbols according to prescribed rules' (Herreman 2000, 26). Ore said that definition was already out of date, but it has appeared as recently as (Mac Lane 1988, 29). It was current in the 1890s and Poincaré's homology was explicitly algebraic in this sense as he manipulated the formal sums of curves and of higher-dimensional cycles.

Further, Poincaré knew his cycles formed groups. The 1-dimensional cycles of a given space, with their formal sums, formed today's 1-dimensional homology group of that space. The 2-dimensional cycles formed today's 2-dimensional homology group. The point is controversial because Poincaré did not write of 'groups' in homology. But homology groups are commutative, and Poincaré systematically avoided referring to commutative groups. For him, 'group' meant a Lie group or at least a transformation group. Even then, if he knew a given Lie group was commutative he preferred not to call it a 'group.' Among his great projects was, in our terms, to find all the commutative subgroups of Lie groups. He worked on them with all the tools of group theory at that time. But he did not call them 'groups.' He called them the *sheaves* (faisceaux) of Lie groups (Poincaré 1916–1956, vol. 5, passim). He felt they were too elementary to be called groups. Detailed evidence of how much Poincaré knew about groups in homology is below in the appendix.

From today's viewpoint, each homology group of a suitable space is *generated* by some finite list of cycles. The 1-dimensional homology of the torus is generated by C_α, C_β as above. Every 1-dimensional cycle on the torus is homologous to a unique sum of multiples of C_α and C_β. This is another way to say the 1-dimensional Betti number of the torus is 2. Poincaré described homology in terms of Betti numbers while he used group theory to calculate with the cycles.

Vietoris rightly wrote to Mac Lane:

> Without doubt H. Poincaré and his contemporaries knew that the Betti numbers and the torsion coefficients were invariants of groups, whose elements were cycles under the operation of addition... Then one worked with the numerical invariants rather than with the invariant groups. That was a matter of 'taste'. (Letter of 16 March 1985 quoted in (Mac Lane 1986, 307).)

Hugo Gieseking in 1912 wrote of 'the Abelian group' of a surface to mean its 1-dimensional homology group (Vanden Eynde 1992, p. 177). Oswald Veblen first published the term 'homology group,' also for the 1-dimensional groups. He lectured on Poincaré's topology in 1916. For the war he became Captain Veblen and eventually published a 'thoroughly revised . . . more formal' account (Veblen 1922, iii). He gave Poincaré's favourite way to calculate the first Betti number of a polyhedron. The routine naturally produces a group that Veblen said 'may well be called the homology group' (Veblen 1922, 141) noted in (Mac Lane 1978, p. 11). Others took no notice because they already knew that homology cycles formed groups in all dimensions.[6]

Weyl spoke of the group of cycles of a (simplicial) space, and the group of boundaries (Weyl 1923, 393). He spoke group theoretically to say that homology looks at the cycles modulo the boundaries. But he conspicuously avoided forming the quotient groups, the homology groups.[7] That makes all the difference, because it prevented him from saying that continuous (simplicial) maps of spaces induce homomorphisms of homology groups. Of course everyone working in homology had known this fact in some way, and used it, since Poincaré. But no one before Noether saw how powerful, simplifying, and unifying it would be to use the groups in the right way.

2 Noether's set-theoretic foundations

Alexandroff relates Noether's set-theoretic algebra to her influence on topology:

> my theory of continuous partitions of topological spaces arose to a large extent under the influence of conversations with her in December and

---

[6] Thanks to Ralf Krömer for pointing me to Gieseking.

[7] In Poincaré's terminology cycles were identified modulo boundaries in the first place. Today we say Weyl's 'cycles' are curves while Poincaré's are equivalence classes of curves (and analogously for higher dimensional homology). Both senses of 'cycle' are used colloquially today. I thank Erhard Scholz for pointing out that Weyl had a philosophic reason to avoid forming the quotient groups in that he disliked forming sets of infinite sets.

> January of 1925–1926 when we were in Holland together. It was also at this time that Noether's first ideas on the set-theoretic foundations of group theory were developing. She lectured on these in the summer of 1926. Although she returned to them several times later, these ideas were not developed further in their initial form, probably because of the difficulty of axiomatizing the concept of a group by taking coset decomposition as the basic notion. But the idea of set-theoretic analysis of the concept of a group turned out to be fruitful, as shown by the subsequent work of Ore, Kurosh, and others. (Alexandroff 1981, 108) cited (Corry 1996, 247)

The next four sections explain Alexandroff's claims. This one describes the set-theoretic conception and its key tool, the *homomorphism theorems*. The next section quotes Noether's own treatment of these theorems and quotes her one published application to topology. Section 6 compares and contrasts Dedekind on these theorems, before Section 6 returns to Alexandroff and topology in Laren.

Consider the ordinary integers $\mathbb{Z}$ and the integers modulo 3, written $\mathbb{Z}/3$.[8] One natural way to compare them is just to say which integers count as 0 modulo 3. That is the subgroup of integer multiples of 3:

$$3\mathbb{Z} = \{\cdots - 3, 0, 3, 6, 9 \ldots\}.$$

Another natural way is to take the homomorphism $q\colon \mathbb{Z} \to \mathbb{Z}/3$ taking each integer n to its remainder modulo 3:

$$q(3i) = 0, \quad q(3i+1) = 1, \quad q(3i+2) = 2 \quad \text{for all } i \in \mathbb{Z}.$$

The homomorphism determines the subgroup since $3\mathbb{Z}$ is just the set of integers i such that $q(i) = 0$. And the subgroup determines the homomorphism. First the subgroup has a pair of *cosets*:

$$3\mathbb{Z} + 1 = \{\cdots - 2, 1, 4, 7, 10 \ldots\}$$

and:

$$3\mathbb{Z} + 2 = \{\cdots - 1, 2, 5, 8, 11 \ldots\}.$$

These partition $\mathbb{Z}$ into three disjoint, exhaustive subsets. So define q by the rule

$$q(3\mathbb{Z} + i) = i.$$

Then $\mathbb{Z}/3$ is called a *factor group* of $\mathbb{Z}$ because it comes from dividing $\mathbb{Z}$ up along the lines of the subgroup $3\mathbb{Z}$.

The homomorphism theorem for commutative groups says every onto homomorphism is, up to isomorphism, projection to a factor group. A textbook version from the Noether school was:

Theorem. Whenever a commutative group $\overline{\mathfrak{G}}$ has a homomorphism onto it from a group $\mathfrak{G}$, it is isomorphic to a factor group $\mathfrak{G}/\mathfrak{e}$. Here $\mathfrak{e}$ is the subgroup of $\mathfrak{G}$ whose

---

[8] So $\mathbb{Z}/3 = \{0, 1, 2\}$ with addition given by $1 + 1 = 2$, $1 + 2 = 0$, $2 + 2 = 1$.

elements correspond to the 0 of $\overline{\mathfrak{G}}$. *(van der Waerden 1930, adapted for commutative groups from vol. 1 p. 35).*

The Noether school contrasted a 'set-theoretic' conception of algebra to what they considered an 'arithmetic' conception in Dedekind's work. Dedekind had noted that divisibility relations between (algebraic) integers can be expressed in terms of *ideals*.[9] To say that 3 divides 6 is the same as saying that the principal ideal (3) contains the principal ideal (6).

$$3|6 \qquad\qquad (6) \subseteq (3).$$

Dedekind generalized the divisibility relation to ideals so that whenever one ideal contains another, $\mathfrak{b} \subseteq \mathfrak{a}$, he calls $\mathfrak{a}$ a *divisor* of $\mathfrak{b}$.

Wolfgang Krull said Dedekind's terminology is 'better adapted to the needs of arithmetic, but experience shows it seems strange to every beginner.' The larger ideal is divisor of the smaller one and, worse, the 'greatest common divisor' of two ideals is the smallest ideal that contains them both. Krull said his terminology and exposition will always 'keep in mind (berucksichtigen) the group- and set-theoretic viewpoint.' Given $\mathfrak{a} \subseteq \mathfrak{b}$, Krull called $\mathfrak{a}$ a *subideal* (Unterideal) of $\mathfrak{b}$, or $\mathfrak{b}$ a *superideal* (Oberideal) of $\mathfrak{a}$.[10]

Noether's idea was genuinely deep. She described 'purely set-theoretic' methods as 'independent of any operations' (Noether 1927, 47).[11] These methods do not look at addition or multiplication of the elements of a ring (or a group,etc.). They look at selected subsets and the corresponding homomorphisms. The homomorphism theorem for commutative rings correlates homomorphisms to all ideals. In groups in general, homomorphisms are correlated only to *normal* subgroups. So she looked at groups in terms of their normal subgroups and homomorphisms. She looked at a ring in terms of its ideals and ring homomorphisms, since those correspond in the homomorphism theorem for rings.

Of course, these selected subsets were themselves defined in terms of the addition and multiplication operators, so her methods were not purely 'set-theoretic',

---

[9] An ideal in a ring $\mathfrak{R}$ is a non-empty subset $\mathfrak{a} \subseteq \mathfrak{R}$ closed under subtraction (i.e. any additive subgroup of $\mathfrak{R}$) such that $y \in \mathfrak{a}$ implies $xy \in \mathfrak{a}$, for every $x \in \mathfrak{R}$. Each $x \in \mathfrak{R}$ gives a *principal ideal* (x) containing all the $\mathfrak{R}$-multiples of x. The relation $(x) \subseteq (y)$ says every multiple of x is a multiple of y, thus x is a multiple of y, or y divides x. In the ring $\mathbb{Z}$ of integers every ideal is a principal ideal (n). This is not true in every ring. But it means that in $\mathbb{Z}$ Dedekind's theory of ideal divisors agrees entirely with the classical theory of integer divisors.

[10] All quotes are (Krull 1935, 2). Wolfgang Krull (1899–1971) was strongly influenced by Noether, in 1920, at the start of her long concentration on algebra.

[11] A key element of this was the *ascending chain condition* (Noether 1921). This is called the Noetherian condition on a ring or module in any abstract algebra text today. It describes the ordering of ideals of a ring, or submodules of a module, with no (explicit) reference to operations on the ring or module. Noether established its power in abstract algebra, number theory, and the factorization of polynomials.

but that was her direction. Ore eventually took this up:[12]

> In the discussion of the structure of algebraic domains, one is not primarily interested in the *elements* of these domains but in the relations of certain *distinguished subdomains* ((Ore 1935, 406), quoted in (Corry 1996, 272))

But Ore's results, like the results of Kurosh mentioned by Alexandroff above, have ceased to be of central concern to mathematicians. They focused too much on the order relations among distinguished subsets of a single ring or module, while keeping the idea of homomorphism in the background.[13]

Noether used the homomorphism theorems to prove *isomorphism theorems*, which show that certain relations among the subsets imply that certain morphisms are isomorphisms. These and other ideas served Noether's well-known goal, in Krull's terms:

> Noether's principle: base all of algebra so far as possible on consideration of isomorphisms. (Krull 1935, 4)

Van der Waerden captured the method exactly: To understand ring ideals is to understand their analogy with normal subgroups, for which 'we proceed from the concept of homomorphism!' Over the next two pages he introduced and proved the homomorphism theorem for rings (van der Waerden 1930, 55–7).[14]

3 Noether's 'Abstrakter Aufbau der Idealtheorie'

Noether gave her first clear statement of the 'set-theoretic' conception in (Noether 1927).[15] She was explicit that there are different kinds of algebraic structure, that each has its own homomorphism theorem, and that each homomorphism theorem implies isomorphism theorems for that same structure. We should note that she did not use the term 'homomorphism theorem' although she italicized

---

[12] Alexandroff's claim that she began this in the mid-1920s is confirmed by (Noether 1924), (Noether 1925), (Noether 1926), and the submission date of (Noether 1927). See (Corry 1996, 265 ff.) for the early twentieth-century history of approaches to algebra through substructures and especially Ore's relation to Noether.

[13] Reinhold Baer led Mac Lane to look at transcendence degree of field extensions, and related examples, in these terms in (Mac Lane 1938).

[14] Mac Lane later canonized the homomorphism and isomorphism theorems in his theory of *Abelian categories*. His idea was redone and simplified by Grothendieck into a standard tool in homology theory. The Abelian category axioms are more or less exactly what it takes to state and prove the homomorphism and isomorphism theorems. Indeed, one might say that Grothendieck gave the first purely 'set-theoretic' foundation, in Noether's sense, for any practically important part of algebra.

[15] The paper was written in 1925 and abstracted in (Noether 1924).

the statement. She used the name, and used the theorem even more cleanly, in (Noether 1929, esp. p. 647).[16]

In her paper (Noether 1927) Noether dealt with modules, and spoke of 'remainder class modules' where we have spoken above of 'factor groups.'[17] She said one module M is homomorphic to another $\overline{\mathsf{M}}$ when each element of M corresponds to a unique element of $\overline{\mathsf{M}}$ in a way that respects the addition law in the modules and the law of ring multiplication. More precisely, where she wrote $\mathsf{M} \sim \overline{\mathsf{M}}$ we would specify an onto homomorphism $f\colon \mathsf{M} \to \overline{\mathsf{M}}$. Noether wrote $\overline{\beta}$ for our $f\beta$, and $(\beta - \gamma) \sim (\overline{\beta} - \overline{\gamma})$ for our $f(\beta - \gamma) = f\beta - f\gamma$. Noether also retained the arithmetic terminology, so that a divisor of a submodule is any submodule containing it. Guided by Krull's remarks quoted above, we shall rewrite her theorems in modern notation without significantly altering their meaning.

Noether said that two modules were *isomorphic* if there are one-to-one onto homomorphisms between them in each direction. She began by noting that the isomorphism theorems 'assume only ring- or respectively module-properties and no further axioms.' She then observed that

> If $\mathfrak{A}$ is any $\mathfrak{R}$-module contained in M then one gets a module $\overline{\mathsf{M}}$ homomorphic to M — the remainder class module $\mathsf{M}|\mathfrak{A}$ — by taking congruence modulo $\mathfrak{A}$ as the equality relation. Each element of M is thereby coordinated with all and only the ones equal to it in $\overline{\mathsf{M}}$. To pass from this equality relation in $\overline{\mathsf{M}}$ to the identity relation, means to collect all the equal elements of $\overline{\mathsf{M}}$ into one class—a *remainder class*—and to conceive these remainder classes as the elements of $\overline{\mathsf{M}}$.
>
> *Every homomorphism is generated by such passage to a module of remainder classes*; because if $\mathsf{M} \sim \overline{\overline{\mathsf{M}}}$ and $\mathfrak{A}$ is the module coordinate to the zero element of $\overline{\overline{\mathsf{M}}}$ then as shown above, $\overline{\overline{\mathsf{M}}}$ *is isomorphic to the remainder class module* $\mathsf{M}|\mathfrak{A}$.

In modernized language, which changes only one word, her first isomorphism theorem then states: *Let* $\overline{\mathsf{M}}$ *be the remainder class module* $\mathsf{M}|\mathfrak{A}$ *and* $\mathfrak{C}$ *a module containing* $\mathfrak{A}$. *Then there is an isomorphism* $\overline{\mathsf{M}}|\overline{\mathfrak{C}} \simeq \mathsf{M}|\mathfrak{C}$. Her proof suppresses all mention of the ring or module operations. It deals only with the equality relation:

> This is because congruence modulo $\mathfrak{A}$ implies congruence modulo $\mathfrak{C}$. Elements equal modulo $\mathfrak{A}$ thus remain equal modulo $\mathfrak{C}$. So one can form the remainder class module $\mathsf{M}|\mathfrak{C}$—and so equality modulo $\mathfrak{C}$—by first setting the elements equal modulo $\mathfrak{A}$, that is passing to $\overline{\mathsf{M}}$, and then collecting the ones equal modulo $\mathfrak{C}$ which expresses equality modulo $\overline{\mathfrak{C}}$ in $\overline{\mathsf{M}}$, and thus the formation of $\overline{\mathsf{M}}|\overline{\mathfrak{C}}$.

---

[16] Van der Waerden follows the 1929 proofs, especially proving isomorphism theorems from homomorphism theorems in (van der Waerden 1930, vol. 1, p. 136).

[17] A module M over a ring $\mathfrak{R}$ is a commutative group M acted on by $\mathfrak{R}$. Readers not familiar with modules may suppose $\mathfrak{R}$ is the ring of integers $\mathbb{Z}$. Every commutative group M is a module over $\mathbb{Z}$ where each integer acts by multiplication: For example, $3\alpha = \alpha + \alpha + \alpha$, and $-1\alpha = -\alpha$ for every $\alpha \in \mathsf{M}$.

Her second isomorphism theorem states that *If* $\mathfrak{B}$ *and* $\mathfrak{A}$ *are modules contained in* M, *and* $(\mathfrak{B}, \mathfrak{A})$ *is the smallest submodule containing both, then there is an isomorphism*: $(\mathfrak{B}, \mathfrak{A})|\mathfrak{A} \simeq \mathfrak{B}|[\mathfrak{B} \cap \mathfrak{A}]$. Noether proves this by the homomorphism theorem:

> The module $\mathfrak{B}$ becomes homomorphic to $\overline{\mathfrak{B}}$ when $(\mathfrak{B}, \mathfrak{A})$ is set equal to $\overline{\mathfrak{B}}$; and since this makes all elements of $[\mathfrak{B}, \mathfrak{A}]$ and only those correspond to the zero element of $\mathfrak{B}$ the above comment (i.e. the homomorphism theorem) yields the isomorphism.

She quickly repeated the reasoning for commutative rings and adapted it to non-commutative rings (Noether 1927, 39–40).[18]

Noether's footnote to a major theorem explains how it is 'purely set-theoretic.' The theorem is:

> Theorem I. *Assuming the ascending chain condition, every ideal of* $\mathfrak{R}$ *can be presented as the intersection of finitely many irreducible ideals, that is ideals which cannot be presented as intersections of any proper ideals.*

Her proof again ignores ring operations in favour of the order relation on ideals: if the result is not true, one can construct an infinitely long chain of ascending ideals, contrary to the a.c.c.[19] The footnote says:

> Theorem I obviously holds in just this way for modules, when the ascending chain condition is assumed for systems of modules. That is to say, it has a purely set-theoretic character—at the same time it is the only one we prove using the well ordering of ideals. It deals, independently of any operations, with the following set-theoretic concepts:
>
> Let a set $\mathfrak{M}$ have a distinguished subset Σ of the power set—that is a system of subsets. Assume Σ is well ordered; the elements of Σ may be called Σ-sets. Assume the *ascending chain condition* in Σ: Every chain of Σ-sets $\mathfrak{A}_1, \mathfrak{A}_2, \ldots \mathfrak{A}_\nu, \ldots$, such that each $\mathfrak{A}_\nu$ is a proper subset of $\mathfrak{A}_{\nu+1}$, is finite. A Σ-set $\mathfrak{A}$ is called *reducible* when it is the intersection of two Σ-sets, each properly greater than $\mathfrak{A}$, and in the contrary case irreducible. Then the above translates to: *Every* Σ*-set can be presented as the intersection of finitely many irreducible* Σ*-sets.* (Noether 1927, 46–7)

The homomorphism and isomorphism theorems come in again for a theorem, trivial in itself but crucial to polynomial algebra: every irreducible ideal is primary (Noether 1927, 47).[20]

In Göttingen on the 27 January 1925 Noether gave a talk in which she proved the structure theorem for finitely generated commutative groups by abstract methods, and then derived the elementary divisor theorem of arithmetic from

---

[18] Her rings are not required to have a unit. So every ideal is a ring and the second isomorphism theorem gives a ring-isomorphism $(\mathfrak{B}, \mathfrak{A})|\mathfrak{A} \simeq \mathfrak{B}|\mathfrak{B} \cap \mathfrak{A}$.

[19] Here Noether did not assume the well ordering theorem (or axiom of choice) in general but explicitly assumed a well ordering on the elements of a given ring $\mathfrak{R}$. This gives a lexicographic order on finite subsets of $\mathfrak{R}$. Since $\mathfrak{R}$ is assumed to satisfy the a.c.c. every ideal has a finite basis so this gives a well ordering on ideals.

[20] She also proved this in (Noether 1921, 39). By 1925 the proof was simpler and the theory of ideals as a whole was radically simpler and farther reaching.

it—which says that an integer matrix can always simplify into a standard form (a diagonal matrix where each entry on the diagonal divides the next one along) by certain arithmetic steps. She precisely reversed the argument of (Frobenius and Stickelberger 1879) who had more or less founded the organized study of finitely generated commutative groups by deriving the structure theorem from the arithmetic one. Noether claimed her order is clearer and (Lang 1993, p.153) among others today agrees. But she did more than reverse the order.

She made the structure theorem itself less central. Her basic methods apply uniformly to all commutative groups, all rings, etc. So today, even when it is vital that the relevant groups are finitely generated (e.g. in homology theory of manifolds as in (Vick 1994)) the structure theorem may never be mentioned. The finiteness is invoked as little as possible, and replaced as far as possible by simpler general methods.

Noether never published her proof of the structure theorem for finitely generated commutative groups. She published an abstract of it, and its last sentence is her only published reference to homology:

> So the theorem on groups is the simpler one; in applications of the theorem on groups—e.g. Betti numbers and torsion numbers in topology—no recourse to the elementary divisor theorem is needed. (Noether 1926, 104)

She assumed that everyone knows there are groups in homology. Her point is that the groups are simpler than the arithmetic.

4 Dedekind to Noether

Noether's influence on homology was closely tied to what she took from Dedekind on the homomorphism and isomorphism theorems, and to the advances she made from there. In the course of editing his collected works she found that Dedekind's early personal notes on group theory give 'such a sharp work-up of the homomorphism theorem as has only lately again become usual' (Dedekind 1930–1932, vol. 3, p. 446). That is to say, only with herself and her school. But he did it without any name for homomorphisms. The closest he came to defining homomorphisms in these notes is when he said:

> Let M be a group with m objects; and let each object θ in M correspond to an object θ_1 in such a way that every product $\theta\phi\psi\ldots\lambda$ of objects $\theta, \phi, \psi, \ldots, \lambda$ contained in M corresponds to the product $\theta_1\phi_1\psi_1\ldots\lambda_1$ of the corresponding objects $\theta_1, \phi_1, \psi_1, \ldots, \lambda_1$. The complex of m_1 mutually distinct objects θ_1 will be called M_1. (Dedekind 1930–1932, 440)

Then he proved M_1 is a group by proving it is closed under products. Dedekind discussed only finite groups. What Noether (quite correctly) recognized as the homomorphism theorem is Dedekind's description of M_{I} in terms of representative elements from M; and it assumes that M_{I} is finite.

Noether saw in Dedekind an understanding of everything she intended in her version of the homomorphism theorem for groups. He described the kernel of a homomorphism, its cosets, and the group of those cosets. The restriction to finite groups is incidental to the concepts even if it is thoroughly embedded in the notation and the statements. These ideas had entered mathematical folklore to some extent but (Corry 1996) and the textbooks discussed there support Noether's claim that from Dedekind's time to hers the homomorphism theorem was rarely if ever put this well.

The gap widens around the isomorphism theorems. Noether said that the isomorphism theorems are first found 'in a somewhat more special conception' in Dedekind (Noether 1927, 41n). In fact, Noether there stated the theorems for 'modules' in our current sense—commutative groups acted on by a ring. Her term was 'Modulbereich' and would soon be 'Modul.' Dedekind had a far narrower sense of module, namely as a set of complex numbers closed under subtraction—that is an additive subgroup of the complex numbers. Notice that a module in this sense, if it contains more than 0, cannot be finite and so Dedekind's stated version of the homomorphism theorem cannot apply to it.

This could have been trivial. Dedekind remarked that the isomorphism theorems and other elementary results on modules apply to 'any group,' and his gloss shows that by this he means what we call any commutative group (Dedekind 1996, 82, cf. pp. 65–6). But it relates to a crucial difference: Dedekind did not use quotient groups for his isomorphism theorems because quotient groups are not modules in his sense. A coset of complex numbers is not a complex number. So, while he has the notion of a quotient group, he has a specific reason not to apply it here. For a similar reason (Noether 1921) does not form quotient modules. She does so by 1925.

Dedekind gave much weaker statements of the theorems than Noether. Let us adapt Dedekind's isomorphism theorems to commutative groups. He did not speak of isomorphisms at all but only counted the cosets of various subgroups. Given a commutative group H and subgroup K, let us define with Dedekind (H, K) to be the number of cosets of K in G if that is finite, and 0 otherwise.[21] Let M be another subgroup of G containing K, and I another that need not contain K. Dedekind wrote explicitly numerical equations:

$$(H, K) = (H, M).(M, K) \quad \text{and} \quad (I + K, K) = (I, I \cap K).$$

These simply say $0 = 0$ if any of the subgroups has infinitely many cosets. Noether formed quotient groups H/K of groups by subgroups and wrote group

---

[21] Compare (Dedekind 1930–1932, vol. 3, p. 76), and especially Dedekind's most mature version of the isomorphism theorems (Dedekind 1900, 382 and 384).

isomorphisms

$$(H/K)/(M/K) \simeq (H/M) \quad \text{and} \quad (I+K)/K \simeq I/(I \cap K).$$

These express a great deal whether the quotients are finite or infinite.

Even for finite groups Dedekind's arithmetic captures less information than group isomorphisms, as Dedekind knew. Finite groups may differ widely, while they have the same number of elements. So Dedekind's version is easier and more elegant for some applications, notably in arithmetic, while it is too weak for others. Dedekind never actually lost that information. It has a plain arithmetic meaning, he kept track of it in practice, and indeed all number theorists of the time kept track of such information in one way or another. But he did not express it in his theorems on groups or modules. Noether did.

Noether entirely reconceived the scope of the theorems and the relation between them. Let me be very clear that I do not say these further ideas have no precedent in Dedekind. Noether was famous for saying her ideas *were* somehow already in Dedekind. But she went on in ways that are never stated in Dedekind. They are not brought together as one theme in Dedekind's work, and no one before Noether saw them there.

Dedekind gave the homomorphism theorem as a way of constructing one finite group from another. He stated the isomorphism theorems as a way of counting cosets of his 'modules'—infinite additive subgroups of the complex numbers. Noether stated isomorphism theorems as dealing with isomorphisms and gave a uniform method of proving them from homomorphism theorems for many categories of structures—all groups, commutative groups, groups or commutative groups with a given domain of operators, all rings, commutative rings, rings with operators, and more. The 'set-theoretic' presentation quoted above (from (Noether 1927, note 27, p. 46–7)) shows she foresaw wide application. She later listed six kinds of examples, still not covering all that she used in practice, with an inconclusive tone that makes it clear she could not yet tell the reach of this idea (Noether 1929, 645–7). These examples were all *categories* in today's mathematical sense, with specified structures and specified homomorphisms as objects and arrows, although the term 'category theory' was not yet coined.

When Dedekind hinted at generalizing the homomorphism theorem he suggested just one generalization, which from his point of view was all-inclusive: the case of all groups. But Noether saw that, for example, the homomorphism theorem for rings is *not* included in that for groups. A ring homomorphism is *not* just a group homomorphism that happens to go between rings. It has further properties and the homomorphism theorem for rings addresses those properties.[22]

Noether was more aware of homomorphisms. She not only described algebraic structures 'up to isomorphism' but expressly described structures by way of homomorphisms. She wrote $\mathfrak{M} \sim \overline{\mathfrak{M}}$ where today we write $\mathfrak{M} \to \overline{\mathfrak{M}}$ and her

---

[22] Ring homomorphisms also preserve multiplication. So the homomorphism theorem for rings says onto ring homomorphisms correspond not to all additive subgroups of rings but specifically to ideals.

proofs were built around these homomorphisms. What categorists call 'arrow theoretic thinking' was 'tilde theoretic' in Noether. In fact, by a prescient typographical error one tilde in the first edition of van der Waerden *Moderne Algebra* was printed as an arrow (van der Waerden 1930, vol. 1, p. 33).

5 Blaricum/Laren 1926–27

By the 1920s Professor Brouwer was rarely in Amsterdam. He spent most of his time in Blaricum, then a town of under 3000 people some 15 miles east of Amsterdam. Blaricum adjoins the larger town of Laren, population some 6500 at that time, where Brouwer's friends would stay. The towns were an artistic and Bohemian center surrounded by heath and farmland. Brouwer and Noether had probably first met in Karlsruhe at a 1912 German Mathematical Union (DMV) meeting, and at any rate they were friends by 1919 (information from Dirk van Dalen).

In the academic year 1925–1926 all the promising young topologists of Europe visited Brouwer, including Hopf, Hurewicz, Menger, Ulam, and Urysohn. Of interest to us: Alexandroff arrived in May 1925 and Vietoris that summer. Both stayed through Winter Semester 1926. During this time Brouwer lectured on homology theory (Alexandroff 1969, 117). No trace of the lectures seems to have survived.

One of Brouwer's devices especially attracted Alexandroff, namely the idea of replacing a continuous curve in the Euclidean plane by a finite chain of small steps (Brouwer 1912). The effect of continuity is achieved by taking infinite sequences of such chains, where the lengths of the steps decrease to 0. Before arriving in Holland Alexandroff generalized this idea into a 'Foundation for *n*-dimensional set theoretic topology' (Alexandroff 1925*b*). Like Brouwer, Alexandroff assumes a continuous topological space in which the points will lie.[23] But for Alexandroff this continuous space is an abstract topological space, whereas Brouwer had looked at the real co-ordinate plane.

Intuitively, Brouwer approximated a curve by a finite chain of intervals, and treated an interval as merely a set of two endpoints. Alexandroff approximated a surface by a finite set of triangles (pasted together along their edges) and treated a triangle as merely a set of three vertices. He approximated a 3-dimensional space by tetrahedra pasted together, where a tetrahedron is a set of four vertices, and so on in higher dimensions.[24]

---

[23] This has nothing to do with eliminating continuity in favour of finite sets.

[24] Vietoris used these ideas for *Vietoris homology* that we are about to see (see also (Hocking and Young 1961, 346)). Eduard Čech would use them another way to get *Čech homology* (Hocking and Young 1961, 320ff.).

Alexandroff used this kind of idea to pursue theorems on topological images, that is on continuous onto maps $f : T \to T^*$.[25] These theorems also led to Alexandroff's theory of **continuous decompositions**:

> My theory of continuous decompositions of topological spaces was created in large part under the influence of my conversations with [Noether] in December and January of 1925–1926 when we were in Holland together.[26] (Alexandroff 1981, 108) cited (Corry 1996, 247)

Alexandroff wrote $T^* = f(T)$ to say f is an image, and said:

> Every map $T^* = f(T)$ determines a decomposition of the space T into disjoint subsets $X = f^{-1}(x^*)$ where x^* is any point of the image space T^*. If the map is continuous then all the sets X are closed.
> To approach this we can investigate a priori the decompositions $T = \Sigma X$ of a space into disjoint closed subsets X. (Alexandroff 1926, 556)

So f partitions T into disjoint closed subsets

$$f^{-1}(y) = \{x \in T \mid f(x) = y\}$$

for each $y \in T^*$. An onto group homomorphism $f : G \to H$ partitions G into disjoint cosets

$$K + y = \{x \in G \mid f(x) = y\}$$

for each $y \in H$, where $K \subseteq G$ is the subgroup of all x with $f(x) = 0$. The homomorphism theorem for groups says that the onto homomorphisms from G correspond uniquely (up to isomorphism) to the coset partitions of G. Alexandroff wanted to show that every topological image of T corresponds uniquely (up to isomorphism) to a *continuous decomposition* of T.[27] In fact he can do it only for certain kinds of images. 'Thus one sees that the concepts of continuous map and of continuous decomposition do not correspond to each other in a satisfactory way' (Alexandroff 1926, 559). A satisfactory way would have been a 'homomorphism theorem' for topology. Alexandroff does not use that name but neither did Noether in 1925 (published as (Noether 1927)).

Meanwhile:

> Emmy Noether arrived [in Holland] in mid December. She stayed the whole Christmas holiday in Blaricum and more. At this time she was thoroughly absorbed by her group theoretic lectures in Göttingen' (Alexandroff 1969, 120).

---

[25] To be explicit $f : T \to T^*$ is an image when each $y \in T^*$ has at least one $x \in T$ with $f(x) = y$.

[26] See (Alexandroff 1925*a*) and (Alexandroff 1926). The first was communicated to the Royal Academy in Amsterdam by Brouwer on 28 November 1925. The second is datelined 'Blaricum bei Amsterdam November 1925' and says 'The first, abstract part of this work is closely related to Frl. E. Noether's recent investigations in general group theory, and partly suggested by them' (Alexandroff 1926, 505). Alexandroff misremembered the timing, but it was Noether's influence.

[27] Precisely defined on (Alexandroff 1926, 557).

Alexandroff was absorbed in her ideas. And Vietoris was talking with Alexandroff if he did not talk with her himself.

Vietoris had a paper communicated to the Royal Academy of the Netherlands by Brouwer on 27 February 1926. In it he cited (Alexandroff 1925*a*) and (Alexandroff 1926) on images, and an 'oral communication' from Alexandroff on a theorem in topology. In a long footnote Vietoris mentioned a modified version of one theorem, plus 'the proof which was given to me orally in its main points, along with the theorem, by Prof. Brouwer' (Vietoris 1926*a*, 1008). In this paper, and its more detailed *Mathematische Annalen* version, both written in May 1926, Vietoris generalized Alexandroff's 'set-theoretic foundation' to all metric spaces, and made the first published reference to 'homology groups' beyond the single use by Veblen (Vietoris 1926*a*, 1010) and (Vietoris 1927, 457).

Like Alexandroff, Vietoris began with the by-then classical homology of polyhedra pasted together from triangles, tetrahedra, and so on. Then he depicted this inside any compact metric space R. In this sense a 'triangle' in R is any three points even if none can be connected to any other by a line in R, a 'tetrahedron' is any four points, and so on.

Since a triangle in R is just any three points it need not have any relation to the topology of R. But the topology of R does determine whether or not it is possible to *subdivide* a triangle $\{a, b, c\}$ in R into six triangles in R each with shorter sides than the original. Is there a 'centre point' with the right distance relations? The picture is obvious although the sides need not exist as lines in R. Vietoris used this to define **fundamental sequences** of polyhedra in R. These are infinite sequences of polyhedra where the side lengths converge to 0. The fundamental sequences in a space R reveal a great deal of its topology, and Vietoris transferred homology to R in this way. Compact metric spaces are much more general than polyhedra. A compact metric space R may have infinitely many holes in it and infinitely many twists. The homology groups need not be finitely generated, and so they reveal much more than the matrix presentations and arithmetic methods that worked for finitely generated groups.

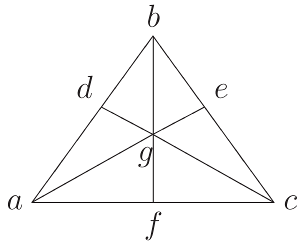

Fig. 2.

Besides that, Vietoris proved theorems on maps $f : M \to M'$. Even if M and M' have finite Betti numbers, the numbers are of little use in studying the map f. Topologists had always studied maps to some extent but in *ad hoc* ways. By treating f as a homomorphism of homology groups, Vietoris made a lot of apparently trivial details overtly trivial in a uniform way. So he cleared up access to the real information.

In correspondence with me (8.2.96) Vietoris emphasized his independence from Noether. She certainly left no trace of using homology groups. Vietoris is certainly the first to publish work with them and I believe he was the first to work with them in any serious way. Just as certainly, whatever the influences on him, Vietoris was original in creating this homology. But he did so in Blaricum/Laren, within months of Noether's visit, in constant contact with Alexandroff. Vietoris (Vietoris 1926*a*) grew from his (Vietoris 1926*b*), which is explicitly concerned with the same problems as (Alexandroff 1925*a*), which was explicitly inspired by Noether's group theory.

Personal conflicts complicate the picture. Alexandroff made a campaign of slighting (Vietoris 1927) as 'a direct result' of Brouwer's remarks.[28] He never mentioned Vietoris's homology groups. Hopf later said of his own 1928 work 'I do not actually know if the concept 'homology group' had ever before appeared in black and white in the literature' (Hopf 1966, 185). And yet in 1928 he must have known Vietoris's papers. Vietoris spoke on them with Alexandroff as chair at the 1926 Annual meeting of the DMV in Düsseldorf. (Vietoris, letter of 8.2.96, mentions that Noether was in Düsseldorf but he does not know if she heard his talk.)

Alexandroff and Hopf went to Princeton for the academic year 1927–28, on Rockefeller grants arranged by Weyl on Noether's advice. They heard lectures by Alexander, Veblen, and Lefschetz. Hopf said:

> By far the most important to us was Lefschetz—on one hand because he was Alexandroff's ally in the struggle for using algebraic methods in set theoretic topology, and on the other hand since my work, on fixed-point theorems was tied to his groundbreaking work. (Hopf 1966, 185)

Lefschetz was not to use homology groups for several more years. But Hopf became the second to work with them, the following summer. He first published what he called a 'generalized Euler–Poincaré formula' not using homology groups but with an added note:

> As this was being typeset, I found a way to fundamentally simplify the proof of the central theorem, Thm. I, specifically following a suggestion by Fräulein E. Noether. The simplified proof appears in (Hopf 1928) and acquaintance with that note will allow the reader to skip the first three sections of this work. (Hopf 1929, 494)

The simplified version using homology groups actually came out in print sooner, saying:

> I was able to put my original proof of this generalized Euler–Poincaré formula into fundamentally clearer simpler form during a series of lectures I gave in Göttingen in Summer 1928, by introducing group theoretic ideas under the influence of Fräulein E. Noether. (Hopf 1928, 127)

---

[28] (Alexandroff 1927, 552), (Alexandroff 1928, 324), (Alexandroff 1969, 119)

Hopf laid out the algebraic machinery so systematically that the whole idea becomes clear. He showed how to take any of several notions of cycle, and of boundary, and look at cycles modulo boundaries. Of course, the particular results depend on the particular definitions. The method stays the same.

6 The Rise of Functors

In less than a decade Noether's ideas were the textbook basis for homology (Seifert and Threlfall 1934) and (Alexandroff and Hopf 1935). All topologists knew n-dimensional boundaries were a subgroup of n-dimensional cycles, $B_n(T) \subseteq Z_n(T)$. Noether naturally used the homomorphism theorem to replace the subgroup inclusions by the corresponding homomorphisms

$$Z_n(T) \longrightarrow Z_n(T)/B_n(T) = H_n(T)$$

onto homology groups. A continuous map $f : T \rightarrow T'$ takes cycles in T to cycles in T', and boundaries to boundaries. Noether used the first isomorphism theorem to replace these relations among subgroups by homomorphisms between homology groups.[29] More fully, for each n, there is a homomorphism of n-dimensional homology groups

$$H_n(f) \ : \ H_n(T) \ \rightarrow \ H_n(T').$$

Brouwer's world of spaces and maps was interwoven with Noether's world of groups and homomorphisms. Two decades after Noether's work, each world would be called a *category* and the interweaving would be called a series of *functors*.

The event suggests two related points bearing on structuralism in the philosophy of mathematics: First, morphisms matter more than structures. Homology does not interweave group structure with topological structure.[30] It interweaves continuous maps and group homomorphisms. And, second, morphisms cannot be defined as structure-preserving functions. Many different classes of functions preserve different aspects, of topological structure—or *reflect* different aspects, which is not the same thing as preserving them. The continuous maps are those that reflect one particular aspect.

---

[29] Here was a glitch because these homomorphisms are generally not onto. Group theorists into the 1950s generally defined homomorphisms as onto, as Noether did in the 1920s. So Noether's students became the first to define homomorphisms as we do today, as in (Alexandroff and Hopf 1935). See (McLarty 1990, 355).

[30] The theory of continuous groups does that.

Brouwer's focus on maps $f: T \to T'$ ran against the trend of the time that focused on topological spaces T and subspaces $S \subseteq T$.[31] Yet Brouwer continued an older tradition. The 'Betti numbers' of surfaces were first defined in pursuit of Riemann's complex analysis where analytic functions were studied as continuous maps between Riemann surfaces before continuity on Riemann surfaces even had a clear definition.[32] Topologists long gave names to individual maps, as Alexandroff wrote $T^* = f(T)$ to say f is a continuous onto map. Even the arrow notation $f: T \to T^*$ came from topology (Mac Lane 1998, 29).

Algebraists focused very much more on individual groups, rings, etc. and their substructures than on homomorphisms. Dedekind went rather against this trend but we have seen that even he did not use a name for 'group homomorphisms,' let alone name specific homomorphisms $f: G \to H$. Noether named many distinct kinds of homomorphisms: homomorphisms of groups, of modules acted on by a group, etc. She still did not name specific homomorphisms or isomorphisms. She would write $\mathfrak{M} \sim \mathfrak{N}$ and say '$\mathfrak{M}$ is homomorphic to $\mathfrak{N}$;' or write $\mathfrak{M} \simeq \mathfrak{N}$ and say '$\mathfrak{M}$ is isomorphic to $\mathfrak{N}$.' This could suggest she thought of homomorphy and isomorphy as relations between structures, and that she merely defined structures 'up to isomorphism' as we say today.[33] But in fact she always referred to specific homomorphisms $f{:}\mathfrak{M} \sim \mathfrak{N}$ and isomorphisms $i{:}\mathfrak{M} \simeq \mathfrak{N}$ between structures. Specific homomorphisms and isomorphisms are tools in her proofs.

Algebraists looked to structures more than homomorphisms because the homomorphisms of algebra *are* defined as structure-preserving functions. A ring homomorphism $f: R \to R'$ is defined as a function that carries the operations and constants of R forward to those of R':

$$f(0_R) = 0_{R'} \quad f(1_R) = 1_{R'}$$
$$f(x +_R y) = f(x) +_{R'} f(y) \quad f(x \cdot_R y) = f(x) \cdot_{R'} f(y).$$

The same holds for other kinds of homomorphisms and so it could seem that structure was more basic than morphism.

Continuous maps $g: T \to T'$ are more subtle. They do not preserve distance—they can famously 'stretch or contract' parts of a space as in 'rubber sheet geometry.' For a time, notably around Riemann's time, it seemed continuous

---

[31] See much of Poincaré and *inter alia* (Schoenflies 1900), (Dehn and Heegaard 1907), (Schoenflies 1913), (Hausdorff 1914).

[32] See notably (Weyl 1913). Weyl has much to say about the history of the subject and especially Brouwer's role.

[33] It is wide of the mark to say that mathematicians today treat each isomorphism as an identity. Galois theory measures precisely how many ways an isomorphism can fail to be an identity. The practice of defining objects up to isomorphism is more subtle and particularly requires keeping track of morphisms.

maps could be defined as those that preserve limits of infinite series. But already by 1900 more general spaces were in use where that definition did not work. When 1920s topologists converged on the now standard definition, it did not say continuous functions *preserve* anything.

Today a topological space is a set T with certain subsets $U \subseteq T$ called **open**, and the **continuous** functions $g: T \to T'$ reflect open subsets. That is, they carry open subsets of T' back to open subsets of T:

$$\text{If} \quad V \subseteq T' \quad \text{is open in } T' \text{ then the}$$
$$\text{inverse image} \quad f^{-1}V \subseteq T \quad \text{is open in T.}$$

Here structure was not prior to morphisms. The open set structure was first identified *because* the desired morphisms reflect it. Preserving it is another matter: maps that preserve the open sets are called **open**, and they are also used in topology but are not the morphisms of most interest; the continuous maps are. Long before these issues were clear, topologists had to be clear about which maps they used.

Noether emphasized homomorphisms, and her influence on homology forced topologists and algebraists to bring their methods together. So algebra shifted to an ever-greater focus on homomorphisms. All this algebra looked too abstract to a tough-minded geometer like Solomon Lefschetz. In the 1920s he made tremendous but obscure progress applying homology in algebraic geometry. It was a major source of his reputation for never stating a false theorem or giving a correct proof. Realizing that his arguments needed serious improvement he went to work on the homology of topological spaces. His first book on it says:

> The connection with the theory of abstract groups is clear. . . . Indeed everything that follows in this section can be, and frequently is, translated in terms of the theory of groups. It is of course a mere question of a different terminology. (Lefschetz 1930, 29)

But Poincaré had said, in promoting *analysis situs*, that we must not 'fail to recognize the importance of well constructed language' (Poincaré 1908, 180). Whether or not Lefschetz noticed this passage he shared the thought.

Soon his topology was heavily algebraic (Lefschetz 1942). He kicked off the spate of articles on what became category theory when he asked for an appendix by Eilenberg and Mac Lane (Eilenberg and Mac Lane 1942). This became the standard foundation for algebraic topology and for the huge proliferation of (co-homology theories in the 1950s and since. The unprecedentedly vast machinery gave unprecedented power in solving concrete problems from topology to abstract algebra to number theory. Lefschetz wrote:

> As first pointed out by Emmy Noether, the proper and only adequate formulation of the relation between chains, cycles, . . . requires group theory. (Lefschetz 1949, 11, Lefschetz's ellipsis)

In fact it required Noether's formulation of group theory, and that soon required functors.

7 Appendix: Poincaré on groups in homology

Two basic kinds of evidence show that Poincaré knew of homology groups. First, his homology uses the textbook methods of commutative group theory. Second is his own comparison of homology to **fundamental groups** in homotopy questions. He worked on fundamental groups by a textbook method for producing new groups and he showed the result was homology. He said the rules of homology differ from the homotopy group rules only in that homology is commutative. He knew that his friends, and the textbooks of the time, spoke of 'commutative groups' but he preferred not to call something a group if it was commutative.

Poincaré used algorithms with linear forms and matrices to calculate with homology cycles throughout (Poincaré 1899–1904). This was the standard presentation of finitely generated commutative groups at least since (Frobenius and Stickelberger 1879). Poincaré's algorithm for calculating Betti numbers was Frobenius and Stickelberger's algorithm for calculating group invariants (Poincaré 1899–1904, Complément 2) and (Frobenius and Stickelberger 1879, section 10).[34] All these methods are in the standard textbook (Weber 1899).

Poincaré defined the **fundamental group** of a space U, also called the **first homotopy group** of U today, as the group of paths in U from a fixed base point x_0 back to x_0. Two paths are considered the same if they have a *homotopy* between them: each can be continuously deformed into the other while keeping the base point fixed. The product of one path with another is just the first followed by the second (Poincaré 1895, 239). This is the textbook definition today. He quickly proved it isomorphic to a transformation group permuting the values of integrals or multivalued functions on U. He often described the fundamental group of a given space by generators and relations, which was then the standard presentation of finitely generated (not necessarily commutative) groups. He would find a short list of paths that generate the group (every path is homotopic to some product of ones on the list), and a short list of relations (a few homotopies among products of these paths), which by general group theory imply all the homotopies that hold in the group.

He used fundamental groups to calculate Betti numbers. Given generators and relations for the fundamental group of a space, he would adjoin new relations making the generators commute, then count how many generators remained linearly independent, and that number was the 1-dimensional Betti number (e.g. (Poincaré 1895, 243ff.)).[35] He knew the standard textbook method of his time: If you present a group by generators and relations, and impose further

---

[34] The algorithm was the elementary divisor theorem, which Noether proposed to eliminate from topology by her group theory, as discussed in Section 6.

[35] In later terms, he used the Hurewicz isomorphism of the first homology group with the Abelianization of the fundamental group.

relations you get a new group. But if the new relations imply commutativity Poincaré preferred not to call it a group.

He contrasted homotopy to 1-dimensional homology in precisely these terms: 'in homologies the terms compose according to the rules of ordinary addition; in [the fundamental group] the terms compose according to the same rules as the substitutions of a group.'[36] By the rules of ordinary addition he meant associativity, commutativity, existence of zero and of negatives. He knew these were the 'commutative group' rules. Yet he reserved the term 'group' for groups of substitutions, as in the quote, and he would not call even a substitution group by that name if he knew it was commutative.[37]

References

Alexandroff, Paul (1925*a*), 'Über stetige Abbildungen kompakter Räume', *Proceedings of the Section on Sciences, KNAW* **28**, 997–99.

Alexandroff, Paul (1925*b*), 'Zur Begründung der n-dimensionalen mengentheoretischen Topologie', *Mathematische Annalen* **94**, 296–308.

Alexandroff, Paul (1926), 'Über stetige Abbildungen kompakter Räume', *Mathematische Annalen* **96**, 555–71.

Alexandroff, Paul (1927), 'Review of Vietoris 1927', *Jahrbuch über die Fortschritte der Mathematik* **53**, 552.

Alexandroff, Paul (1928), 'Zusammenhangszahlen abgeschlossener Mengen', *Nachrichten, Gesellschaft der Wissenschaften zu Göttingen* pp. 323–29.

Alexandroff, Paul (1932), *Einfachste Grundbegriffe der Topologie*, Julius Springer, Berlin. Translated as *Elementary Concepts of Topology*. New York: Dover, 1962.

Alexandroff, Paul (1969), 'Die Topologie in und um Holland in den Jahren 1920-1930', *Nieu Archief voor Wiskunde* **XXVII**, 109–27.

Alexandroff, Paul (1981), *In memory of Emmy Noether*, Marcel Dekker, New York, pp. 99–114. This 1935 eulogy at the Moscow Mathematical Society is also in N. Jacobson ed. *Emmy Noether Collected Papers*, Springer Verlag, 1983, 1–11; and Dick 1970, 153–80.

Alexandroff, Paul and Heinz Hopf (1935), *Topologie*, Julius Springer, Berlin. Reprinted 1965, New York: Chelsea Publishing.

Brouwer, Luitzen (1911), 'Über Abbildungen von Mannigfaltigkeiten', *Mathematische Annalen* **71**, 97–115. Reprinted as 1911D in Arendt Heyting ed. *Brouwer, Collected Works*, North Holland, 1975, vol. 2, pp. 454–74.

---

[36] (Poincaré 1899–1904, 449–50), see also (Poincaré 1895, 241ff) and (Poincaré 1983, 325).

[37] (Poincaré 1901) talked about the '*groupe*' of rational points on an elliptic curve and today we know these points form a finitely generated Abelian group. But Poincaré's operation did not make the curve a group in the sense of algebra. It was not associative and had no neutral element except in trivial cases. Today we alter his operation to get a group in the usual sense. See (Schappacher and Schoof 1996, 62–3).

Brouwer, Luitzen (1912), 'Beweis der Invarianz der geschlossen Kurve', *Mathematische Annalen* **72**, 422–25. Reprinted as 1912L in Arendt Heyting ed. *Brouwer, Collected Works*, North Holland, 1975, vol. 1, pp. 523–26.

Byers, N. (1996), The life and times of Emmy Noether, *in* H.Newman and T.Ypsilantis, eds, 'History of original ideas and basic discoveries in particle physics', Plenum Press, New York, pp. 945–64.

Corry, Leo (1996), *Modern Algebra and the Rise of Mathematical Structures*, Birkhäuser.

Dedekind, Richard (1900), 'Über die von drei Moduln erzeugte Dualgruppe', *Mathematische Annalen* **53**, 371–403.

Dedekind, Richard (1930–1932), *Gesammelte mathematische Werke*, Friedr. Vieweg & Sohn, Braunschweig. Three volumes.

Dedekind, Richard (1996), *Theory of Algebraic Integers*, Cambridge University. Originally published in French 1877.

Dehn, Max and Paul Heegaard (1907), Analysis situs, *in* 'Encyklopadie der mathematischen Wissenschaften III A B **3**', B.G. Teubner, Leipzig, pp. 153–220.

Dieudonné, Jean (1984), 'Emmy Noether and algebraic topology', *J. Pure Appl. Algebra* **31**, 5–6.

Eilenberg, Samuel and Saunders Mac Lane (1942), Appendix A: On homology groups of infinite complexes and compacta, *in* 'Algebraic Topology', American Mathematical Society, pp. 344–49.

Ferreirós, José Domínguez (2005), Notice on Dedekind and Peano, *in* I.Grattan-Guinness, ed., 'Landmark Writings in Western Mathematics 1640–1940', Elsevier, pp. 613–626.

Frobenius, Ferdinand and Ludwig Stickelberger (1879), 'Ueber Gruppen von vertauschbaren Elementen', *Journal für die reine und angewandte Mathematik* **86**, 217–62.

Hausdorff, Felix (1914), *Grundzuge der Mengenlehre*, Von Veit, Leipzig.

Herreman, Alain (2000), *La topologie et ses signes: Éléments pour une histoire sémiotique des mathématiques*, L'Harmattan, Paris.

Hilbert, David and S. Cohn-Vossen (1932), *Anschauliche Geometrie*, Julius Springer, Berlin. Translated to English as *Geometry and the Imagination*, New York: Chelsea Publishing, 1952.

Hocking, John and Gail Young (1961), *Topology*, Addison-Wesley.

Hopf, Heinz (1928), 'Eine Verallgemeinerung der Euler-Poincaréschen Formel', *Nachrichten, Gesellschaft der Wissenschaften zu Göttingen* pp. 127–36.

Hopf, Heinz (1929), 'Über die algebraische Anzahl von Fixpunkten', *Mathematische Zeitschrift* **29**, 493–524.

Hopf, Heinz (1966), 'Eine abschnitt aus der Entwicklung der Topologie', *Jahresberichte der DMV* **68**, 182 (96)–192 (106). The same lecture is printed as 'Einige Persönliche Erinnerungen aus der Vorgeschichte der heutigen Topologie', *Colloque de Topologie Tenu a Bruxelles Centre Belge de Recherches Mathématique*, 9–20. Paris: Gauthier Villars, 1966.

Jacobson, Nathan, ed. (1983), *E. Noether: Gesammelte Abhandlungen*, Springer Verlag.

Krull, Wolfgang (1935), *Idealtheorie*, Julius Springer, Berlin.

Lang, Serge (1993), *Algebra*, Addison-Wesley.

Lefschetz, Solomon (1930), *Topology*, American Mathematical Society, Providence.

Lefschetz, Solomon (1942), *Algebraic Topology*, American Mathematical Society, Providence.

Lefschetz, Solomon (1949), *Introduction to Topology*, Princeton University Press.

Mac Lane, Saunders (1938), 'A lattice formulation for transcendence degrees and p-bases', *Duke Mathematical Journal* **4**, 455–68.

Mac Lane, Saunders (1978), 'Origins of the cohomology of groups', *Enseignement Mathématique* **24**, 1–29.

Mac Lane, Saunders (1986), 'Topology becomes algebraic with Vietoris and Noether', *J. Pure Appl. Algebra* **39**, 305–307.

Mac Lane, Saunders (1988), 'Group extensions for 45 years', *Mathematical Intelligencer* **10**, 29–35.

Mac Lane, Saunders (1998), *Categories for the Working Mathematician*, 2nd edn, Springer-Verlag.

McLarty, Colin (1990), 'The uses and abuses of the history of topos theory', *British Journal for the Philosophy of Science* **41**, 351–75.

Noether, Emmy (1921), 'Idealtheorie in Ringbereichen', *Mathematische Annalen* **83**, 2466. In (Jacobson 1983), 354-96.

Noether, Emmy (1924), 'Abstract of a talk "Abstrakter Aufbau der Idealtheorie im algebraischen Zahlkörper"', *Jahresbericht DMV* **33**, 104. In (Jacobson 1983), 482.

Noether, Emmy (1925), 'Abstract of a talk "gruppencharaktere und idealtheorie"', *Jahresbericht DMV* **34**, 144. In (Jacobson 1983), 484.

Noether, Emmy (1926), 'Abstract', *Jahresbericht DMV* **34**, 104.

Noether, Emmy (1927), 'Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionenkörpern', *Mathematische Annalen* **96**, 26–91. I cite this from (Jacobson 1983). 493–528.

Noether, Emmy (1929), 'Hyperkomplexe Grössen und Darstellungstheorie', *Mathematische Zeitschrift* **30**, 641–92. I cite this from (Jacobson 1983), 563–614.

Noether, Emmy and Jean Cavaillès, eds (1937), *Briefwechsel Cantor-Dedekind*, Hermann, Paris.

Ore, Öystein (1935), 'On the foundations of abstract algebra, I', *Annals of Mathematics* **36**, 406–37.

Poincaré, Henri (1895), 'Analysis situs', *Journal de l'école Polytechnique* **1**, 1–121. Reprinted in Oeuvres vol. 6, 193–288.

Poincaré, Henri (1899–1904), 'A series of five "compléments à l'analysis situs"'. Extending and correcting Poincaré 1895. Collected in *Oeuvres* vol. 6, 289–498.

Poincaré, Henri (1901), 'Sur les propriétés arithmétiques des courbes algébriques', *Journal des Mathématiques* **7**, 161–233. Reprinted in *Oeuvres* V, 483-550.

Poincaré, Henri (1908), L'avenir des mathématiques, *in* 'Atti del IV Congresso Internazionale dei Matematici', Accademia dei Lincei, pp. 167–182. Translated in heavily edited form as 'The future of mathematics' in G. Halstead ed. 1921: *The Foundations of Science*. New York: The Science Press, 369–82.

Poincaré, Henri (1916-1956), *Oeuvres de Henri Poincaré*, Gauthier-Villars. In 11 volumes.

Poincaré, Henri (1983), Analyse de ses travaux scientifiques, *in* F.Browder, ed., 'The mathematical heritage of Henri Poincaré', Vol. 2, American Mathematical Society, pp. 257–357. A description of his own work written in 1901 for Mittag Lefler. Published in *Acta Mathematica* **38** (1921) 36–135.

Pont, Jean-Claude (1974), *La topologie algébrique des origines à Poincaré*, Presses Universitaires de France, Paris.

Sarkaria, K. (1999), the topological work of Henri Poicaré, *in* I. James, ed., 'History of Topology', Elsevier, pp. 123–68.

Schappacher, Norbert and René Schoof (1996), 'Beppo Levi and the arithmetic of elliptic curves', *Mathematical Intelligencer* **18**, 57–69.

Schoenflies, Arthur (1900), 'Die Entwicklung der Lehre von den Punktmannigfaltigkeiten', *Jahresberichte DMV* **82**, 1–250.

Schoenflies, Arthur (1913), *Entwickelung der Mengenlehre und ihrer Anwendungen*, B.G Teubner, Leipzig.

Seifert, H. and W. Threlfall (1934), *Lehrbuch der Topologie*, Teubner, Leipzig. Translated as *A Textbook of Topology*. New York: Academic Press, 1980.

van der Waerden, Bartel L. (1930), *Moderne Algebra*, J. Springer, Berlin.

Vanden Eynde, Ria (1992), 'Historical evolution of the concept of homotopic paths', *Archive for History of Exact Sciences* **29**, 127–88.

Veblen, Oswald (1922), *Analysis Situs*, American Mathematical Society, New York.

Vick, James (1994), *Homology Theory: An Introduction to Algebraic Topology*, Springer-Verlag.

Vietoris, Leopold (1926*a*), 'Ueber den höheren Zusammenhang von kompakten Räumen', *Proceedings of the Section on Sciences, KNAW* **29**, 1008–13.

Vietoris, Leopold (1926*b*), 'Ueber stetige Abbildungen einer Kugelfläche', *Proceedings of the Section on Sciences, KNAW* **29**, 443–53.

Vietoris, Leopold (1927), 'Ueber den höheren Zusammenhang von kompakten Räumen', *Mathematische Annalen* **97**, 454–72.

Weber, Heinrich (1899), *Lehrbuch der Algebra*, 2nd edn, F. Vieweg und Sohn, Braunschweig.

Weyl, Hermann (1913), *Die Idee der riemannschen Fläche*, B.G. Teubner, Leipzig, Berlin.

Weyl, Hermann (1923), 'Análisis situs combinatorio', *Revista Matematica Hispano-Americana* **5**, 209–18, 241–48, 278–79. I cite this from (Weyl 1968) vol. 2, 390–415.

Weyl, Hermann (1939), 'Invariants', *Duke Mathematical Journal* **5**, 489–502. Also in (Weyl 1968) vol. 3, 670–83.

Weyl, Hermann (1968), *Gesammelte Abhandlungen*, Springer-Verlag.