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SAUNDERS MAC LANE AND THE UNIVERSAL IN MATHEMATICS

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> The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features.
>
> Eliakim Hastings Moore (1910, p. 1)

There is no place to begin to describe the achievements of Saunders Mac Lane. Over his long career he linked every achievement to every other. It was not so at first. During his first 10 years as a mathematician, while he worked in many subjects, the unity was more abstractly philosophical and logical than mathematical. It was based more on the philosophy behind Russell and Whitehead's *Principia Mathematica* than on experience in geometry, algebra, or analysis. Yet he gained experience. Then came his collaboration with Samuel Eilenberg on one specific problem. They worked to explain how some pure algebra by Mac Lane had arrived at the cohomology of the p-adic solenoid, an infinitely tangled compact metric space. To do this they created category theory. They organized the functorial basis for homology and cohomology in topology. And they created group cohomology in its full functorial form.

Over the next 60 years these tools gave precise mathematical unity to Mac Lane's work. His 1930s work in arithmetic algebraic geometry took on new importance in light of his homological algebra of the 1950s. These ideas joined with his earliest interest in logic in his last mathematical book (Mac Lane & Moerdijk 1992). Mac Lane expressed all of this experience, plus what he learned from Emmy Noether and Hermann Weyl when he was a student in the last great days of David Hilbert's Göttingen, in his reflections on "mathematical truth and beauty" (Mac Lane 1986, p. 409).

As an undergraduate at Yale University from 1926 to 1930 Mac Lane was attracted to the new symbolic logic, especially *Principia Mathematica*, and he studied Hausdorff's set theory and topology. He also learned that Emmy Noether was producing great new mathematical results by thinking about the foundations of algebra although he did not yet learn much of her work. As a graduate student at the University of Chicago he was deeply impressed by the founder of the mathematics department, Eliakim Hastings Moore. Moore followed the Chicago tradition by taking a strong interest in education as well as research. Mac Lane would do the same. Moore had an ambitious project to draw the greatest benefits for advanced mathematics from the new logic and set theory. He created a "form of general analysis" using Peano's logical concepts and Cantor's theory of ordered point sets to simplify and unify theorems from many parts of analysis: multi-variable calculus, calculus of variations in one variable, and infinite dimensional vector spaces. This work led Moore to state his principle on analogies and general theories quoted above.

Mac Lane tells how Moore assigned him Zermelo's paper on an axiomatic proof of the well-ordering theorem using the axiom of choice. Mac Lane explained the paper to a seminar. Afterwards:

> Moore took me aside and explained what the paper was really all about and what I should have said. That was an occasion on which I learned a great deal thanks to Professor Moore: I learned from him how to give a talk on mathematics, and I learned about sets as a foundation for mathematics. He was an amazing professor. (Mac Lane 2005, p. 37)

Yet there was no way to earn a doctorate in logic at Chicago. Moore had studied in Germany. Hilbert was developing logic there as Noether was developing her algebra. Mac Lane went to Göttingen.

There he found two difficulties. Göttingen mathematicians were less interested in logic than he had hoped. Still he got his doctorate with a dissertation on how to simplify formal logical proofs and make them more usable in practice. He also found Noether a confusing lecturer because she was creating her ideas as she spoke. He said he did not understand her ideas of "factor sets."

Within a few years, though, Mac Lane and O.F.G. Schilling made factor sets basic to their search for a non-Abelian class field theory. The two also worked together on valuations and factorization in algebraic number theory and algebraic geometry. Perhaps Mac Lane thought of Moore's principle. Certainly he believed the deepest analogy between central parts of arithmetic and geometry lay in the theory of valuations. He wanted to find the general theory. He promoted Lefschetz's idea that algebraic geometry could avoid analytic methods, and so work over other fields than the complex numbers, and yet still treat important results like the Riemann-Roch theorem in classically geometrical terms. He promoted some of Zariski's ideas as progress in this direction. None of this work is known today, for an obvious reason: It was all swept away by modern functorial cohomology and the *Abelian category* methods that Mac Lane introduced.[1]

The arithmetic algebraic geometry was not enough to keep Mac Lane busy. From 1934 to 1942 he wrote on many aspects of fields of finite characteristic, on the topology of graphs embedded in the plane (or sphere), on combinatorics and projective geometry, more than two dozen reviews in logic for the *Journal of Symbolic Logic*, and the famous algebra textbook (Birkhoff & Mac Lane 1941).[2] Through it all Mac Lane affirmed that abstraction serves concrete ends:

> The rapid exploitation of the new techniques of abstract algebra and the introduction of many new types of algebraic systems, though perhaps superficially confusing, has actually centered about several

[1](Mac Lane & Schilling 1939*a*, Mac Lane & Schilling 1939*b*, Mac Lane & Schilling 1940). For valuations, Lefschetz and Zariski see (Mac Lane 1939, pp. 4–6). Mac Lane introduced the term "Abelian category," and the germ of the idea which Grothendieck later took much farther, in (Mac Lane 1948, Mac Lane 1950).

[2]He wrote one philosophy article based on his dissertation (Mac Lane 1935).

> well defined lines of investigation: function-fields, linear algebras, p-adic fields, Lie algebras, matrices. (Mac Lane 1939, p. 3)

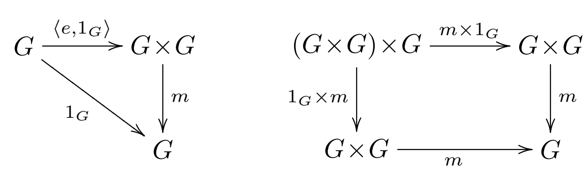
Rather than recount the often told collaboration with Eilenberg, let us focus on the most famous lemma in category theory. Many aspects of Mac Lane's thought are captured in the history and the mathematics of this result. Mac Lane was passionate about organizing and building the knowledge of category theory. He knew Nobuo Yoneda's work in homology and so when they met in Paris Mac Lane eagerly talked with him about his wider, unpublished perspective on the methods. Mac Lane's care as a historian of mathematics shows in his account of learning this lemma from Yoneda on a platform of the Gare du Nord waiting for Yoneda's train (Mac Lane 1998*b*).

In its simplest form the Yoneda lemma says every representable functor h_X on a category $\mathbf{C}$ has a universal element, namely the identity arrow 1_X on the representing object X. But this is only the beginning. Trivially, any natural transformation $\nu : h_X \to \mathbf{F}$ is determined by its value $\nu_X(1_X)$ at the universal element which can be any element of $\mathbf{F}(X)$. So natural transformations $h_X \to \mathbf{F}$ correspond to elements of $\mathbf{F}(X)$. The dual calculation shows this is natural in X so that $h_{(\_)}$ is a full and faithful functor from $\mathbf{C}$ to the category of contravariant set-valued functors on $\mathbf{C}$. The two naturalities together say every contravariant set-valued functor is a canonical colimit of representable functors (Mac Lane & Moerdijk 1992, p. 41).

Here is a case of the "protean nature" of mathematics which so appealed to Mac Lane (Mac Lane 1996). Essentially one fact takes three forms. But there is a catch. You must ignore foundations or else find some foundation more sophisticated than standard set theory to apply the lemma to large categories $\mathbf{C}$, that is categories with a proper class of objects such as all groups or all topological spaces. Mac Lane was adept at ignoring foundations as well as finding sophisticated ones. He saw the role of both already in the first paper on general category theory (Eilenberg & Mac Lane 1945, p. 246).

To say the least, the Yoneda lemma expresses the wide scope of categorical thinking as Mac Lane saw it in the 1950s. It applies to every category and lies at the confluence of universal elements, representable functors, and adjunctions. It appears in all advanced technical applications: for Mac Lane in the 1950s that meant primarily homological algebra and homotopy. But more, the lemma makes a quick link between advanced technical applications and the wider more fundamental uses of categories that Mac Lane promoted in the 1960s especially after meeting William Lawvere.

The lemma is central to the diagrammatic thinking which became natural and even delightful to Mac Lane. One powerful intuition says a *group* in any category $\mathbf{C}$ is an object G equipped with a unit $e : 1 \to G$, multiplication $m : G \times G \to G$, and group inverse $(\_)^{-1} : G \to G$ making the relevant diagrams commute as for example these express the left unit law and associativity of multiplication:



$e \cdot x = x$ $\qquad\qquad$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Another intuition says $G, e, m, (\_)^{-1}$ is a group in $\mathbf{C}$ if the set-valued representable functor h_G takes each object of $\mathbf{C}$ to a group in a natural way. The Yoneda lemma is the proof that the two intuitions agree.

Mac Lane's great textbook on categories shows how every concept of universality is equivalent to every other: universal elements are terminal objects, limits are universal elements, adjoints are limits, while Kan extensions are Yoneda ends, and both are adjoints, always supposing we can construct the categories and functors used to recast each universal as the other (Mac Lane 1998*a*, esp. chapter X). And the most powerful calculating devices of mathematics are clarified by using universals, as for example spectral sequences in his great textbook on homology (Mac Lane 1963, chapter XI). The theory is born out of vast specific calculations, as for example calculating the cohomology of Eilenberg-Mac Lane spaces (Eilenberg & Mac Lane 1986). Mac Lane's early work in class field theory, algebraic geometry, and algebraic number theory went into his collaboration with Eilenberg, and has been entirely surpassed since then by Tate, Serre, Grothendieck and others using the tools that Eilenberg and Mac Lane produced. Mac Lane's student vision of an easy, agile logic producing great mathematics, has become the reality.

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