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FAILURE OF CARTESIAN CLOSEDNESS IN NF

COLIN MCLARTY

On any reasonable definition of functions, neither the category of sets nor the category of small categories is cartesian closed in New Foundations (NF). The latter category is sometimes proposed as a foundation for category theory since it is among its own objects. Our result shows it is a poor one.

In NF, as in other set theories, a "function" f from a set A to a set B is defined to be a set f of ordered pairs $\langle x, y\rangle$ with x in A and y in B, such that (a) if $\langle x, y\rangle \in f$ and $\langle x, y'\rangle \in f$ then $y = y'$, and (b) for every x in A there is some y in B with $\langle x, y\rangle \in f$. But in NF different definitions of ordered pairs give significantly different functions. I say a *reasonable* definition must give:

1. The formula $z = \langle x, y\rangle$ is stratifiable.
2. For every set S there is a set $\{\langle x, x\rangle \mid x \in S\}$.
3. If f is a function from A to B, and g one from B to C, there is a set $\{\langle x, z\rangle \mid (\exists y)\langle x, y\rangle \in f \;\&\; \langle y, z\rangle \in g\}$.

Principles 2 and 3 are needed for identity functions and composites. By principle 1, any sets A and B have a set $A \times B$ of all ordered pairs $\langle x, y\rangle$ with x in A and y in B, but it does not follow that functions exist making $A \times B$ a categorical product of A and B. The principles do imply that, for any functions f and g both from A to B, there exist a set $\{x \mid x \in A \;\&\; f(x) = g(x)\}$ and functions making it an equalizer of f and g.

Principles 1–3 imply subsets have monic inclusion functions, the restrictions of identities; so every set has a monic to the set V of all sets. They also imply that subsets of B have preimages along any function to B and these are pullbacks of the inclusions along that function. Further, the set $\{\varnothing, \{\varnothing\}\}$ is a subset classifier, as any subset S of a set A has a classifying function defined by

$$\{\langle x, y\rangle \mid x \in A \;\&\; (x \notin S \rightarrow y = \varnothing) \;\&\; (x \in S \rightarrow y = \{\varnothing\})\}$$

stratifiable by principle 1 (define away the occurrences of $\varnothing$). Yet the category of sets is not a topos:

LEMMA. *Only a trivial topos has an object T such that every object has a monic to T.*

PROOF. If the power object Ω^T has a monic i to T, then define the subobject R of T containing all $x \in T$ such that no subobject S of T has $i(S) = x$ and $x \in S$. Then both $\sim(i(R) \in R)$ and $i(R) \in R$ are internally true, so the topos is trivial. ∎

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The lemma uses the internal language of a topos, not set-membership in NF, so neither stratifiability nor the relation of power objects to NF powersets is an issue.

Thus the category of sets is not cartesian closed. If it has products it lacks exponentials. In fact, on Kuratowski's or Quine's definition of ordered pairs the set $A \times B$ is a categorical product of A and B and there is a set B^A of all functions from A to B. But then, if there is an evaluation function ev: $B^A \times A \to B$ with $\mathrm{ev}(f, x) = f(x)$, every function to B^A has a transpose. Thus not all sets A and B have such functions.

Each definition of functions determines a category of small categories and functors. Taking sets as discrete categories shows that, for reasonable definitions of ordered pairs as described above, the category of small categories is not cartesian closed.

DEPARTMENT OF PHILOSOPHY
CASE WESTERN RESERVE UNIVERSITY
CLEVELAND, OHIO 44106