

BOOK REVIEWS

Mathematics: Form and Function. SAUNDERS MACLANE. New York: Springer-Verlag, 1986. 476 p. \$42.00.

This book, by an eminent mathematician and coinventor of category theory, urges richer interaction between the philosophy and the practice of mathematics than we have seen in fifty years. It illustrates as much as states a philosophy of mathematics, and it lays out the background MacLane believes the philosophy of mathematics requires. He makes high demands, and has a high opinion of what philosophy of mathematics could be if it drew on the actual wealth of mathematics. He finds too many theories of foundations “are anchored almost exclusively in the most elementary parts of Mathematics—numbers and continuity” (4), and deprecates Wittgenstein’s work “where the actual mathematical content rarely rises above third grade arithmetic, while the actual concern is less with Mathematics than with its use to illustrate some purely philosophical issue” (444).

So most of the book presents various branches of mathematics. These clear, extensive descriptions include well motivated, nearly rigorous proofs for an array of major theorems from linear algebra to complex analysis. The book is not entirely self-contained. MacLane assumes the reader “has at hand some of his own familiar texts for possible reference” (5). But a reader who knows a bit of set theory, groups, linear algebra, and calculus will learn a great deal; and one who knows more will learn more.

The book may attract philosophers by offering the most accessible introduction to yet category theory and topoi. This section has no special prerequisites, although the examples draw on other subjects. MacLane gives detailed definitions of the basic tools of category theory and of all the terms used in the topos axioms. He gives examples of topoi based on sets. He does not advocate the radically categorial foundations of mathematics associated with F. William Lawvere, who has worked with him.

MacLane’s basic example of a topos is the class of all sets posited by ZF set theory with Choice, plus the functions between them. But, as he notes, a general topos need not be much like a universe of classical sets. So he immediately adds axioms for a well-pointed topos with natural numbers and Choice. These axioms are essentially a categorial version of Zermelo set theory, and MacLane offers them as alternative foundations for mathematics. He says this approach “has nowhere been carried out in great detail” (402), but this is so

close to set-theoretic foundations that there is a fairly routine translation.¹ He ignores other extensions of the topos axioms which are inconsistent with classical set theory but give neatly tailored foundations for specific branches of mathematics, such as axioms for synthetic differential geometry.²

MacLane is also less committed than Lawvere to categorial methods in the practice of mathematics. He takes the fact that "categories and functors are everywhere in topology and parts of algebra, but they do not yet relate well to most of analysis" (407) as showing a weakness in category theory. Lawvere takes the same fact as showing the infelicity of those parts of analysis most affected by set-theoretic technicalities. Lawvere does not take categorial foundations as the last word either, but as a more powerful latest word than does MacLane.

MacLane is nevertheless a major architect of the categorial attitude. It shows throughout his survey of mathematics, even in the wide range of material he takes as essential to philosophy of mathematics. If mathematics were only arithmetic and basic calculus there would be no use for categories or categorial foundations. Samuel Eilenberg and MacLane invented category theory in the early 1940s as a tool for homology theory. Many central theorems of topology are proved by assigning to each topological space various groups, called its *homology groups*. In 1940 this powerful method was accessible only to insiders, and its basis was obscure even to them. Category theory clarified the situation, so homology theory became accessible to any serious mathematician and was itself fruitfully extended. Category theory arose from a practical need to relate two great subjects: group theory and topology, a need remote from traditional interests of philosophers of mathematics.

Chapter Five typifies MacLane's style and purpose in surveying mathematics. It begins with conventional set-theoretic treatment of functions plus a mention of category theory, then passes through transformation groups to a clear five-page sketch of Galois theory. It describes some constructions on groups and then the finite simple groups. The final section asks why group theory is so rich in comparison to other axiomatic theories. This section draws on all the themes I have listed and more; yet it can be read as quite banal unless the reader actively applies it to the earlier material. MacLane only opens the question and gives a framework for answers.

¹ This should be clear from the discussion of the category of sets in W. S. Hatcher, *The Logical Foundations of Mathematics* (New York: Pergamon Press, 1982).

² Synthetic differential geometry was conceived by Lawvere and can be seen in A. Kock, *Synthetic Differential Geometry* (New York: Cambridge, 1981).

On calculus MacLane gives scant attention to the well-worn topic of infinitesimals versus limits, and stresses topological ideas and relations to physics. The chain rule quickly leads to gradients, tangent spaces, cotangent spaces, and duality. This may be more than the reader wanted to know about the chain rule, but later it is crucial to MacLane's own geometric derivation of Hamiltonian dynamics. Then he discusses tricks, such as the apparently ad hoc substitution of variables usually used to derive Hamiltonian dynamics, versus ideas, such as the geometric ideas he claims are actually behind the derivation. This gives a more accurate idea of practice in calculus and differential geometry, and a more accurate sense of their history, than the common fascination with Dedekind cuts and ϵ - δ definitions.

The introduction and the last chapter are more purely philosophical, with some twists. MacLane asks about the wealth of mathematics surveyed: "How does it illuminate the philosophical questions as to Mathematical truth and beauty and does it help to make judgements about the direction of Mathematical research?" (409) He knows that few philosophers ever ask about mathematical beauty and few mathematicians care for judgments about the direction of research informed by a philosophy of mathematics. But he studied mathematics at Göttingen 1931–1933. He heard Emmy Noether lecture on algebra, and Hermann Weyl on geometry, algebra, and the philosophy of mathematics. He stayed in Weyl's house for a time and wrote up Weyl's lectures on philosophy of mathematics. He did a dissertation on logic under Paul Bernays. All this in an atmosphere shaped more by the recently retired David Hilbert than by anyone else. In Göttingen philosophic debate over formalism, logicism, and intuitionism had been a part of mathematical research for some decades. And if "On the Infinite" is usually cited as a source of ideas on metamathematics, remember that Hilbert wrote it as a defense of beautiful mathematics—of "paradise." MacLane knows how far mathematics has come since then and has no desire to revive old philosophies; but he keeps the standards he learned in those days.

He summarizes his "formal functionalism" this way: "Mathematics aims to understand, to manipulate, to develop, and to apply those aspects of the universe which are formal" (456). By "universe" MacLane means all there actually is around us. So this is not a philosophy of pure mathematics leaving applications as another question. And yet he says that mathematics is irrefutable. It is irrefutable because it deals with forms and not facts; yet it is applicable because it deals with forms of facts (414). On the other hand, he says that any truth must be falsifiable in experience, so "Mathematics is not true, but its correct results are certain" (442). This is not the Kantian doctrine

that we know truths about form a priori. Nor is it positivist, since MacLane treats forms as being objectively in the universe and formal knowledge as far from meaningless. Throughout the book he points out alternative proofs for various results, then singles out one as giving the real reason for each theorem (145, 189, 427, 455).

MacLane says a number of things about form. He draws on earlier chapters to show how many ways the basic ideas of geometry have been formalized and how each formalization has unforeseen applications. He says no single formalism such as set theory or category theory can serve as foundation for all mathematics, because it would "preclude the novelty which might result from the discovery of a new form" (455). And yet mathematics cannot proceed without formalizing its ideas: "Before that an idea can be pretty nebulous; it is 'up in the air.' That may be why it is hard to convey to others some new intuition about Mathematics" (444). The remarks, though only suggestive, are richly suggestive in the context of MacLane's contribution to mathematics and formal foundations.

He claims, since mathematics can be correct but cannot be true, that it makes no ontological commitments and there is no epistemological question of mathematical knowledge. There is only the question of how to recognize correct proofs, and this is internal to mathematics. "But this does not dispose of the hard questions about the philosophy of mathematics; they are merely displaced" (444). They are removed from epistemology and ontology and embedded firmly in the practice and history of mathematics. He raises such questions as "What are the characteristics of a Mathematical idea?" and "How does Mathematical form arise from human activities or scientific questions?" and argues they must be approached with specific, often advanced, knowledge of mathematics (444–446).

It may be jarring to find, twelve pages from the end of a lengthy book, that the hard questions have been not solved but only displaced. MacLane does not begin to answer all the questions he raises and does not finish the answers he begins. He aims to open up a field neglected since the Nazis ended the great days of Göttingen—philosophy of mathematics as a whole, mathematics at its best, not only everyday arithmetic and the basics of calculus, and not only advanced set theory or meta-mathematics. It is an exciting try. Nonetheless a philosopher reading this book has to bridge a cultural gap between philosophy and mathematics. Such a gap depends more on different experiences of learning and research than on specific differences in subject matter or technique. It must be evoked rather than described. This one is epitomized in a phrase which runs between the lines of MacLane's book without appearing in print, a phrase indis-

pensable in mathematics though it would read as a joke in most philosophical writing: "Details of the proof are left to the reader."

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Language and Logic in Ancient China. CHAD HANSEN. Ann Arbor: The University of Michigan Press. 1983. 207 p. \$25.00.

This book has, in a sense, fallen through a crack. Analytical philosophers have not picked it up because, by and large, they are not interested in Chinese philosophy; and the more traditional intellectual historians and others who *are* interested in Chinese philosophy have not picked it up because they have been put off by the tight, aggressive argumentation Hansen marshals and the bold, even iconoclastic, interpretations he introduces and defends. Yet the book has something to offer each camp. For the modern philosopher, Hansen's sophisticated interpretations and careful comparisons with modern theories of meaning and reference bring a number of neglected Chinese classics back to life by making apparent their genuine philosophical interest. For the more traditional scholars of Chinese thought, Hansen's book presents a strong methodological challenge; for to defend the traditional interpretations against Hansen's alternatives, a new approach will be needed: Hansen's arguments will have to be met head on, and at a level as least as rigorous, analytical, and philosophically informed as Hansen's.

The book begins with a chapter called "Methodological Reflections" in which Hansen presents his ideas on how to develop and evaluate an interpretation (and translation) of a text. Hansen's views owe much to W. V. Quine's theory of radical translation, but he also emphasizes Richard Grandy's "principle of humanity" according to which people will make roughly the same sorts of inferences if they start from the same assumptions. Thus, for Hansen, it will not be enough for an interpretation to make a text look more or less true; we must interpret it in a way consistent with our understanding of an author's assumptions, motives, and background knowledge generally. On one level the book can be seen as an attempt to present such a theory of interpretation and illustrate how it works. In Chapter Five, for example, Hansen develops an interpretation and translation of Kung-sun Lung's "White Horse Paradox" in which Kung-sun Lung is taken as saying something which, although false, is neverthe-