Saunders Mac Lane *Saunders Mac Lane: A Mathematical Autobiography.* Wellesley MA: A K Peters, 2005. Pp. xvi + 358. ISBN 1-56881-150-0.

We are used to seeing foundations linked to the mainstream mathematics of the late 19th century: the arithmetization of analysis, non-Euclidean geometry, and the rise of abstract structures in algebra. And a growing number of case studies bring a more philosophy of science viewpoint to the latest mathematics, as by Carter [2005], Corfield [2006], Krieger [2003], Leng [2002]. Mac Lane's autobiography is a valuable bridge between these recounting his experience of how the mid and late 20th century mainstream grew especially through Hilbert's school.

An autobiography at age 95 obviously has a lot of ground to cover. Mac Lane entered Yale in 1926 to study chemistry as a good practical career field but by the end of his first year he had won a $50 prize in mathematics and learned that you could make a living with it—as an actuary. He decided to do that. The next year he was excited to learn from his philosophy professor F.S.C. Northrop that you can also pursue new mathematical discoveries! Northrop had studied with A.N. Whitehead and sold Mac Lane on the excitement of *Principia Mathematica*. In a pattern that foreshadowed his career Mac Lane bought and annotated a copy of the first volume of *Principia* but his planned tutorial study of the book turned into a study of Hausdorff [1914] applying set theory to topology and analysis. As a graduate student at Chicago he was powerfully influenced by E.H. Moore who taught the new axiomatic methods as tools for unifying and advancing the most classical analysis [see Moore, 1910]. Then he went to Göttingen. He heard Hilbert lecture on philosophy. He studied mathematics and its philosophy with Weyl. He followed Noether's lectures on abstract algebra for number theory. He had gone there to study logic and he wrote a dissertation on it [Mac Lane, 1934]. He published an article in the *Monist* on his project of bringing formal logic closer to the working practice of mathematics [Mac Lane, 1935]. But he would not explicitly return to the philosophy he learned in Göttingen until [Mac Lane, 1986]. Through the 1930s he worked largely on algebra for very advanced number theory and geometry [see references in McLarty, 2005, 2006].

The autobiography tells how all of this felt to Mac Lane at the time and how others reacted to it. He tells of street life and mathematical life in Germany as the Nazis took power—and speculates on how it could be that a doctoral student in logic there and then would not know of Gödel's work (p. 51). He describes what it was like bringing the new German algebra back to the US and teaching it at Harvard and co-authoring the first influential English language account of it as Birkhoff and Mac Lane [1941]. He tells the now well-known events of his co-creation of category theory with topologist Eilenberg during the 1940s. Then in the most extensive mathematical discussion in the book he describes a topic much less known to philosophers, which was crucial to category theory, and is still central to current number theory, and which advanced Noether's vision. He first calls it "crossed product algebras" following Noether (pp. 93–9) and later "cohomology of groups" as the full form of it is called today (pp. 127–31). He shows how to "cross" the complex numbers with the complex conjugation operator to get the quaternions, and relates this to the basic idea of Galois theory. This approachable example points towards extremely serious mathematics. For group cohomology, he briefly relates Galois theory to simple topology. It will not teach you group cohomology but it will show you roughly what the subject does and why it needs categorical apparatus.

Some material has been published before but in widely scattered small notes. This includes his thoughts on applied mathematics and war work (i.e. World War II), university administration, government support of research, and decades of thought on math education, as well as Mac Lane's experience with Bourbaki.

Some of the most deeply felt material, and most interesting to philosophers, is on the rise of category theory through the 1950s and 60s. Mac Lane traces many of the people involved, the university contexts, and especially the conferences that made category theory a subject in its own right. For Mac Lane that did not mean an isolated subject. In fact the preface by Eisenbud describes Mac Lane's value as a dissertation advisor to himself and many others who were not at all categorists, working on subjects with little category theory in them.[1] But category theory itself was Mac Lane's passion for the last 40 years of his 70 year career. Not by coincidence this period began at roughly the time Mac Lane met graduate student F.W. Lawvere. Lawvere's work brought Mac Lane back to questions of foundations, logic, and philosophy that Mac Lane had not focussed on since Göttingen. Unfortunately there are many historical errors in this part of the book.

We come to an unhappy fact. This is not exactly the last book Mac Lane wrote. It is the book he did not finish writing. The acknowledgments written by publisher Klaus Peters say that by the time the manuscript was done "Saunders' health did not allow him to express all the thanks that are due" to those who completed it (p. xv). Nor had he been able to check all the efforts to complete it. There are many errors of detail. It is a great overview of the history as Mac Lane saw it but cannot be taken as an historical authority.[2]

Anyone who knew Saunders will recognize his voice in many passages. And philosophers ought to hear the voices of great mathematicians. When we ask what the theorems of mathematics can mean we ought to attend, among other things, to what they do mean to great practitioners. Listen to Mac Lane on concepts changing before his eyes. First an example from 1930s Göttingen:

> From the contrast between Göttingen and Chicago I learned a great deal. In Chicago (and at Yale) a vector in an n-dimensional vector space (over the real numbers) was an n-tuple $(y_1, \ldots, y_n)$ of such numbers. At Göttingen a vector was an arrow and a vector space consisted of objects (vectors) which could be suitably added and multiplied by scalars.... (p. 50)

Then compare his own rejection of Lawvere's categorical set theory before he learned the axioms and his appreciation of it once he did:

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[1] Eisenbud has never been a categorist per se although he uses enough of it when he has reason to. Compare the now standard commutative algebra text [Eisenbud, 1995].

[2] Some corrections I know are: Eilenberg left Indiana University years before Lawvere arrived, and Lawvere began studying category theory there on his own (p. 191). Lawvere did not lose his fellowship at Columbia, and his dissertation presents the category of categories as a foundation not the category of sets (p. 192). Lawvere began his approach to differential geometry at the ETH in Zurich influenced by the Grothendieck school before he began teaching with Mac Lane in Chicago (p. 244).

> Lawvere had used the basic idea of a category in ways going far beyond the original intentions; in particular, it included this striking different way of providing a foundations for mathematics. Neither Sammy [Eilenberg] nor I had contemplated any such direction. To this day many logicians, and others, have failed to understand this approach, which has now been expressed using topos theory. (p. 192)

For philosophers this once controversial reconception of vectors probably seems standard (though I have made no survey to see how many do take it as standard) while categorical set theory may seem new or risky. For Mac Lane they are parallel events in a long mathematical life.

Naturalists who hold that mathematical concepts can properly change for mathematical reasons, but ought not to change for philosophical reasons, could try to sort philosophy from mathematics in this book. Mac Lane sees his motives as entirely mathematical, entirely concerned to prove more and better theorems. The autobiography illustrates this throughout, and it is frankly stated in Mac Lane's contribution to Atiyah et al. [1994]. Yet he sees this *as being* philosophical, a view he shared with Weyl. The autobiography often mentions philosophy and never contrasts it to mathematics.

Structuralists in philosophy of mathematics can look to this book more than any other of Mac Lane's writings, including [Mac Lane, 1986], for a working view of how and why structural methods are actually used. It is not for their ontological consequences per se. They are used as a vehicle for substantial, mathematical insight. Examples are throughout the book but one concise statement links the ideas to Göttingen on one hand and to the shaping of postwar American mathematics on the other:

> The use of axioms to describe algebraic objects was a basic element in Emmy Noether's view of algebra. In her view, algebra should deal with concepts and not just manipulation. It was only in Göttingen that I came to understand these things well—an understanding that was important to my later exposition of modern algebra in my joint book with Garrett Birkhoff, *A Survey of Modern Algebra*. (p. 50)

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