

Probabilités/Probability Theory

Minima of sequences of Gaussian random variables  
Minima des suites des variables aléatoires Gaussiennes

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**Abstract.** For a given sequence of real numbers  $a_1, \dots, a_n$  we denote the  $k$ -th smallest one by  $k$ - $\min_{1 \leq i \leq n} a_i$ . We show that there exist two absolute positive constants  $c$  and  $C$  such that for every sequence of positive real numbers  $x_1, \dots, x_n$  and every  $k \leq n$  one has

$$c \max_{1 \leq j \leq k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \leq \mathbb{E} k\text{-} \min_{1 \leq i \leq n} |x_i g_i| \leq C \ln(k+1) \max_{1 \leq j \leq k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where  $g_i \in N(0, 1)$ ,  $i = 1, \dots, n$ , are independent Gaussian random variables. Moreover, if  $k = 1$  then the left hand side estimate does not require independence of the  $g_i$ 's. Similar estimates hold for  $\mathbb{E} k\text{-} \min_{1 \leq i \leq n} |x_i g_i|^p$  as well.

**Résumé.** Pour une suite  $a_1, \dots, a_n$  des nombres réels on denote le  $k$ -ième plus petit membre par  $k$ - $\min_{1 \leq i \leq n} a_i$ . On demontre qu'il existe deux constants positives  $c$  et  $C$  telle que pour toute suite  $x_1, \dots, x_n$  des nombres réels et pour tout  $k \leq n$  on a

$$c \max_{1 \leq j \leq k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \leq \mathbb{E} k\text{-} \min_{1 \leq i \leq n} |x_i g_i| \leq C \ln(k+1) \max_{1 \leq j \leq k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i}.$$

$g_i \in N(0, 1)$ ,  $i = 1, \dots, n$ , sont des variables aléatoires Gaussiennes indépendentes. En plus, si  $k = 1$ , on n'a pas besoin de l'indépendence des  $g_i$ 's pour obtenir l'inégalité du gauche. On demontre également les inégalités correspondentes pour  $\mathbb{E} k\text{-} \min_{1 \leq i \leq n} |x_i g_i|^p$ .

For a given sequence of real numbers  $(a_i)_{i=1}^n$  we denote its non-decreasing rearrangement by  $(k\text{-}\min_{1 \leq i \leq n} a_i)_{k=1}^n$ , thus  $1\text{-}\min_{1 \leq i \leq n} a_i = \min_{1 \leq i \leq n} a_i$ ,  $2\text{-}\min_{1 \leq i \leq n} a_i$  is the next smallest, etc.

Given  $A \subset \mathbb{N}$  we denote its cardinality by  $|A|$ . We say that  $(A_j)_{j=1}^k$  is a partition of  $\{1, 2, \dots, n\}$  if  $\emptyset \neq A_j \subset \{1, 2, \dots, n\}$ ,  $j \leq k$ ,  $\cup_{j \leq k} A_j = \{1, 2, \dots, n\}$ , and  $A_i \cap A_j = \emptyset$  for  $i \neq j$ . The canonical

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Euclidean norm and the canonical inner product on  $\mathbb{R}^n$  we denote by  $|\cdot|$  and  $\langle \cdot, \cdot \rangle$ . By  $1/t$  we mean  $\infty$  if  $t = 0$  and  $0$  if  $t = \infty$ .

In this note we present two theorems. The first one investigates the behavior of the expectation of the minimum of symmetric Gaussian random variables.

**Theorem 1** *Let  $p > 0$ . Let  $(x_i)_{i=1}^n$  be a sequence of real numbers. Let  $g_i \in N(0, 1)$ ,  $i \leq n$ , be Gaussian random variables. Then*

$$\frac{1}{1+p} \left(\frac{\pi}{2}\right)^{p/2} \left(\sum_{i=1}^n |x_i|^{-1}\right)^{-p} \leq \mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p.$$

Moreover, if the  $g_i$ 's are independent then

$$\mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p \leq \Gamma(1+p) \left(\frac{\pi}{2}\right)^{p/2} \left(\sum_{i=1}^n |x_i|^{-1}\right)^{-p}.$$

An immediate consequence of this theorem is the following Corollary.

**Corollary 2** *Let  $p > 0$ . Let  $(x_i)_{i=1}^n$  be a sequence of real numbers. Let  $f_i \in N(0, 1)$ ,  $i \leq n$ , be Gaussian random variables and  $g_i \in N(0, 1)$ ,  $i \leq n$ , be independent Gaussian random variables. Then*

$$\mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p \leq \Gamma(2+p) \mathbb{E} \min_{1 \leq i \leq n} |x_i f_i|^p.$$

**Remark.** This inequality is connected to the Mallat-Zeitouni problem ([2]). In fact, to prove a particular case of Conjecture 1 from [2] it is enough to prove our Corollary for  $p = 2$  and with factor 1 instead of  $\Gamma(2+p)$  ([3]). Thus we provide the solution of this case up to constant 6.

Next theorem deals with the moments of  $k$ -min of independent symmetric Gaussian variables.

**Theorem 3** *Let  $p > 0$ . Let  $2 \leq k \leq n$ . Let  $0 < x_1 \leq x_2 \leq \dots \leq x_n$ . Let  $g_i \in N(0, 1)$ ,  $i \leq n$ , be independent Gaussian random variables. Then*

$$c_p \max_{1 \leq j \leq k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \leq \left(\mathbb{E} k\text{-min}_{1 \leq i \leq n} |x_i g_i|^p\right)^{1/p} \leq C(p, k) \max_{1 \leq j \leq k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where  $c_p = \frac{1}{2e} \sqrt{\frac{\pi}{2}} \left(1 - \frac{1}{4\sqrt{\pi}}\right)^{1/p}$  and  $C(p, k) = 4\sqrt{\pi} \max\{p, \ln(k+1)\}$ .

**Remark.** Theorem 3 shows that we may evaluate sums of the form  $\sum_{k \in I} \mathbb{E} k\text{-min}_{1 \leq i \leq n} |x_i g_i|^p$ , where  $I \subset \{1, 2, \dots, n\}$  is any subset of integers. Related inequalities, though in a different context, were developed initially in [1].

Theorems 1 and 3 are consequences of the following Lemmas, which are of independent interest.

**Lemma 4** *Let  $0 < x_1 \leq x_2 \leq \dots \leq x_n$ . Let  $g_i \in N(0, 1)$ ,  $i \leq n$ , be Gaussian random variables. Let  $a = \sqrt{2/\pi} \sum_{i=1}^n 1/x_i$ . Then for every  $t > 0$*

$$\mathbb{P} \left\{ \omega \left| \min_{1 \leq i \leq n} |x_i g_i(\omega)| \leq t \right. \right\} \leq at.$$

Moreover, if the  $g_i$ 's are independent then for every  $t > 0$

$$\mathbb{P} \left\{ \omega \left| \min_{1 \leq i \leq n} |x_i g_i(\omega)| > t \right. \right\} \leq e^{-at}.$$

**Lemma 5** Let  $1 \leq k \leq n$ . Let  $0 < x_1 \leq x_2 \leq \dots \leq x_n$ . Let  $g_i \in N(0, 1)$ ,  $i \leq n$ , be independent Gaussian random variables. Let

$$a = \frac{e}{k} \sqrt{\frac{2}{\pi}} \sum_{i=1}^n \frac{1}{x_i}.$$

Then for every  $0 < t < 1/a$  one has

$$\mathbb{P} \left\{ \omega \left| k\text{-} \min_{1 \leq i \leq n} |x_i g_i(\omega)| \leq t \right. \right\} \leq \frac{1}{\sqrt{2\pi k}} \frac{(at)^k}{1 - at} \quad (1)$$

In the rest of this note we provide proofs of Theorems 1 and 3. Proofs of all lemmas will be shown in the forthcoming paper.

**Proof of Theorem 1.** Let us note that if  $x_i = 0$  for some  $i$  then the expectation is 0 and the Theorem is trivial. Therefore, without loss of generality, we assume that  $x_i > 0$  for every  $i$ .

Denote

$$A = \left( \sqrt{\frac{2}{\pi}} \sum_{k=1}^n 1/x_k \right)^{-p}.$$

Then, by the first estimate in Lemma 4, we have

$$\mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p = \int_0^\infty \mathbb{P} \left\{ \omega \left| \min_{1 \leq i \leq n} |x_i g_i(\omega)| > t^{1/p} \right. \right\} dt \geq \int_0^A \left( 1 - t^{1/p} A^{-1/p} \right) dt = \frac{A}{1 + p},$$

which proves the first estimate.

Now assume that the  $g_i$ 's are independent and use the second estimate of Lemma 4. We obtain

$$\mathbb{E} \min_{1 \leq i \leq n} |x_i g_i|^p = \int_0^\infty \mathbb{P} \left\{ \omega \left| \min_{1 \leq i \leq n} |x_i g_i(\omega)| > t^{1/p} \right. \right\} dt \leq \int_0^\infty \exp \left( -t^{1/p} A^{-1/p} \right) dt = Ap \Gamma(p),$$

which implies the desired result.  $\square$

To prove Theorem 3 we need also the following combinatorial lemma.

**Lemma 6** Let  $1 \leq k \leq n$ . Let  $(a_i)_{i=1}^n$ , be a nonincreasing sequence of positive real numbers. Then there exists a partition  $(A_l)_{l \leq k}$  of  $\{1, 2, \dots, n\}$  such that

$$\min_{1 \leq l \leq k} \sum_{i \in A_l} a_i \geq a := \frac{1}{2} \min_{1 \leq j \leq k} \frac{1}{k+1-j} \sum_{i=j}^n a_i.$$

**Remark.** In fact one can show that the  $A_l$ 's can be taken as intervals, i.e.  $A_l = \{i \mid n_{l-1} < i \leq n_l\}$ ,  $l \leq k$ , for some sequence  $0 = n_0 < 1 \leq n_1 < n_2 < \dots < n_k = n$ .

**Proof of Theorem 3.** First we show the lower estimate. Since for every sequence  $(a_i)_{i=1}^n$  and every  $r < k$  one has

$$k\text{-} \min(a_i)_{i=1}^n \geq (k-r)\text{-} \min(a_i)_{i=r+1}^n,$$

it is enough to show that for every  $k$  we have

$$c_p k \left( \sum_{i=1}^n 1/x_i \right)^{-1} \leq \left( \mathbb{E} k\text{-} \min_{1 \leq i \leq n} |x_i g_i|^p \right)^{1/p}. \quad (2)$$

Let  $a$  be as in Lemma 5 and  $t = (2a)^{-p}$ . Then, by Lemma 5 and since  $k \geq 2$ , we have

$$\mathbb{P} \left\{ \omega \left| k\text{-} \min_{1 \leq i \leq n} |x_i g_i(\omega)|^p \geq t \right. \right\} \geq 1 - \frac{1}{\sqrt{2\pi k}} \frac{(at^{1/p})^k}{1 - at^{1/p}} \geq 1 - \frac{1}{4\sqrt{\pi}}.$$

Therefore (2) follows from the standard estimate

$$\mathbb{E} k\text{-}\min_{1 \leq i \leq n} |x_i g_i|^p \geq t^p \mathbb{P} \left\{ \omega \mid k\text{-}\min_{1 \leq i \leq n} |x_i g_i(\omega)| \geq t \right\}.$$

Now we prove the upper bound. Let  $(A_j)_{j \leq k}$  be the partition given by Lemma 6 for sequence  $a_i = 1/x_i$ ,  $i \leq k$ . The number  $q$ ,  $q \geq 1$ , will be specified later. It is easy to see that  $k\text{-}\min_{1 \leq i \leq n} |x_i g_i|^p \leq \max_{j \leq k} \left\{ \min_{i \in A_j} |x_i g_i|^p \right\}_{j \leq k}$ . Therefore, using Theorem 1, we get

$$\begin{aligned} \left( \mathbb{E} k\text{-}\min_{1 \leq i \leq n} |x_i g_i|^p \right)^{1/p} &\leq \left( \mathbb{E} \left( \sum_{j \leq k} \left( \min_{i \in A_j} |x_i g_i|^p \right)^q \right)^{1/q} \right)^{1/p} \leq \left( \mathbb{E} \sum_{j \leq k} \min_{i \in A_j} |x_i g_i|^{pq} \right)^{1/(pq)} \\ &\leq \sqrt{\frac{\pi}{2}} \left( \Gamma(1 + pq) \sum_{j \leq k} \left( \sum_{i \in A_j} 1/x_i \right)^{-pq} \right)^{1/(pq)} \leq \sqrt{\frac{\pi}{2}} (k \Gamma(1 + pq))^{1/(pq)} \max_{j \leq k} \left( \sum_{i \in A_j} 1/x_i \right)^{-1}. \end{aligned}$$

Applying Lemma 6, we obtain

$$\left( \mathbb{E} k\text{-}\min_{1 \leq i \leq n} |x_i g_i|^p \right)^{1/p} \leq \sqrt{2\pi} (k \Gamma(1 + pq))^{1/(pq)} \max_{1 \leq j \leq k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i}.$$

To complete the proof we choose  $q = \frac{\ln(k+1)}{p}$  if  $p \leq \ln(k+1)$ ,  $q = 1$  otherwise, and apply Stirling's formula.  $\square$

**Remark.** Finally we would like to note that our results can be extended to the case of general distributions satisfying certain conditions. Namely, fix  $\alpha > 0$ ,  $\beta > 0$  and say that a random variable  $\xi$  satisfies an  $(\alpha, \beta)$ -condition if for every  $t > 0$  one has

$$\mathbb{P} (|\xi| \leq t) \leq \alpha t \quad \text{and} \quad \mathbb{P} (|\xi| > t) \leq e^{-\beta t}.$$

Then Theorems 1, 3 and Lemmas 4, 5 hold for  $g_i$ 's satisfying an  $(\alpha, \beta)$ -condition (even not identically distributed), with constants depending on  $\alpha, \beta$ . More precisely, in the estimates of Theorem 1,  $(\pi/2)^{p/2}$  should be substituted by  $\alpha^{-p}$  and  $\beta^{-p}$  correspondingly; in Theorem 3,  $\sqrt{\pi/2}$  should be substituted by  $1/\alpha$  and, in the upper estimate,  $4\sqrt{\pi}$  by  $4\sqrt{2}/\beta$ ; in Lemma 5 and in the first estimate of Lemma 4,  $\sqrt{2/\pi}$  should be substituted by  $\alpha$ ; in the second estimate of Lemma 4,  $\sqrt{2/\pi}$  should be substituted by  $\beta$ .

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## References

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