Corrections to *Toposes, Triples and Theories*

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The corrections are listed by page number. The name in parentheses after the page number shows who told us of the error.

GENERAL COMMENT Our text is intended primarily as an exposition of the mathematics, not a historical treatment of it. In particular, if we state a theorem without attribution we do not in any way intend to claim that it is original with this book. We note specifically that most of the material in Chapters 4 and 8 is an extensive reformulation of ideas and theorems due to C. Ehresmann, J. Bénabou, C. Lair and their students, to Y. Diers, and to A. Grothendieck and his students. We learned most of this material second hand or recreated it, and so generally do not know who did it first. We will happily correct mistaken attributions when they come to our attention.

p. 9 (Peter Johnstone). Exercise (SGRPOID) is incorrect as it stands; a semilattice without identity satisfies (i) through (iii) but is not a category. Condition (iii) must be strengthened to read: Say an element e has the **identity property** if $e \circ f = f$ whenever $e \circ f$ is defined and $g \circ e = g$ whenever $g \circ e$ is defined. Then we require that for any element f, there is an element e with the identity property for which $e \circ f$ is defined and an element e' with the identity property for which $f \circ e'$ is defined.

p. 26 (D. Čubrić). Property (ii) of Exercise (SUBF) should read "If $f : A \longrightarrow B$, then $F(f)$ restricted to $G(A)$ is equal to $G(f)$."

p. 27 (Dwight Spencer), second line from bottom: $T : S \longrightarrow T$ should be $t : S \longrightarrow T$.

pp. 39-40 (Peter Johnstone). It should be noted that the product of an empty collection of objects in a category must be a terminal object. Then the phrase after the comma on line 4 of p. 40 should read, "which by an obvious inductive argument is equivalent to requiring that the category have a terminal object and that any two objects have a product."

p. 43 (Peter Johnstone). Exercise (PROD)(b) should read: "Show that if a category has a terminal object and all products of pairs of objects, then it has all finite products."

p. 49 (Peter Johnstone). Exercise (FCR) uses the concept of small category without defining it. It is used in the main body of the text on page 66 and later, and "small sketch" occurs on p. 146. A graph or a category is **small** if its arrows constitute a set. A sketch is small if its graph is small and its cones and cocones constitute a set. In connection with the discussion of foundations on page ix of the Preface, no matter what set theory is used, one is going to have to deal with categories and graphs whose arrows do not constitute sets.

p. 49 Closing parenthesis missing at end of Exercise (EAPL)(a).

p. 53 (Dwight Spencer) The display in the middle of the page should be: $i(T, LA)(L(f \circ h)) = \eta A \circ f \circ h = RLf \circ \eta D \circ h = RLf \circ i(T, LD)(Lh) = i(T, LA)(Lf \circ Lh)$

In diagram (7), just below, the vertical arrows should be pointing upward.

p. 54 lines 5 from the bottom: (Dwight Spencer) change "arrow for" to "element of the functor". Add to the last sentence "(A universal element of $\mathrm{Hom}(A, R(-))$ is called a **universal arrow** for R and A.)"

p. 55 line 4: (Dwight Spencer) change RLA to RWA.

p. 55 line 14: (Dwight Spencer) Change Ry to y.

p. 64 (Dwight Spencer) The diagram at the bottom should be labeled (1). (I don't think it is actually referred to, but the diagram numbering in this section begins with (2).)

p. 68 (D. Čubrić). The end of line 5 should be $\mathrm{Sub}(F \times E)$.

p. 69 (V. Pratt) The reference to Section 1.4 (line 13) should be to Subsection, "Global elements" on page 24, more precisely the second paragraph.

p. 72 (D. Čubrić). In the next to last line, $p : LF \longrightarrow X$.

p. 75 Geometric morphisms are discussed in Chapter 6, not Chapter 5.

p. 90 (D. Čubrić). In diagram (1), the leftmost arrow should be labeled $T\mu$.

p. 94 (D. Čubrić). In the next to last line, $R : \mathrm{Set}\,/|X| \longrightarrow \mathrm{Sh}(X)$.

p. 95 (D. Čubrić). The end of Proposition 2 should say, "is a cotriple on $\mathcal{B}$".

p. 100 (D. Čubrić). Line 3 should have $U^{\mathsf{T}} : \mathcal{C}^{\mathsf{T}} \longrightarrow \mathcal{C}$, $F^{\mathsf{T}} : \mathcal{C}^{\mathsf{T}} \longrightarrow \mathcal{C}$.

p. 110 (D. Čubrić). In Diagram (19), the arrow $\eta C : TC \longrightarrow T^2C$ should be $T\eta C : TC \longrightarrow T^2C$.

p. 125 Third line from bottom. The word "morphism" is repeated.

p. 126 (Felipe Gago-Couso). Proposition 1 has an omitted hypothesis. We include here a complete restatement of the proposition and its proof:

The following proposition gives one method of constructing morphisms of triples. We are indebted to Felipe Gago-Couso for finding the gap in the statement and proof in the first edition and for finding the correct statement.

Proposition 1. *In the notation of the preceding paragraphs, let $\sigma : TT' \longrightarrow T'$ be a natural transformation for which*

$$\begin{array}{ccc} T' & \xrightarrow{\eta T'} & TT' \\ & \searrow^{\mathrm{id}} & \downarrow{\sigma} \\ & & T' \end{array}$$

and

$$\begin{array}{ccc} TTT' & \xrightarrow{\mu T'} & TT' \\ {\scriptstyle T\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\ TT' & \xrightarrow[\sigma]{} & T' \end{array} \qquad \begin{array}{ccc} TT'T' & \xrightarrow{T'\mu} & TT' \\ {\scriptstyle \sigma T'}\downarrow & & \downarrow{\scriptstyle \sigma} \\ T'T' & \xrightarrow[\mu']{} & T' \end{array}$$

(a) (b)

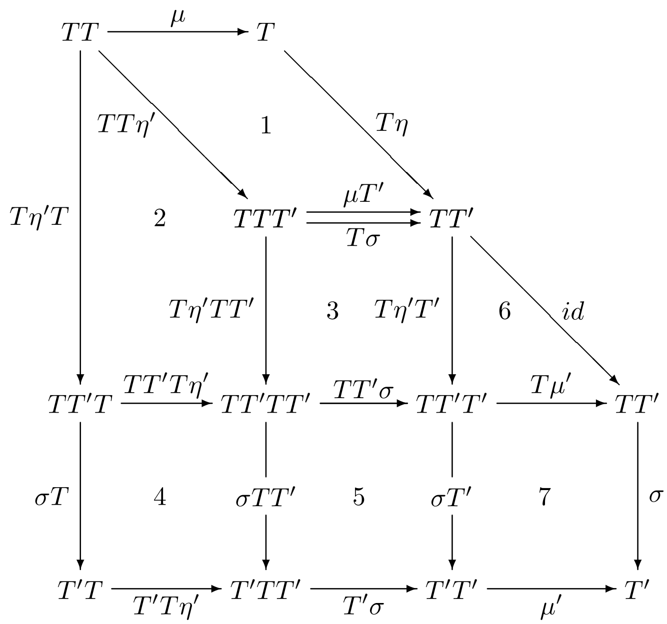
commute. Let $\alpha = \sigma \circ T\eta' : T \longrightarrow T'$. Then α is a morphism of triples.

Proof. That (1) commutes follows from the commutativity of

$$\begin{array}{ccc} \mathrm{id} & \xrightarrow{\eta} & T \\ {\scriptstyle T'}\downarrow & & \downarrow{\scriptstyle T\eta'} \\ T' & \xrightarrow[\eta']{} & TT' \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \sigma} \\ & & T' \end{array}$$

In this diagram, the square commutes because η is a natural transformation and the triangle commutes by (3).

The following diagram shows that (2) commutes.



In this diagram, square 1 commutes because μ is a natural transformation, squares 2 and 3 because $T\mu'$ is and squares 4 and 5 because σ is. The commutativity of square 6 is a triple identity and square 7 is diagram 4(b). Finally, diagram 4(a) above says that $\sigma \circ \mu T' = \sigma \circ T\sigma$ which means that although the two arrows between TTT' and TT' are not the same, they are when followed by σ, which makes the whole diagram commute.

Squares 1 through 5 of diagram (6) are all examples of part (a) of Exercise (GOD), Section 1.3. For example, to see how square 1 fits, take $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of the exercise to be $\mathcal{C}$, $F = \mathrm{id}$, $G = T'$, $H = T^2$ and $K = T$, and $\kappa = \eta'$, $\mu = \mu$. Then square 1 is $\eta'\mu$.

Corollary 2. *With T and T' as in Proposition 1, suppose $\sigma : T'T \longrightarrow T'$ is such that $\sigma \circ T'\eta = \mathrm{id}$, $\sigma \circ \sigma T' = \sigma \circ \eta T : T \longrightarrow T'$ is a morphism of triples.*

Proof. This is Proposition 1 stated in $\mathbf{Cat}^{\mathrm{op}}$ (which means: reverse the functors but not the natural transformations).

p. 134 (Colin McLarty). In the second through fourth paragraphs of the proof of (a), every occurrence of "L" should be "W".

p. 137 (Jim Otto). Theorem 5 has too few hypotheses and Proposition 4 too many to apply the latter in the proof of the former. There are various possibilities of getting it right, including adding the finite limits and colimits to the hypotheses of Theorem 5. Nevertheless, Theorem 5 is true as it stands. The problem occurs with the last sentence of Theorem 5. Probably the best approach is to point out first that only equalizers and coequalizers are needed and then only the U-contractible ones. Then, if we suppose only the U-contractible equalizers exist (any that exist will be preserved by a functor that has a left adjoint), that is enough to do Theorem 5 (Exercise, using the fact that the underlying functor of a tripleable functor creates limits) and from that will follow that the U-contractible equalizers exist.

As for Proposition 4, if we suppose that $\mathcal{B}$ has U-contractible equalizers, the dual of Theorem 5 would imply that U-contractible equalizers exist. Hence sufficient for Proposition 4 is that *either* U-contractible equalizers or U-contractible coequalizers exist. Does anyone want to find an example to show that any completeness hypothesis is necessary?

p. 139 (Samuel Eilenberg) The story told at the end of the second paragraph is not exact. In fact the word was chosen as the manuscript was in final preparation. Nonetheless, the main point of the story, the care that went into choosing the name of an important concept, remains.

p. 139 (F-J de Vries) l. -3: "substantially" should read like "substantially"

p. 140 (F-J de Vries) l. 9: "Kiesler" should be "Keisler"

General comment about chapters 4 and 8 (C. Lair). In many places we state that some extension of a functor is unique, when in fact it is only unique up to isomorphism of functors in the functor category.

These occur on p. 153 (Theorem 4), p. 156 (Theorem 2), and implicitly in p. 293, Theorem 2 and p. 294, Theorem 1.

p. 146. C. Lair has told us that Ehresmann proved a more general form of Kennison's Theorem in Ehresmann [1967a], [1967b].

p. 162 (C. Lair). The sketch for LE categories constructed here has LE categories with specified limits of finite diagrams as models, and morphisms of models are functors which preserve the specified limits. A similar remark should be made about the sketch for toposes on p. 165.

p. 168 (F-J de Vries) l. -15: "Kiesler" should be "Keisler"

p. 172 Second sentence: **P** having a left adjoint does not imply T does. The sentence and following one should be combined as follows: "Since T is the composite of a functor and its right adjoint, it is the functor part of a triple $(T, \eta : 1 \longrightarrow T, \mu : T^2 \longrightarrow T)$."

p. 110 (D. Čubrić). Second line in the proof of Theorem 1 should begin "map $\mathcal{E}/A \longrightarrow \mathcal{E}$.

p. 179 , Proposition 2 (D. Čubrić). This proposition should say that $\mathbf{P}' \circ L$ is naturally isomorphic to $L^{\mathrm{op}} \circ \mathbf{P}$.

p. 183 (F-J de Vries) l. -7: Kock [1982] should be Kock [1981] (SDG)

p. 197 Lemma 3 (Francisco Marmolejo). Dropping the subscript on the arrows leaves it too ambiguous. Part (a) should read, "$D \wedge (B \Rightarrow_A C) = D \wedge B \Rightarrow_D D \wedge C$."

p. 200 , line -4 (Francisco Marmolejo). The sentence should be expanded to, "Observe that any sheaf is $\supset$-closed inside any separated presheaf. (**Separated** isn't defined until page 233, but it simply means that in the definition of sheaf in the middle of page 69, the unique existence is replaced by the assumption that there is at most one element x such that $X|U_i = x_i$.)"

p. 205 (Francisco Marmolejo). In the last line of Exercise TPPB, change "induce" to "constitutes".

p. 213 , last line (Francisco Marmolejo). Should read, "products commute with reflexive coequalizers." In fact, a product of a fixed object with any coequalizer is a coequalizer. This fact follows from Lemma 6 on page 158-159.

p. 214 The reference, third line from bottom, to section 6.4 should be to section 7.3.

p. 215 (D. Čubrić). In Corollary 7, both occurrences of "E" should be "$\mathcal{E}$".

p. 225 (D. Čubrić). In Exercise (SEP), "given" should be "give".

p. 233 "Epimorphic family" should be boldface and indexed.

p. 234 second paragraph (Jürgen Koslowski): Change Corollary 5 of Section 5.4 to Corollary 8 of Section 5.3. Also, Exercise CANON doesn't exist; delete the reference.

p. 236 (D. Čubrić). In Proposition 4, we should have noted that (a) and (b) are equivalent by definition. The last sentence of the proof should read "Hence (b), and therefore (a), is true."

p. 238 (B. Howard). Sums in a category are **disjoint** if for any objects A and B, the commutative diagram

$$\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ A & & B \\ \searrow & & \swarrow \\ & A+B & \end{array}$$

(in which the arrows to $A + B$ are the canonical injections) is a pullback, and the canonical injections are monic.

p. 242 "Cocontinuous", in Theorem 12, was not defined. A cocontinuous functor is one which preserves all colimits.

p. 250 Fourth line is broken.

p. 250 Theorem 7 is referred to several times elsewhere as Freyd's embedding theorem, and should be named as such here.

p. 261 second line from bottom: Freyd's Theorem is Theorem 7 of 7.1, not Theorem 5 of 7.2.

p. 267 (F-J de Vries) l. 267: "Mikkelson" should be "Mikkelsen"

p. 293 (Peter Johnstone). In Exercise (TOTO), the maps should be strictly increasing rather than nondecreasing.

p. 294 We should point out the connections between Theorem 1 here and Theorem 12, p. 242 and Theorem 2, p. 263.

p. 295 second line before exercises. "Function" misspelled.

p. 295 (Peter Johnstone). The description of the realizability topos is completely incorrect; in particular, the realizability topos is not a classifying topos, so the reference does not belong here at all. The reference which *does* belong here is to Mulry [1982].

p. 296 Same change for Exercise (DLO) as for Exercise (TOTO) above.

p. 297 (Peter Johnstone and many others). Theorem 1 omits the very important fact that models of geometric theories have filtered colimits.

p. 300 The statement on line 6 that filtered colimits of regular functors are regular deserves some discussion, or at least should be made an exercise!

p. 301 In connection with the first sentence beginning on this page, we now know that the category of orthodox semigroups and their morphisms is the category of models of an LE-sketch and is regular, but is not the category of models of an FP-sketch. (An orthodox semigroup is one in which the product of idempotents is an idempotent.)

p. 302 (Peter Johnstone). Because models of geometric theories preserve filtered colimits (see correction to p. 297), the answer to Exercise (CYCGRP)(c) is easily seen to be: No.

p. 307 diagram (5). The two rightmost arrows lack labels. The one from UB to C is c and the one from C to UB is s.

p. 318 (Colin McLarty). Exercise (DL) should say that all composites *of length three* are the identity.

p. 325 (Peter Johnstone). In line 15, $(R : C)$ is not a full subcategory of the comma category (R, C).

p. 337 (F-J de Vries) "Mikkelson" should be "Mikkelsen"

p. 338 (F-J de Vries) "J.Shonfield" should read as "J.R. Shoenfield"

INDEX, pp. 342ff. Some omissions:
Beck's Tripleability Theorem, 112.
Butler's Theorem, 135.
epimorphic family, 233.
Freyd's Representation Theorem, 246ff.
Freyd's characterization of natural number objects, 273.

Supplemental Bibliography

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