ANSWERS TO ALL THE PROBLEMS in \CATEGORY THEORY FOR COMPUTING SCIENCE"

(Second Edition)

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Remarks on teaching from `Category Theory for Computing Science'

The book is intended for students and researchers in computing science, although it may be studied pro¯tably by students in mathematics as well. Such students have varied backgrounds and the authors have taken some pains to ensure that the book is self-contained. The ideas from mathematics and theoretical computing science used in the book are introduced at least brie°y in the book, even down to the level of sets and functions.

The essential prerequisite is some previous experience with abstract mathematics, so that the student understands the mathematician's technique of de¯ning abstract concepts formally and proving theorems about them using the de¯nitions and previous theorems. This is the same sort of background necessary to understand theoretical computing science, for example in the study of grammars or complexity. The book is not entirely, or even mostly, about theorems in category theory and their proofs, but axiomatic mathematics is a fundamental tool and the book is written assuming the reader has some facility with that tool.

We recommend that even sophisticated students look brie°y at Chapter 1. The concept of function used herein (with the mandatory speci¯cation of the codomain) is standard in category theory and in many other modern texts in mathematics and computing science, but the older de¯nition is still widespread. Our de¯nition of graph is not the one used in most graph theory courses and should therefore be noted as well. Students with less background (for example computing science students in North America who have had only a discrete mathematics course) should spend at least two class periods on this chapter.

Chapters 2 and 3 give the basic de¯nitions of category and functor, a great many examples, and some elementary properties. We recommend that computing science students pass fairly quickly through the material on algebraic and other structures (sections 2.3 through 2.5); it is easy to get bogged down there. An instructor may want to omit certain examples in the interests of progressing rapidly. Indeed it is desirable for a class to go through these chapters at a fairly fast pace; doing this sets an appropriate level of di±culty for the course since the concepts are relatively easy in these chapters and they get harder later.

It would be perfectly reasonable to postpone Section 3.4 (equivalences) until Section 4.3 (natural transformations between functors), using Proposition 4.3.4 as a de¯nition.

Chapter 4 (particularly the ¯rst ¯ve sections) introduces the student to a complex of ideas involving diagrams and natural transformations that are at the heart of category theory. These ideas, along with adjoints (Chapter 9) make categories more than just another slightly odd algebraic structure; they are what allow a category to be a universe of mathematical discourse and provide the basics of the categorical approach to model theory. The heart of hearts is in Section 4.5 (the Yoneda Lemma and its relatives). That section is distinctly more di±cult than the preceding material, so the instructor should be prepared to slow down. As mentioned in the text, Sections 4.6 through 4.8 are optional. Section 4.6 and 4.7 should be covered if you are going to do any more work with sketches. 2-categories (Section 4.8) have become important in applications recently. They are quite complicated and will require more than one lecture to do them justice.

Chapters 5 and 8 cover limits and colimits, which make category theory an expressive language that allows the uniform formulation of many di®erent elementary concepts in various branches of mathematics. Our recommendation is a concentration on Sections 5.1 through 5.4 because the ideas there are repeated with variations in the general discussions of limits and colimits in Chapter 8. Sections 5.5 through 5.8 and 8.7 are particular topics that have become important in computing science. We believe that it is important for computing science students to emphasize the way one uses limits to describe certain constructions by specifying their relationship with other structures rather than by saying `what they are'. This is closely analogous with the idea of a program speci¯cation that says what the program must do rather than saying how it should be constructed.

Chapter 6, cartesian closed categories, is a special topic in category theory that has become fundamental to the study of programming language semantics. You can stop after 6.2 and the student will know the basics of the subject. Sections 6.3 through 6.5 provide a translation between a type of category and a type of language. That kind of translation is characteristic of much of the literature connecting category theory with computing science and is one of only two examples in the text (the other is in Section 7.7 and requires some study of ¯nite product sketches), so it might be desirable to study them as an example of the kind of thing a student with a theoretical bent will meet again. Section 6.6 gives another special topic that has been important in computing science.

Chapter 7 introduces ¯nite product sketches, the categorist's way of doing universal algebra. It is completely optional but has been the source of some suggested applications in computing science. You can expect the student to have trouble with the many levels of abstraction in this topic. For example, a model of the sketch for monoids can be thought of as a functor or as a monoid, and the theory of a sketch can easily be confused with the initial model of the sketch.

Adjunctions, the subject of Chapter 9, provide the main connection between category theory and logic and are involved in most applications of category theory. Many category theorists (including the authors) feel that you have not learned category theory until you understand adjunctions. A detailed look at the many examples in Section 9.2 is important to give the student some of the ways that adjunctions can appear. Going through the proof of Theorem 9.3.2 also provides good practice with the concept.

Many instructors will want to cover Sections 9.1 through 9.3 immediately after Section 4.5 (you will have to ignore some of the comments involving limits and colimits). Doing this makes considerable demands on the student, but it gets to adjunctions more rapidly.

Chapters 10 and 11 introduce some special topics that have become important in applications in recent years. Triples (Sections 10.3 and 10.4) are also a fundamental tool in category theory itself. Closed monoidal categories (Chapter 11) have a di®erent °avor from the other topics. They are categories with extra structure rather than categories with speci¯c properties (2-categories are like that as well) and it is desirable to make that point early in discussing the topic.

Solutions to the exercises

Solutions for Chapter 1

Section 1.2

1. Suppose h is surjective, and suppose $\mathrm{Hom}(h, T)(f) = \mathrm{Hom}(h, T)(g)$. Then $f \circ h = g \circ h$. Let $x \in S$, and let $w \in W$ satisfy $h(w) = x$ (using surjectivity of h). Then $f(x) = f(h(w)) = g(h(w)) = g(x)$. Since x was arbitrary, this shows that $f = g$, so that $\mathrm{Hom}(h, T)$ is injective. Conversely, suppose that h is not surjective. This means there is a particular $x \in S$ which is not $h(w)$ for any $w \in W$. Let $t, u \in T$ be two distinct elements of T. Let $f : S \to T$ be the constant function with value t. Let $g : S \to T$ take x to u and all other elements to t. Then $\mathrm{Hom}(h, T)(f) = \mathrm{Hom}(h, T)(g)$, so $\mathrm{Hom}(h, T)$ is not injective.

2. a. To show that the mapping is injective, suppose $\langle f, g\rangle = \langle f', g'\rangle$. Then for any $x \in X$,

$$(f(x), g(x)) = \langle f, g\rangle(x) = \langle f', g'\rangle(x) = (f'(x), g'(x))$$

The coordinates of these pairs must be the same, so $f(x) = f'(x)$ and $g(x) = g'(x)$ for all $x \in X$. Hence $f = f'$ and $g = g'$, so $(f, g) = (f', g')$. To get surjectivity, suppose that $q : X \to S \times T$ is any function. Define $f : X \to S$ by requiring that $f(x)$ be the first coordinate of the pair $q(x)$ for all $x \in X$; in other words, $f(x) = \mathrm{proj}_1(q(x))$. Similarly set $g(x) = \mathrm{proj}_2(q(x))$. Then the mapping of the problem takes (f, g) to q, so it is surjective.

b. The pair $(\mathrm{proj}_1, \mathrm{proj}_2)$.

3. a. Define $\psi : \mathrm{Hom}(S \cup T, V) \to \mathrm{Hom}(S, V) \times \mathrm{Hom}(T, V)$ by $\psi(h : S \cup T \to V) = (h|S, h|T)$, where $h|S$ denotes the restriction of h to the subset S of $S \cup T$. It is easy to see that ψ is the inverse function of ϕ.

b. It is (i, j), where i is the inclusion of S in $S \cup T$ and j is the inclusion of T in $S \cup T$.

4. a. Define $\psi : \mathrm{Hom}(A, \mathscr{P}B) \to \mathrm{Rel}(A, B)$ by

$$\psi(f : A \to \mathscr{P}B) = \{(a, b) \mid b \in f(a)\}$$

Then ψ is inverse to ϕ.

b. The map $a \mapsto \{a\}$.

c. The opposite of the 'element of' relation: If Y is a subset of B and $b \in B$ then Y is related to b if and only if $b \in Y$.

Section 1.3

1. The graph

$$a \underset{t}{\overset{s}{\rightrightarrows}} n \xrightarrow{\text{label}} l$$

expresses the concept. Interpret n as nodes and l as labels; then the arrow called `label' takes a node to its label.

2. By definition, $\mathscr{G}$ is simple if and only if for any two distinct arrows f and g, either $\text{source}(f) \neq \text{source}(g)$ or $\text{target}(f) \neq \text{target}(g)$. Since for any arrow x,

$$\langle \text{source}, \text{target} \rangle(x) = (\text{source}(x), \text{target}(x))$$

$\mathscr{G}$ is simple if and only if for any two distinct arrows f and g,

$$\langle \text{source}, \text{target} \rangle(f) \neq \langle \text{source}, \text{target} \rangle(g)$$

But that says that $\langle \text{source}, \text{target} \rangle$ is injective by definition.

Section 1.4

1. Let $\phi : \mathscr{G} \to \mathscr{H}$ be a graph homomorphism with $\mathscr{H}$ simple. Let $f : a \to b$ be an arrow of $\mathscr{G}$. Then $\phi_1(f)$ has to be an arrow from $\phi_0(a)$ to $\phi_0(b)$. But there is at most one such arrow because $\mathscr{H}$ is simple. Thus $\phi_1(f)$ is determined by $\phi_0(a)$ and $\phi_0(b)$.

2. We must show that if $u : m \to n$ is an arrow of $\mathscr{G}$, then

$$\psi_1(\phi_1(u)) : \psi_0(\phi_0(m)) \to \psi_0(\phi_0(n))$$

in $\mathscr{K}$. Since ϕ is a graph homomorphism, $\phi_1(u) : \phi_0(m) \to \phi_0(n)$ in $\mathscr{H}$. The required equation then follows because ψ is a graph homomorphism.

3. a. Let $h : c \to d$ be an arrow in $\mathscr{H}$. We must show that $\psi_1(h) : \psi_0(c) \to \psi_0(d)$ in $\mathscr{G}$. Suppose $\psi_1(h) : a \to b$. Then, by definition of ψ and the fact that ϕ is a graph homomorphism,

$$h = \phi_1(\psi_1(h)) : \phi_0(a) \to \phi_0(b)$$

in $\mathscr{H}$. Thus $\phi_0(a) = c$ and $\phi_0(b) = d$, so $\psi_0(c) = a$ and $\psi_0(d) = b$ as required.

Here is a second proof that fits a pattern we use over and over in this book in showing that if an invertible arrow has a certain property, then its inverse does too.

$$\begin{aligned} \text{source}(\psi_1(h)) &= \psi_0(\phi_0(\text{source}(\psi_1(h)))) \\ &= \psi_0(\text{source}(\phi_1(\psi_1(h)))) \quad = \quad \psi_0(\text{source}(h)) \end{aligned}$$

and similarly for target.

b. By definition, we must prove the following equations:

$$\psi_0 \circ \phi_0 = (\mathrm{id}_{\mathscr{G}})_0$$
$$\psi_1 \circ \phi_1 = (\mathrm{id}_{\mathscr{G}})_1$$
$$\phi_0 \circ \psi_0 = (\mathrm{id}_{\mathscr{H}})_0$$
$$\phi_1 \circ \psi_1 = (\mathrm{id}_{\mathscr{H}})_1$$

For any graph $\mathscr{F}$, $(\mathrm{id}_{\mathscr{F}})_0 = \mathrm{id}_{F_0}$ and $(\mathrm{id}_{\mathscr{F}})_1 = \mathrm{id}_{F_1}$ (see Example 1.4.3). The result then follows from the fact that $\psi_i = (\phi_i)^{-1}$ for $i = 1, 2$.

Solutions for Chapter 2

Section 2.1

1. Functional composition is associative, and the identity functions satisfy C–3 and C–4, so it is only necessary to show that the identity functions are injective and that composite of injective functions is injective. If S is a set and $x, y \in S$ with $x \neq y$, then $\mathrm{id}_S(x) = x \neq y = \mathrm{id}_S(y)$, so id_S is injective. Let $f : S \to T$ and $g : T \to V$. If $x, y \in S$ and $x \neq y$, then $f(x) \neq f(y)$ because f is injective. But then $g(f(x)) \neq g(f(y))$ because g is injective. Hence $g \circ f$ is injective.

2. It is necessary to show that id_S is surjective for each set S and that if $f : S \to T$ and $g : T \to V$ are surjective then so is $g \circ f$. If $x \in S$, then $\mathrm{id}_S(x) = x$ so id_S is surjective. If $v \in V$, then there is a $t \in T$ for which $g(t) = v$ since g is surjective. There is $x \in S$ for which $f(x) = t$ since f is surjective. Then $g(f(x)) = v$, so $g \circ f$ is surjective.

3. Let α be a relation from A to B, β a relation from B to C and γ a relation from C to D. By definition of composite, for $x \in A$ and $u \in D$, $(x, u) \in (\gamma \circ \beta) \circ \alpha$ if and only if there is an element $y \in B$ for which $(x, y) \in \alpha$ and $(y, u) \in \gamma \circ \beta$. But $(y, u) \in \gamma \circ \beta$ if and only if there is $z \in C$ such that $(y, z) \in \beta$ and $(z, u) \in \gamma$. Thus $(x, u) \in (\gamma \circ \beta) \circ \alpha$ if and only if there are elements $y \in B$ and $z \in C$ such that $(x, y) \in \alpha$, $(y, z) \in \beta$ and $(z, u) \in \gamma$.

On the other hand, (x, u) is in $\gamma \circ (\beta \circ \alpha)$ if and only if there is $z \in C$ such that $(x, z) \in \beta \circ \alpha$ and $(z, u) \in \gamma$. But $(x, z) \in \beta \circ \alpha$ if and only if there is $y \in B$ such that $(x, y) \in \alpha$ and $(y, z) \in \beta$. Thus $(x, u) \in \gamma \circ (\beta \circ \alpha)$ if and only if there are $y \in B$ and $z \in C$ such that $(x, y) \in \alpha$, $(y, z) \in \beta$ and $(z, u) \in \gamma$. Hence $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$.

4. a. $\mathrm{id}_A \circ u = u$ by definition of id_A, but $\mathrm{id}_A \circ u = \mathrm{id}_A$ by assumption. Hence $u = \mathrm{id}_A$. This is an example of a useful heuristic: when a property is given in terms of all arrows and you want to prove something about the property, see what it says for identity arrows.

b. $u \circ \mathrm{id}_A = u$ by definition of id_A, but $u \circ \mathrm{id}_A = \mathrm{id}_A$ by assumption. Hence $u = \mathrm{id}_A$.

Section 2.2

1. This requires an arrow $\mathtt{nonzero} : \mathsf{NAT} \to \mathsf{BOOLEAN}$ and the following equations: $\mathtt{nonzero} \circ 0 = \mathtt{false}$ and $\mathtt{nonzero} \circ \mathtt{succ} = \mathtt{true}$. Since an arrow to NAT has to be a composite ending in 0 or succ, this takes care of all possibilities.

Section 2.3

1. A must be empty or have only one element.

2. Composition is certainly a binary operation on $\mathrm{Hom}(A, A)$ since all the arrows involved have the same source and target, so that every pair of arrows is composable. It is associative by C–2 and id_A is an identity by C–4.

3. If a semigroup S has two identity elements e and e', then $e = ee' = e'$. The first equation is true because e' is an identity and the second one is true because e is an identity.

Section 2.4

1. a. If (S, α) is a set with relation, then $\mathrm{id}_S : (S, \alpha) \to (S, \alpha)$ is a homomorphism and satisfies C–3 and C–4. We must show that the composite of homomorphisms is a homomorphism; since they are set functions, C–1 and C–2 will follow. Suppose $f : (S, \alpha) \to (T, \beta)$ and $g : (T, \beta) \to (V, \gamma)$ are homomorphisms. If $x \alpha y$, then $f(x) \beta f(y)$ because f is a homomorphism. Then $g(f(x)) \gamma g(f(y))$ since g is a homomorphism. Hence $g \circ f$ is a homomorphism.

b. This is an immediate consequence of the definition of monotone in 2.4.2.

2. The identity functions are clearly continuous and strict (if relevant), so we need only show that the composite of continuous functions is continuous and the composite of strict functions is strict. Let $f : S \to T$ and $g : T \to V$ be continuous functions. Let s be the supremum of a chain $\mathscr{C} = (c_0, c_1, c_2, \ldots)$ in S. Then the image $f(\mathscr{C}) = \{f(c_i) \mid i \in \mathbf{N}\}$ is a chain in T because a continuous function is monotone. Moreover, $f(s)$ is the supremum of $f(\mathscr{C})$. Since g is continuous, $g(f(s))$ is the supremum of $g(f(\mathscr{C}))$, so that $g \circ f$ is continuous. Finally, if f and g are strict, so is $g \circ f$, since then $g(f(\bot)) = g(\bot) = \bot$.

3. The nonnegative integers form a chain in $\mathbf{R}^+$. There is no element in $\mathbf{R}^+$ satisfying SUP–1, since such an element would be a real number which is bigger than any integer.

4. Let $\mathscr{C} = (C_0, C_1, C_2, \ldots)$ be a chain in $(\mathscr{P}(S), \subseteq)$. Let V be the union $\bigcup_{i=1}^{\infty} C_i$. Then for every i, $C_i \subseteq V$, so SUP–1 is satisfied. Let $C_i \subseteq W$ for every i and suppose $v \in V$. Then $v \in C_i$ for at least one i since V is the union of all the C_i. Thus $v \in W$, which proves that $V \subseteq W$ so that SUP–2 holds. The bottom element is $\emptyset$.

5. Let $S = \mathbf{Z} \cup \{\top, \top'\}$, where $\top$ and $\top'$ are distinct and are not integers. Let the ordering $\leq$ on S be the usual ordering on $\mathbf{Z}$, with $\top \leq \top'$ and for any integer n, $n \leq \top$ and $n \leq \top'$. Then $(S, \leq)$ is an ω-CPO. The function $f : (S, \leq) \to (S, \leq)$ for which $n \mapsto n$ for $n \in \mathbf{Z}$, $\top \mapsto \top'$ and $\top' \mapsto \top'$ is monotone, but not continuous, since it does not preserve the sup of the chain $\mathbf{Z}$: that sup is $\top$ but $f(\top) = \top'$.

6. For a partial function h, define $\psi(h)(0) = 1$, and require that $\psi(h)(n)$ is defined if and only if $h(n-1)$ is defined, and then $\psi(h)(n) = 2h(n-1)$. Then the unique fixed point of ψ is g.

7. For a partial function h, define $\psi(h)$ by requiring that $\psi(h)(0) = \psi(h)(1) = 1$ and that for $n > 1$, if $h(n-1)$ and $h(n-2)$ are defined, then $\psi(h)(n)$ is defined and $\psi(h)(n) = h(n-1) + h(n-2)$. Then the Fibonacci function is the unique fixed point of ψ.

Section 2.5

1. Let $f : S \to T$ and $g : T \to V$ be semigroup homomorphisms. Then for any two elements $s, s' \in S$, $g(f(s))g(f(s')) = g(f(s)f(s')) = g(f(ss'))$, the first equation because g is a homomorphism and the second because f is a homomorphism. Thus $g \circ f$ is a homomorphism.

In the monoid case, we also have $g \circ f(1) = g(f(1)) = g(1) = 1$, so that again $g \circ f$ is a homomorphism.

2. For $m \in \mathbf{Z}_k$, $m = m + 0 = 0 \cdot k + m$, so $m +_k 0 = m$ and similarly $0 +_k m = m$. To verify the associative law let m, n and $p \in \mathbf{Z}_k$. Define r_i, q_i, $i = 1, 2, 3, 4$ by

$$\begin{aligned} m + n &= q_1k + r_1 \\ n + p &= q_2k + r_2 \\ r_1 + p &= q_3k + r_3 \\ m + r_2 &= q_4k + r_4 \end{aligned}$$

and $0 \leq r_i < k$ for $i = 1, 2, 3, 4$. We must show that $r_3 = r_4$. It follows from the equations just given that $r_3 - r_4 = (q_2 + q_4 - q_1 - q_3)k$. Since $0 \leq r_i < k$ for each i, the difference $r_3 - r_4$ has to be between $-k$ and k not inclusive. Since it is a multiple of k, it must be 0; hence $r_3 = r_4$.

3. Call the function ϕ, so that for some q, $n = qk + \phi(n)$ and $0 \leq \phi(n) < k$. Since $0 = 0 \cdot k + 0$, $\phi(0) = 0$. For arbitrary m and n in $\mathbf{Z}$, let q_1 and q_2 be the integers for which $m + n = q_1k + \phi(m + n)$ and $\phi(m) + \phi(n) = q_2k + \phi(m) +_k \phi(n)$. Then

$$\phi(m + n) - (\phi(m) +_k \phi(n)) = (q_1 - q_2)k + (m - \phi(m)) + (n - \phi(n))$$

so that the left hand side of that equation is a multiple of k. Since it is the difference of two numbers whose absolute values are less than k, the difference must be 0, so that $\phi(m) +_k \phi(n) = \phi(m + n)$.

4. It is easy to show that there is a one to one correspondence between homomorphisms ϕ from that first monoid to any given monoid M and elements $x \in M$ such that $x^4 = 1$. The correspondence takes such an element x to the homomorphism ϕ which take the elements 0, 1, 2 and 3 to 1, x, x^2 and x^3, respectively. This clearly preserves the identity and the equation $\phi(m + n) = \phi(m)\phi(n)$ is interesting only when there is a carry mod 4, and in that case reduces to the requirement that $x^4 = 1$. A ϕ constructed this way is injective if and only if $x^n \neq 1$ for $n = 1$, 2 or 3. The second monoid is easily seen to have two such elements with that property, namely 2 and 3.

5. Since f^* is a homomorphism by 2.5.7, it is necessary by 2.5.5 only to show that it is bijective. We are given that f is a bijection (isomorphisms in **Set** are bijections). If a and a' are lists in A^* and they are different, then they differ in some coordinate; suppose the ith coordinates a_i and a'_i are different. The ith coordinates of $f^*(a)$ and $f^*(a')$ are then $f(a_i)$ and $f(a'_i)$, which must be different because f is injective. Thus f^* is injective. If b is a list in B, let a be a list in A of the same length for which $f(a_i) = b_i$ for each coordinate i. There is such a list since f is surjective, and then $f^*(a) = b$, so f^* is surjective.

Section 2.6

1. $C(M)^{op}$ has exactly one object $*$ because $C(M)$ does. Define M^{op} to be $\mathrm{Hom}_{C(M)^{op}}(*, *)$. If $x, y, z \in M$ and $xy = z$ in M then $yx = z$ in M^{op}. Clearly $C(M^{op}) = C(M)^{op}$.

2. The ordering in all posets in this answer will be written '$\leq$'. If P is a poset then between any two objects in $C(P)^{op}$ there is at most one arrow because that is true in $C(P)$. Thus $C(P)^{op}$ is the category determined by a poset called P^{op}; $x \leq y$ in P^{op} if and only if $y \leq x$ in P.

3. Let P be the set $\{1, 2\}$ with discrete ordering (no two different elements are related). Let Q be the same set with the usual ordering ($1 \leq 2$). Let R be the set $\{1, 2, 3\}$ with the discrete ordering. $C(P)$ is wide in $C(Q)$ (they have the same elements) but not full ($1 \leq 2$ in Q but not in P). $C(P)$ is full in $C(R)$ because in both $x \leq y$ holds only when $x = y$, but it is not wide since they do not have the same elements.

4. Let M be the two-element monoid $M = \{1, e\}$ with identity element 1 and $ee = e$. Then in $C(M)$, the only object is $*$ and the arrows are $1 : * \to *$ and $e : * \to *$. Consider the category $\mathscr{D}$ with one object $*$ and one arrow e. Note that e is the identity arrow for $*$ in $\mathscr{D}$ but not in $\mathscr{C}$. $\mathscr{D}$ satisfies requirements S–1, S–2 and S–4 but not S–3. Note that $\mathscr{D}$ is even a category; it is just not a subcategory.

Section 2.7

1. We have $f^{-1}\circ g^{-1}\circ g\circ f = f^{-1}\circ f = \mathrm{id}_A$ and similarly $g\circ f\circ f^{-1}\circ g^{-1} = \mathrm{id}_C$. Thus $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$ because the inverse is unique (2.7.3).

2. (a) Let P_1 be the set $\{1,2\}$ with discrete ordering (no two different elements are related). Then $P_1 = P_1^{op}$ since $x\le y$ if and only if $x = y$.

(b) Let P_2 be the same set with the usual ordering $(1\le 2)$. Then $P_2 \neq P_2^{op}$ since $1\le 2$ in P_2 but not in P_2^{op}. However the map $1\mapsto 2$, $2\mapsto 1$ is an isomorphism from P_2 to P_2^{op}.

(c) Finally, let $P_3 = \{\{1\},\{2\},\{1,2\}\}$ ordered by inclusion. Then P_3 is not isomorphic to P_3^{op} since P_3 has an upper bound, namely $\{1,2\}$, but P_3^{op} has no upper bound.

3. First we construct a monoid M for which $M \neq M^{op}$ but M is isomorphic to M^{op}: let $A = \{x,y\}$ and $M = A^*$. Then in M, $(x)(y) = (x,y)$, while in M^{op}, $(x)(y) = (y,x)$ so that $M \neq M^{op}$. On the other hand the function $\phi : M \to M^{op}$ defined by $\phi(a_1,\ldots,a_n) = (a_n,\ldots,a_1)$ is easily seen to be an isomorphism of M with M^{op}.

For a monoid M that is not isomorphic to M^{op}, we use a construction that works for any set S: The 'right monoid' generated by S is defined on the set $S\cup\{1\}$ (1 is some element not in S) by $xy = y$ for $x, y\in S$ and $x1 = 1x = x$ for $x\in S\cup\{1\}$. M^{op} (the 'left monoid') is not isomorphic to M so long as S has two or more elements.

4. Let $(P,\le)$ and $(Q,\le)$ be posets and $\phi : P \to Q$ an isomorphism. Suppose P is totally ordered. Let $q, q'\in Q$. Since P is totally ordered, we may assume (after perhaps renaming) that $\phi^{-1}(q)\le\phi^{-1}(q')$. Then $q = \phi(\phi^{-1}(q)) \le \phi(\phi^{-1}(q')) = q'$ so Q is totally ordered.

5. If x and y are isomorphic objects, then there is an arrow from x to y and one from y to x (the isomorphism and its inverse). Then $x\le y$ and $y\le x$, so $x = y$ by antisymmetry.

6. Let $\phi : S \to T$ be an isomorphism of semigroups. Since it a set function with an inverse, it is a bijection by Proposition 2.7.9. Conversely, suppose that $\phi : S \to T$ is a bijective homomorphism. Then its inverse is a homomorphism by 2.5.5.

7. Suppose f satisfies (a). Then for any $x, y\in A$, $f(x) = k(*) = f(y)$. Conversely, suppose f satisfies (b). If $A = B = \emptyset$, then $\langle\rangle = \mathrm{id}_\emptyset$ and we can take $k = \mathrm{id}_\emptyset$ to get $f = \mathrm{id}_\emptyset = \mathrm{id}_\emptyset\circ\mathrm{id}_\emptyset$. Otherwise, define an element b of B as follows. If $A = \emptyset$, let b be any element of B. If not, let b be the element such that $f(x) = f(y) = b$ for all $x, y\in A$. Define $k : 1 \to A$ to be $*\mapsto b$ where $*$ is the unique element of 1. Then $f = k\circ\langle\rangle$.

8. Let $\mathscr{E}$ be the graph with one node e and one arrow $f : e \to e$. Let $\mathscr{G}$ be any graph. Define $\phi : \mathscr{G} \to \mathscr{E}$ by $\phi_0(g) = e$ for any node g and $\phi_1(u) = f$ for any arrow $f : g \to h$. Then $\phi_1(u) : \phi_0(g) \to \phi_0(h)$, so ϕ is a graph homomorphism. It is clearly the only possible one.

9. Suppose $f : S \to S$ is an idempotent set function. Let x be an element of the image, so there is some $s \in S$ such that $f(s) = x$. Then $f(x) = f(f(s)) = f(s) = x$, so x is a fixed point. Conversely, if $f(x) = x$, then x is in the image of f by definition.

Conversely, suppose the image of f is the same as its set of fixed points. Then for any $x \in S$, $f(f(x)) = f(x)$, so that $f \circ f = f$.

10. a. Let $f : A \to A$ be an idempotent set function. Let B be the image of f and $g : A \to B$ the corestriction of f. Let $h : B \to A$ be the inclusion of B in A. Then for any $x \in A$, $h(g(x)) = g(x) = f(x)$, so $h \circ g = f$. And for any $b \in B$, $g(h(b)) = g(b) = f(b) = b$ by the result of the preceding problem, so $g \circ h = \mathrm{id}_B$.

b. Let $\mathscr{E}$ be the category with one object $\star$ and two arrows 1, e with $1 = \mathrm{id}_\star$ and $e \circ e = e$. There are no arrows g, h for which $h \circ g = e$ and $g \circ h = 1$ since in this category, composition is commutative. (Note that 1 is split. The identity is split in any category.)

11. (i) is false. The category with two distinct objects and only identity arrows is a counterexample.

(ii) is false. Take the category of the preceding example and add one arrow u going from one of the objects, say A, to itself, with $u \circ u = \mathrm{id}_A$.

(iii) is true. Suppose $f : A \to B$ is an arrow of $C(P)$. Its inverse goes from B to A so $A \leq B$ and $B \leq A$ in P. Hence $A = B$. Since there is never more than one arrow with the same source and target in $C(P)$, f must be id_A.

Section 2.8

1. a. Let $f : A \to B$ be a monomorphism in $\mathscr{C}$ and let $\mathscr{D}$ be a subcategory of $\mathscr{C}$. Then if $f \circ g = f \circ h$ in $\mathscr{D}$ the same equation is true in $\mathscr{C}$, so that $g = h$ in $\mathscr{C}$. Hence f is a monomorphism in $\mathscr{D}$.

b. Let $\mathscr{C}$ be the full subcategory of **Set** determined by the sets $A = \{1\}$ and $B = \{1, 2\}$. Let $k : B \to A$ be the (only possible) function. Let $\mathscr{D}$ be the subcategory of $\mathscr{C}$ consisting of id_A, id_B, and k. Then k is a monomorphism in $\mathscr{D}$ since there are not two distinct arrows from B to B, but it is not a monomorphism in $\mathscr{C}$ since k composed with any of the four functions from B to B gives k again.

2. Suppose $(g \circ f) \circ x = (g \circ f) \circ y$. Then $g \circ (f \circ x) = g \circ (f \circ y)$. Because g is a monomorphism, $f \circ x = f \circ y$. Then $x = y$ as required because f if a monomorphism.

3. This happens in **Set**. Let f be the inclusion of $\{1,2\}$ into $\{1,2,3\}$ and $g : \{1,2,3\} \to \{1,2\}$ be $1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 2$. Then $g \circ f$ (which is $\mathrm{id}_{\{1,2\}}$) is monic but g is not.

4. In this answer we use these facts repeatedly:

(a) An isomorphism in the category of sets is a bijection and conversely.

(b) If $i : X \to S$ and $j : Y \to S$ are injective functions and $\beta : X \to Y$ is a bijection for which $j \circ \beta = i$ then $i \circ \beta^{-1} = j$.

(a)(i) Let $s = m(a)$ be in the image of m and let $\beta : A \to B$ be the bijection for which $m = n \circ \beta$. Then $s = n(\beta(a))$ so s is in the image of n. A symmetric argument using β^{-1} shows that the image of n is included in the image of m, so the images are equal.

(a)(ii) Let $m : A \to S$ be an injection in $\mathcal{O}$. Define $\beta : A \to I$ to be the corestriction of m to I. β is injective because m is and surjective because I is the image of m. Hence it is bijective. Clearly $i \circ \beta = m$.

(a)(iii) Let $\beta : J \to I$ be the bijection for which $i \circ \beta = j$. Since i and j are both inclusions, for any $x \in J$, $i(x) = x = j(x) = i(\beta(x))$, so $i \circ \beta = i$. Since i is injective, $\beta(x) = x$ for all $x \in J$. Since β is surjective, $I = J$ and $\beta = \mathrm{id}_I$.

(a)(iv) Immediate from (i), (ii) and (iii).

(b)(i) Immediate from the definition of subobject.

(b)(ii) Immediate from properties of equivalence relations.

(b)(iii) Immediate from (i) and (ii).

5. We must find arrows $k : D \to C$ and $k' : C \to D$ such that $\mathrm{id}_D = h \circ k$ and $h = \mathrm{id}_D \circ k'$. These requirements are satisfied by $k = h^{-1}$ and $k' = h$.

6. Let $m : C \to 0$ be a monomorphism into an initial object. By Proposition 2.8.7, m is an isomorphism. By Problem 5, it is equivalent to id_0, so that the subobject it inhabits is not proper by definition of proper.

7. Let $m : A \to 1$ be monic and let $f, g : B \to A$. Then $m \circ f = m \circ g : B \to 1$ by definition of terminal object, and m is monic, so $f = g$. In particular, for any arrow $h : A \to C$, $h \circ f = h \circ g$ implies that $f = g$; hence h is monic.

8. a. The terminal object is a one-element set. By the remarks in 2.8.12, such a set has two subobjects because it has two subsets: the set itself and the empty set. (Note in connection with the preceding exercise that there is at most one function from any set B to the empty set: none at all if B is nonempty, and the identity function if B is empty.)

b. The terminal graph has one node and one arrow. It has three subgraphs: itself, the graph with one node and no arrows, and the graph with no nodes or arrows. These are in one to one correspondence with the subobjects.

c. The terminal monoid is the monoid with one element. It has only itself as a submonoid.

Section 2.9

1. Let $f : S \to T$ be a surjective monoid homomorphism and let $g, h : T \to V$ be monoid homomorphisms. Suppose that $g \circ f = h \circ f$. Let t be any element of T. Then there is an element $x \in S$ for which $f(x) = t$. Then $g(t) = g(f(x)) = h(f(x)) = h(t)$. Since t was arbitrary, $g = h$ so f is an epimorphism.

2. That ϕ is a homomorphism requires verifying that $\phi(0) = 0$ (which is true by definition) and that $\phi(m +_4 n) = \phi(m) +_2 \phi(n)$ for all $m, n \in \mathbf{Z}_4$: for example, $\phi(2 +_4 3) = \phi(1) = 1$ and $\phi(2) +_2 \phi(3) = 0 +_2 1 = 1$. (The comments in the answer to Exercise 4 of Section 2.5 apply here too.) It is surjective since it has both 0 and 1 as values.

Now suppose $\psi : \mathbf{Z}_2 \to \mathbf{Z}_4$ is a monoid homomorphism satisfying $\phi \circ \psi = \mathrm{id}_{\mathbf{Z}_2}$. Since $\psi(0) = 0$ necessarily, there are only four possibilities for ψ, determined by what it does to 1. If $\psi(1) = 0$ then $\phi(\psi(1)) = \phi(0) = 0 \neq 1$. If $\psi(1) = 1$ then $\psi(1 +_2 1) = \psi(0) = 0$ but $\psi(1) +_4 \psi(1) = 2$, so ψ is not a homomorphism. If $\psi(1) = 2$ then $\phi(\psi(1)) \neq 1$. Finally, if $\psi(1) = 3$, then again ψ is not a homomorphism because $\psi(1 +_2 1) = 0 \neq 2 = \psi(1) +_4 \psi(1)$.

3. a. The statement that $m \in M$ is a monomorphism says that $mn = mp$ implies $n = p$ (m is left cancellable). That it is an epimorphism says that $nm = pm$ implies $n = p$ (m is right cancellable) If m is an isomorphism then m is a monomorphism and an epimorphism by Proposition 2.9.9. To see the converse, let m be a monomorphism. Let $M = \{m_1, m_2, \ldots, m_k\}$ be finite. The elements $mm_1, mm_2, \ldots, mm_k$ are all different, so there are k of them, so one of them, call it mn, is 1. But then $mnm = m = m \cdot 1$ and m is a monomorphism, so $nm = 1$. Hence m is an isomorphism. A symmetric argument shows that an epimorphism is an isomorphism.

Warning: In a finite semigroup, a left cancellable element need not be right cancellable.

b. Every nonzero element of the nonnegative integers with addition as operation is both a monomorphism and an epimorphism but not an isomorphism. (In other words, $m + n = m + p$ implies $n = p$ and similarly on the other side, but if $m \neq 0$ then m has no inverse.)

4. If $f : A \to B$ has a splitting $g : B \to A$, then in P, $A \leq B$ and $B \leq A$ so $A = B$. The only arrow from A to itself is id_A.

5. $h \circ g$ is an idempotent because $h \circ g \circ h \circ g = h \circ \mathrm{id} \circ g = h \circ g$. It is split because h and g fit the requirements for the h and g of the definition in the exercise mentioned: $h \circ g = h \circ g$ and $g \circ h = \mathrm{id}$.

6. (i) $\mathrm{Hom}(A, f)(g) = \mathrm{Hom}(A, f)(h)$ if and only if $f \circ g = f \circ h$ by definition of $\mathrm{Hom}(A, f)$. The result is the immediate from the definitions of injective and monomorphism.

(ii) $\mathrm{Hom}(f, D)(u) = \mathrm{Hom}(f, D)(v)$ if and only if $u \circ f = v \circ f$. Again, the result follows from the definition of injective and epimorphism.

(iii) Suppose f is a split monomorphism with splitting g, so $g \circ f = \mathrm{id}_B$. Suppose $u : B \to D$. Then $\mathrm{Hom}(f, D)(u \circ g) = u \circ g \circ f = u$, so $\mathrm{Hom}(f, D)$ is surjective. Conversely suppose $\mathrm{Hom}(f, D)$ is surjective for every object D. Then there is a $g : C \to B$ such that $\mathrm{hom}(f, B)(g) = \mathrm{id}_B$; that is, $g \circ f = \mathrm{id}_B$, so f is split.

(iv) Suppose f is a split epimorphism, with $f \circ g = \mathrm{id}_C$. Let $u : A \to C$. Then we have $\mathrm{Hom}(A, f)(g \circ u) = f \circ g \circ u = u$ so $\mathrm{Hom}(A, f)$ is surjective. Conversely, suppose $\mathrm{Hom}(A, f)$ is surjective for every A. Let $g : C \to B$ satisfy $\mathrm{Hom}(B, f)(g) = \mathrm{id}_C$, that is, $f \circ g = \mathrm{id}_C$, so f is a split epimorphism.

(v) An isomorphism is a split epimorphism and a split monomorphism because it is split by its inverse. Thus an isomorphism satisfies (a) and (b). In particular an isomorphism is an epimorphism and a monomorphism. It then follows from (i) through (iv) that an isomorphism satisfies (c) and (d). Conversely, suppose f is a split epi, split by $g : C \to B$, so $f \circ g = \mathrm{id}_C$. Then $f \circ g \circ f = f = f \circ \mathrm{id}_B$. If f is also mono then $g \circ f = \mathrm{id}_B$. Hence (a) implies that f is an isomorphism. That also follows from (b) by doing the same proof in the opposite category (note that the dual of the concept of isomorphism is isomorphism). Finally, (c) implies (a) by (i) and (iv) and (d) implies (b) by (ii) and (iii).

7. (i) Suppose an arrow $x : m \to n$ in $\mathscr{H}$ is not in the image of f_1. Let $\mathscr{U}$ be the graph with one node a and two arrows $u, v : a \to a$. Let $g : \mathscr{H} \to \mathscr{U}$ take all nodes to a and all arrows to u, and let h take all nodes to a and all arrows except x to u, with $h(x) = v$. Both g and h are graph homomorphisms. Then $g \circ f = h \circ f$ but $g \neq h$, showing that f is not epi.

If all the arrows of $\mathscr{H}$ are in the image of f_1 but there is a node n not in the image of f_0, then there can be no arrows with n as source or target. Let $\mathscr{V}$ be the graph with two nodes a and b and one arrow $u : a \to a$. Let $g : \mathscr{H} \to \mathscr{V}$ take all nodes to a and all arrows to u; let $h : \mathscr{H} \to \mathscr{V}$ take all nodes except n to a, n to b and all arrows to u. Again, g and h are graph homomorphisms and $g \neq h$ but $g \circ f = h \circ f$.

Conversely, suppose that f_0 and f_1 are both surjective. Let $g, h : \mathscr{H} \to \mathscr{K}$ satisfy $g \circ f = h \circ f$. Then for every node or arrow x of $\mathscr{H}$ there is a node or arrow m of $\mathscr{G}$ for which $f_i(m) = x$ ($i = 0$ if x is a node, $i = 1$ if x is an arrow). Then $g_i(x) = g_i(f_i(m)) = h_i(f_i(m)) = h_i(x)$, so $g = h$ and f is epi.

(ii) Suppose f is monic and suppose u and v are arrows of $\mathscr{G}$ for which $f_1(u) = f_1(v)$. Let $\mathscr{F}$ denote the graph with two nodes a and b and one arrow $x : a \to b$, and define $g : \mathscr{F} \to \mathscr{G}$ to take x to u and $h : \mathscr{F} \to \mathscr{G}$ to take x to v. What g and h do on nodes is then forced. Then $f \circ g = f \circ h$ but $g \neq h$, so f is not monic. If f is monic and f_1 is injective but there are nodes m and n of $\mathscr{G}$ for which $f_0(m) = f_0(n)$, then we carry out the same trick except that we take $\mathscr{F}$ to be the graph with one node and no arrows and let g and h take that node to m and n respectively.

Conversely, suppose f_0 and f_1 are injective. Suppose $g : \mathscr{F} \to \mathscr{G}$ and $h : \mathscr{F} \to \mathscr{G}$ are graph homomorphisms such that $f \circ g = f \circ h$. Let x be any node or arrow of $\mathscr{F}$. Then $f_i(g_i(x)) = f_i(h_i(x))$ ($i = 0$ if x is a node, $i = 1$ if x is an arrow), so because f_i is injective, $g_i(x) = h_i(x)$. Hence $g = h$ so f is monic.

(iii) This is an immediate consequence of Exercise 3 of Section 1.4 and the fact that an isomorphism in the category of sets is a bijection.

Solutions for Chapter 3

Section 3.1

1. Since functors preserve the operations of domain and codomain, the fact that $g \circ f$ is defined implies that the domain of g is the codomain of f. But then the domain of $F_1(g)$, which is F_0 applied to the domain of g, is the same as the codomain of $F_1(f)$. Hence $F_1(g) \circ F_1(f)$ is defined in $\mathscr{D}$.

2. The initial category has no objects and, therefore, no arrows. It clearly has exactly one functor to every other category. The terminal category has just one object and the identity arrow of that object. To any category $\mathscr{C}$ there is just one functor that takes every object to that single object and every arrow to that one arrow.

3. If $f : A \to B$ is a function between sets, the existential powerset functor takes f to f_* defined by $f_*(A_0) = \{f(a) \mid a \in A_0\}$. If $g : B \to C$, then $c \in (g_* \circ f_*)(A_0)$ if and only if there is an $a \in A_0$ such that $c = g(f(a))$. This is the same condition that $c \in (g \circ f)_*(A_0)$. It is evident that when f is the identity, so is f_*.

The universal powerset functor $f_!$ is defined by $b \in f_!(A_0)$ if and only if $b = f(a)$ implies that $a \in A_0$. If f is the identity, then $b = f(a)$ if and only if $b = a$ so that $f_!(A_0) = A_0$ (the identity arrow). If $f : A \to B$ and $g : B \to C$, then $c \in (g_! \circ f_!)(A_0)$ if and only if $c = g(b)$ implies that $b \in f_!(A_0)$, which is true if and only if $b = f(a)$ implies $a \in A_0$. Putting these together, we see that $c \in (g_! \circ f_!)(A_0)$ if and only if $c = f(g(a))$ implies $a \in A_0$. But this is the condition that $c \in (g \circ f)_!(A_0)$.

4. a. Since a functor is determined by its value on objects and arrows, a functor that is injective on objects and arrows is certainly a monomorphism. (Compare Exercise 7 of Section 2.9.) The other way is less obvious. If F is not injective on objects, say $C \neq C'$ but $F(C) = F(C')$ then let $\mathbf{1}$ be the category with one object and only the identity arrow. Let $G : \mathbf{1} \to \mathscr{C}$ take that object to C, while $G' : \mathbf{1} \to \mathscr{C}$ takes it to C'. Then $G \neq G'$, while $F \circ G = F \circ G'$, whence F is not a monomorphism. Next suppose F is injective on objects, but not on arrows.

If $f \neq g$ are two arrows with $F(f) = F(g)$, then $F(f)$ and $F(g)$ have the same domain and the same codomain. Since F is injective on objects, it must be that f and g have the same domain and the same codomain. Say that $f, g : C \longrightarrow C'$. Now let **2** be the category with two objects and one nonidentity arrow between them:

$$0 \longrightarrow 1$$

Let $G : \mathbf{2} \longrightarrow \mathscr{C}$ take 0 to C, 1 to C' and the nonidentity arrow to f, while G' is the same on objects, but takes the nonidentity arrow to g. Clearly $G \neq G'$, while $F \circ G = F \circ G'$.

b. The functor simply forgets the existence of the identity element. Two distinct monoids must differ in either their sets of elements or multiplication and in either case must differ as semigroups. Thus the functor is injective on objects. For similar reasons, it is injective on arrows. We have just seen that a functor that is injective on objects and arrows is a monomorphism.

5. a. It being evident that e is a two-sided identity, it is necessary only to show that the multiplication is associative. In any equation $x(yz) = (xy)z$ as soon as any of the variables is e, both sides reduce to a binary product of the remaining two terms (one or both of which might also be e). If none of them is e, then this is an equation involving terms from S and so is valid because it is in S.

b. If we denote by e_S and e_T the elements added to S and T respectively, then we define $F(f) = f^1 : S^1 \longrightarrow T^1$ by

$$f^1(x) = \begin{cases} f(x) & \text{if } x \in S \\ e_T & \text{if } x = e_S \end{cases}$$

In verifying that $f^1(xy) = f^1(x)f^1(y)$ it is necessary to consider cases. If neither x nor y is e_S, then it follows from the fact that f is a homomorphism of semigroups. If, say, $x = e_S$, then both sides reduce to $f^1(y)$ and similarly if $y = e_S$. Finally, we must show that F is a functor. It is clear that if $f : S \longrightarrow S$ is the identity, then $f^1 : S^1 \longrightarrow S^1$ is the identity as well. It is also clear that if $g : T \longrightarrow R$ is another monoid homomorphism, then

$$(g^1 \circ f^1)(x) = \begin{cases} (g \circ f)(x) & \text{if } x \in S \\ e_R & \text{if } x = e_S \end{cases} = (g \circ f)^1(x)$$

c. Since F is injective on objects and arrows, it is a monomorphism (see preceding exercise).

6. Define $\alpha : \mathrm{Hom}_{\mathbf{Mon}}(F(A), M) \longrightarrow \mathrm{Hom}_{\mathbf{Set}}(A, U(M))$ by $\alpha(g)(a) = g(a)$. The fact that $(\alpha(\beta(f)))(a) = f(a)$ is clear. On the other hand, if $g : F(A) \longrightarrow M$ is a monoid homomorphism, then

$$\begin{aligned}(\beta(\alpha(g)))(a_1, a_2, \cdots, a_n) &= \alpha(g)(a_1)\alpha(g)(a_2)\cdots\alpha(g)(a_n) \\ &= g(a_1)g(a_2)\cdots g(a_n) = g(a_1, a_2, \cdots, a_n)\end{aligned}$$

the last equality coming from the fact that g is a monoid homomorphism. Thus β is invertible and hence bijective.

7. In Exercise 5, we showed that $\gamma(h)$ (which would have been called h^1 there) is a monoid homomorphism. To see that γ is a bijection, we define

$$\delta : \mathrm{Hom}_{\mathbf{Mon}}(F(S), M) \to \mathrm{Hom}_{\mathbf{Sem}}(S, U(M))$$

by $\delta(g)(x) = g(x)$ for $x \in S$. Since a monoid homomorphism is also a semigroup homomorphism, this is well defined. Clearly $\delta(\gamma(h))(x) = h(x)$ for $x \in S$ and $h : S \to U(M)$. To go the other way, suppose $g : F(S) \to M$. For $x \in S$, $\gamma(\delta(g))(x) = \delta(g)(x) = g(x)$. For e_S, we have $\gamma(\delta(g))(e_S) = 1$ by definition of γ, but $g(e_S) = 1$ since g is a monoid homomorphism. Thus $\delta = \gamma^{-1}$ and γ is a bijection.

8. This is an immediate consequence of Exercises 4.b and 5.c.

9. It is obvious that **Set** is a subcategory of the category of partial functions. To get a functor in the other direction, let $F(S) = S \cup \{S\}$. If $f : S \to T$ is a partial function, let $F(f) : F(S) \to F(T)$ be the total function defined by

$$F(f)(x) = \begin{cases} f(x) & \text{if } x \in S \text{ and } f(x) \text{ is defined} \\ T & \text{if } x \notin S \text{ or } f \text{ is not defined at } x \end{cases}$$

It is clear that if $f \neq g$, then $F(f) \neq F(g)$.

10. On the one hand, if $\mathscr{A}$ is discrete, for any function $F_0 : \mathscr{A}_0 \to \mathscr{B}_0$ there is a unique functor $F : \mathscr{A} \to \mathscr{B}$ whose value at the identity of some object A of $\mathscr{A}$ is the identity of $F_0(A)$. To go the other way, suppose $\mathscr{A}$ is not discrete. Suppose first that there is an arrow $f : A \to B$ in $\mathscr{A}$ with $A \neq B$. Let $\mathscr{B}$ be the category with the same objects as $\mathscr{A}$, but with no nonidentity arrows. Then the identity function $F_0 : \mathscr{A}_0 \to \mathscr{B}_0$ cannot be extended to a functor since there is nowhere to send f. Now suppose that all arrows of $\mathscr{A}$ are endoarrows. Let $\mathscr{B}$ have the same objects as $\mathscr{A}$ and let

$$\mathrm{Hom}_{\mathscr{B}}(A, A) = \mathrm{Hom}_{\mathscr{A}}(A, A) \times \mathrm{Hom}_{\mathscr{A}}(A, A)$$

for an object A of $\mathscr{A}$. The identity function $F_0 : \mathscr{A}_0 \to \mathscr{B}_0$ can be extended in at least two ways to a functor. The first way is to take $F(f) = (f, \mathrm{id}_A)$ for $f : A \to A$ in $\mathscr{A}$ and the second is $F'(f) = (\mathrm{id}_A, f)$.

11. We claim that a category is indiscrete if and only if there is exactly one arrow between any two objects. It is obvious that such a category is indiscrete. To go the other way, first suppose that $\mathscr{A}$ has two objects A and B with no arrows between them, then $\mathscr{A}$ cannot be indiscrete. For if **2** denotes the category with two objects 0 and 1 and one arrow between them, then the object function that takes 0 to A and 1 to B cannot be extended. If there is more than one arrow from A to B, then the same object function on **2** has more than one extension to a functor.

Section 3.2

1. If M acts on S, let $\phi : M \to FT(S)$ be defined by $\phi(a)(x) = ax$ for $a \in M$ and $x \in S$. We have $\phi(1)(a) = 1a = a = \mathrm{id}(a)$ so that $\phi(1) = \mathrm{id}$. Also, for $a, b \in M$,

$$\phi(ab)(x) = (ab)x = a(bx) = \phi(a)(\phi(b)(a))$$

so that $\phi(ab) = \phi(a) \circ \phi(b)$. Conversely, if $\phi : M \to FT(S)$, then let M act on S by letting $ax = \phi(a)(x)$. The computations above can be reversed to show that since ϕ is a monoid homomorphism, the M-set identities are satisfied. It is clear that these processes are inverse to each other.

Section 3.3

1. a. Since a functor is faithful if it is injective between hom sets and a monoid has only one hom set, such a functor is faithful if and only if it is injective.

b. Since hom sets are either singleton or empty and a function on such a set is always injective, such a functor is always faithful.

2. A functor is full if it is surjective on hom sets so a functor between monoids is full if and only if it is surjective. As for posets, a functor $f : P \to Q$ between posets is full if and only if whenever $f(x) \le f(y)$, then $x \le y$.

3. No. For example let $(\mathbf{N}; +)$ and $(\mathbf{N}; *)$ denote the monoids of integers with the operations of addition and multiplication, respectively. The function $f : (\mathbf{N}; +) \to (\mathbf{N}; *)$ that is constantly 0 is a semigroup homomorphism that is not a monoid homomorphism since it does not preserve the identity.

4. It is faithful, but not full. Certainly, if $f \ne g : S \to T$, then $F(f) \ne F(g) : S^* \to T^*$ since on strings of length 1, $F(f)$ is essentially the same as f. On the other hand, there are infinitely many homomorphisms from $\mathbf{N} = F(1)$ to itself (take the generating element to any power of itself), but only one function from $\{1\}$ to itself.

5. It is faithful because f can be recovered from $\mathscr{P}(f)$ by its actions on singletons. On the other hand, it cannot be full since, for example, there is only one function from 1 to 1 and four from $\mathscr{P}(1)$ to $\mathscr{P}(1)$.

6. The isomorphism is the functor from $\mathbf{Rel}^{op}$ to $\mathbf{Rel}$ that takes every object to itself and every relation $R \subseteq A \times B$ from A to B in $\mathbf{Rel}$ (hence from B to A in $\mathbf{Rel}^{op}$) to the relation $R^{op} = \{(b, a) \mid (a, b) \in R\}$. The inverse functor is this same functor considered as going from $\mathbf{Rel}$ to $\mathbf{Rel}^{op}$.

7. Let $\mathscr{G}$ be a groupoid. The isomorphism $F : \mathscr{G}^{op} \to \mathscr{G}$ is the function that takes objects to themselves and an arrow $f : X \to Y$ in $\mathscr{G}$ (hence $f : Y \to X$ in $\mathscr{G}^{op}$) to $f^{-1} : Y \to X$ in $\mathscr{G}$. F preserves composition by the Shoe-Sock Theorem (Exercise 1, page 42.) Since $F(F(f)) = (f^{-1})^{-1} = f$ (see 2.7.3), it follows that $F : \mathscr{G}^{op} \to \mathscr{G}$ is the inverse of $F : \mathscr{G} \to \mathscr{G}^{op}$, so that F is an isomorphism.

8. A functor from a monoid to a category takes the single object of the monoid to some object of the category and is a monoid homomorphism from the monoid to the monoid of endoarrows of that object. In other words, its arrow part is a monoid homomorphism. Therefore, if $i : \mathbf{N} \to \mathbf{Z}$ is the inclusion, the nonexistence of monoid homomorphisms $g \neq h$ such that $g \circ i = h \circ i$ implies that there also cannot exist functors with that property.

9. Suppose $f : A \to B$ is a split mono in a category. Then there is a $g : B \to A$ with $g \circ f = \mathrm{id}_A$. For any functor F, $F(g) \circ F(f) = F(g \circ f) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, since functors preserve composition and identities. The argument for split epis is dual.

10. a. There is exactly one semigroup structure on the empty set and also on the one point set and these are the initial, respectively terminal semigroups. Thus the underlying functor preserves, reflects and creates both initial and terminal objects.

b. There is exactly one category structure on the empty graph and that is the initial category, so the underlying functor preserves, reflects and creates the initial object. The terminal category has one object and one arrow, the identity. This is the unique category structure on the graph with one object and one arrow and that is the terminal graph. Thus the terminal object is also preserved, reflected and created.

11. Let $F : \mathscr{C}/B \to (\mathscr{C}/A)/f$ send $u : C \to B$ to $u : f \circ u \to f$ as suggested. If $v : D \to B$ is another object of $\mathscr{C}/B$ and $g : u \to v$ is an arrow (which is to say that $v \circ g = u$), then $f \circ v \circ g = f \circ u$ so that we can simply let $F(g : u \to v) = g : F(u) \to F(v)$. Let $G : (\mathscr{C}/A)/f \to \mathscr{C}/B$ take an object $w : (g : C \to A) \to f$ to $w : C \to B$. Suppose $w' : g' \to f$ is also an arrow of $\mathscr{C}/A$ (hence an object of $(\mathscr{C}/A)/f$) and $x : w' \to w$ is an arrow of $(\mathscr{C}/A)/f$. Then by definition of arrow in a slice category, $w \circ x = w' : g' \to f$ in $\mathscr{C}/A$, hence in $\mathscr{C}$. Thus we can set $G(x : w' \to w) = x : \mathrm{dom}(w') \to \mathrm{dom}(w)$. It is straightforward to check that F and G are functors. It is clear that they are inverse to each other.

Section 3.4

1. Let $F : \mathscr{C} \to \mathscr{D}$ be an isomorphism with inverse G. Then G is a pseudo-inverse for F. We define the arrows u_C required by E–2 to be id_C for each object C, and similarly $v_C = \mathrm{id}_C$. Since $G(F(f)) = f$ and $F(G(g)) = g$ for all arrows f of $\mathscr{C}$ and g of $\mathscr{D}$, requirements E–2 and E–3 are satisfied.

2. Define $F : \mathbf{Pfn} \to \mathbf{Pts}$ by $F(S) = (S \cup \{S\}, S)$. S is chosen as the additional element to guarantee that it is not already an element of S. If $f : S \to T$ is a partial function, let $F(f)$ be defined by

$$F(f)(x) = \begin{cases} f(x) & \text{if } x \in S \text{ and } f \text{ is defined at } x \\ T & \text{otherwise} \end{cases}$$

It is obvious that F preserves identities. As for composition, if $g : T \to R$ is a partial function, then $g(f(x))$ is defined if and only if f is defined at x and g is defined at $f(x)$, which is exactly when $g \circ f$ is defined at x. From this, it is immediate that F preserves composition.

We define a pseudo-inverse $G : \mathbf{Pts} \to \mathbf{Pfn}$ by $G(S, s) = S - \{s\}$ and if $f : (S, s) \to (T, t)$ is an arrow, then

$$G(f)(x) = \begin{cases} f(x) & \text{if } f(x) \neq t \\ \text{undefined} & \text{if } f(x) = t \end{cases}$$

It is easy to show that G is a functor. Now if S is a set, it is clear that $G(F(S)) = S$ and $(S, s) \cong F(G(S, s))$ by an isomorphism v_S which is the identity on S and takes s to S (the latter being the added element of $F(S - \{s\})$). What has to be shown is that for any partial function $f : S \to T$ and any function $g : (S, s) \to (T, t)$, E–2 and E–3 hold. These are a simple matter of considering cases and we omit them.

3. Let $\mathscr{P}$ be the category of preordered sets and $\mathscr{Q}$ the category of small categories as described. We let $F : \mathscr{P} \to \mathscr{Q}$ take a preordered set $(P, \leq)$ to the category $C(P, \leq)$ as described in 2.3.1. If $f : P \to P'$ is a monotone function, then $F(f)$ agrees with f on objects and when $a \leq b$, then we let $F(f)(b, a) = (f(b), f(a))$. In the other direction, let Q be a category in $\mathscr{Q}$ and let $G(Q)$ be the preordered set whose elements are the objects of Q with the preorder that $a \leq b$ if there is an arrow $a \to b$. The composition law in the category makes this relation transitive and the identity makes it reflexive. If $f : Q \to Q'$ is a functor in $\mathscr{Q}$, let $G(f)$ be the object function of f. $G(f)$ is monotone because if $x \leq y$ in Q then f must take the corresponding arrow to an arrow in Q'. It is clear that $G \circ F$ is the identity, so that E–2 is satisfied with $u_C = \mathrm{id}_C$. $F \circ G$ is the identity on the objects and that there will be an arrow $a \to b$ in Q if and only if $a \leq b$ in $G(Q)$ if and only if there is an arrow $a \to b$ in $F(G(Q))$. Thus there is an isomorphism $v_C : Q \to F(G(Q))$ which is the identity on objects and which takes an arrow $a \to b$ in Q to (the only) arrow $a \to b$ in $F(G(Q))$. These isomorphisms must satisfy E–3 because for any arrow g there is only one arrow that $v_{D'} \circ g \circ v_D^{-1}$ can be.

4. There is an functor $F : \mathscr{M} \to \mathscr{L}$ that takes n to the space of n-rowed column vectors and a matrix $A : m \to n$ to the linear transformation of multiplying on the left by A. In case one of the numbers is 0, there is only the 0 linear transformation between them. It is well known that this is a functor (matrix multiplication corresponds to composition of linear transformations) which is full and faithful and that every finite dimensional vector space is isomorphic to a space of column vectors. A pseudo-inverse G is found by choosing, for each space V, a basis B and then letting $G(V)$ be the number of elements of B. If V' is another space with chosen basis B' and $T : V \to V'$ is a linear transformation,

then G(T) is the matrix of T with respect to B and B^0. Although G is uniquely de¯ned on objects, its value on arrows depends completely on the choice, for each vector space V, of a basis for that space. In this case u_m is the m £ m identity matrix, so E{2 is satis¯ed, and E{3 follows from the de¯nition of the linear transformation determined by a matrix, given a basis.

5. a. Let F : Mon ¡! Ooc take the monoid M to the category with one object and with M as its set of endoarrows. If f : M ¡! N is a monoid homomorphism, then F(f)(M) = N on the object and F(f)(a) = f(a) for a 2 M. It is immediate that F is a functor. Let G : Ooc ¡! Mon take the one-object category C to the object of endomorphisms of that single object. It is clear that G ± F is the identity, so E{2 holds. F ± G is the identity on the arrows of the category and is evidently the only possible isomorphism on the singleton set of objects, so E{3 holds because there is no choice possible for the arrow.

b. Suppose f; g : M ¡! N are homomorphisms of monoids. Then since F(f) agrees with f on the arrows of F(M), F(f) = F(g) implies f = g. This shows directly that F is faithful. To see it is full, let h : F(M) ¡! F(N) be a functor. Then the arrow function of h is a monoid homomorphism f from the monoid of endomorphisms of the object of F(M) to the object of F(N). F(f) agrees with h on arrows and certainly does on objects since there is no choice.

Section 3.5

1. Suppose that both CR{1 and CR{2 of 3.5.1 are satis¯ed. Then special cases result from letting h or k be an identity. Using these special cases, we have that

$$g_1 ± f_1 » g_2 ± f_1 » g_2 ± f_2$$

On the other hand, special cases of the diagram here result by setting $f_1 = f_2$ or $g_1 = g_2$. Using them, we have that f » g implies that k ± f » k ± g and that f ± g » f ± h.

2. Suppose that $»_1$ and $»_2$ are congruences. The intersection of two equivalence relations is an equivalence relation on any set. Also if $f_1 »_1 f_2$ and $g_1 »_1 g_2$ implies that $f_1 ± g_1 »_1 f_2 ± g_2$ and if $f_1 »_2 f_2$ and $g_1 »_2 g_2$ implies that $f_1 ± g_1 »_2 f_2 ± g_2$, then for $» = »_1 \backslash »_2$, $f_1 » f_2$ and $g_1 » g_2$ imply that $f_1 ± g_1 » f_2 ± g_2$.

3. A functor F is full if every arrow in Hom(F(A); F(B)) has the form F(f) for some f : A ¡! B. A quotient as described here is surjective on arrows, so the condition of fullness is certainly satis¯ed.

4. a. It is an equivalence because F(f) = F(f), F(f) = F(g) implies F(g) = F(f) and F(f) = F(g) and F(g) = F(h) implies F(f) = F(h). It is a congruence because $F(f_1) = F(f_2)$ and $F(g_1) = F(g_2)$ implies that

$$F(f_1 ± g_1) = F(f_1) ± F(g_1) = F(f_2) ± F(g_2) = F(f_2 ± g_2)$$

b. Suppose $[f],[g]: A \to B$ are arrows of $\mathscr{C}/\sim$ such that $F_0([f]) = F_0([g])$. Then $F(f) = F(g)$, whence $f \sim g$ and $[f] = [g]$.

c. $F = F_0 \circ Q$ and we just seen that F_0 is faithful and that Q is full (previous exercise).

5. It follows from 2.2.5(ii) that : true = false and : false = true. Thus

$$\bar{P}(\ :\text{true}) = \bar{P}(\text{false}) \qquad \text{and} \qquad \bar{P}(\ :\text{false}) = \bar{P}(\text{true})$$

Similarly, $\bar{P}(\text{chr} \circ \text{ord}) = \text{id}$. Thus $f \sim g$ implies $\bar{P}(f) = \bar{P}(g)$ for the generators of the congruence. The set of pairs $\{(f,g)\}$ for which $\bar{P}(f) = \bar{P}(g)$ thus includes the generators and is closed under composition since functors preserve composition. Thus it includes all pairs for which $f \sim g$.

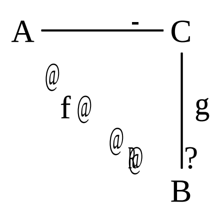
6. a. Such a relation satisfies Definition 3.5.1(a) automatically, since all elements of M are arrows of $C(M)$ with the same domain and codomain. The definition forces it to satisfy (b).

b. Suppose K is a submonoid and $n \sim n'$. Then (m,m) and (n,n') are both in K and $(m,m)(n,n') = (mn,mn')$ so $mn \sim mn'$. Similarly, $nm \sim n'm$. Conversely, suppose $\sim$ is a congruence on M. Let (m,m') and (n,n') be elements of K. Then $m \sim m'$ and $n \sim n'$, so by Exercise 1, $(mn, m'n') \in K$. Finally, $(1,1) \in K$ because the relation is reflexive. Thus K is a submonoid of $M \times M$.

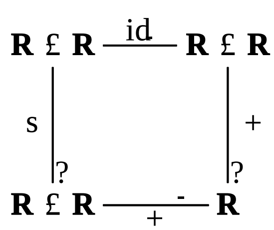
Solutions for Chapter 4

Section 4.1

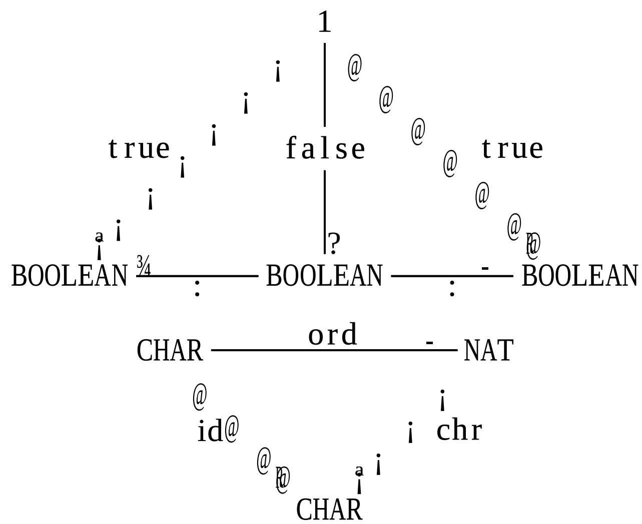
1.



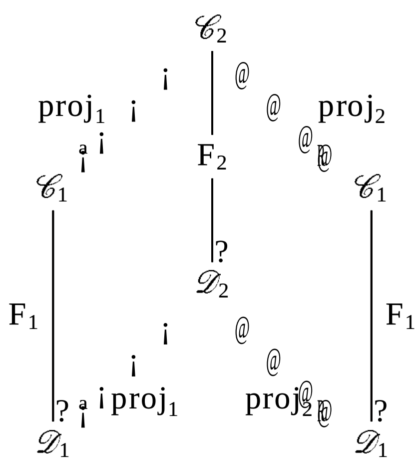
2. Let s be the function $(x,y) \mapsto (y,x)$. Then the diagram is



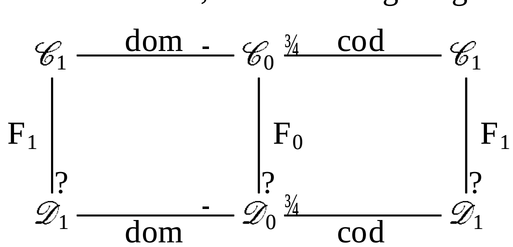
3\.

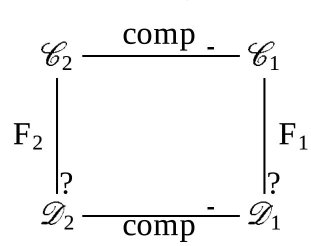


4\. Let $\mathscr{C}$ and $\mathscr{D}$ be categories with sets of objects $\mathscr{C}_0$ and $\mathscr{D}_0$, sets of arrows $\mathscr{C}_1$ and $\mathscr{D}_1$, and sets of composable pairs of arrows $\mathscr{C}_2$ and $\mathscr{D}_2$, respectively. A functor $F : \mathscr{C} \to \mathscr{D}$ consists of functions $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$, $F_1 : \mathscr{C}_1 \to \mathscr{D}_1$ along with the uniquely determined function $F_2 : \mathscr{C}_2 \to \mathscr{D}_2$ such that



commutes. In addition, the following diagrams must commute:





Section 4.2

1. It is the model M defined by $M(n) = \{n, a\}$, $M(a) = \{\text{source}, \text{target}\}$,

$$M(\text{source})(\text{source}) = M(\text{source})(\text{target}) = a$$

and

$$M(\text{target})(\text{source}) = M(\text{target})(\text{target}) = n$$

2. Define an isomorphism $\phi : \text{u-Struc} \to \text{Mod}(\mathscr{U}, \text{Set})$ as follows (these are the constructions in 4.2.15). If (S, f) is a u-structure, $\phi(S, f)$ is the model of $\mathscr{U}$ that takes u_0 to S and e to f. If $h : (S, f) \to (T, g)$ is a homomorphism, then $\phi(h)$ is the natural transformation whose only component (at u_0) is h. The inverse of ϕ takes a model M to $M(u_0)$ and a natural transformation γ to its only component.

An isomorphism $\psi : \text{u-Struc} \to \mathbf{N}\text{-Act}$ can be defined this way: If (S, f) is a u-structure, the action $\alpha : \mathbf{N} \times S \to S$ is defined by $\alpha(k, x) = f^k(x)$, where f^k denotes $f \circ f \circ \cdots \circ f$ (k occurrences of f) as in 2.3.5. This is indeed an action, since $\alpha(0, x) = f^0(x) = x$ and

$$\alpha(k + k', x) = f^{k+k'}(x) = f^k(f^{k'}(x)) = \alpha(k, \alpha(k', x))$$

The inverse to ψ takes an action $\alpha : \mathbf{N} \times S \to S$ to the u-set (S, f), where $f(x) = \alpha(1, x)$.

3. The arrow category has as objects arrows $f : C \to D$ and an arrow from $f : C \to D$ to $f' : C' \to D'$ is a pair of arrows (g, h) where $g : C \to C'$ and $h : D \to D'$ are such that

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C' & \xrightarrow[f']{} & D' \end{array}$$

commutes. The slice $\mathscr{C}/B$ is the subcategory that consists of those objects $f : C \to D$ for which $D = B$ and those arrows (g, h) for which $h = \mathrm{id}_B$. Since it does not include all arrows between objects of the arrow category, it is not full (nor, since it does not include all the objects, is it wide).

4. An object of $\text{Mod}(\mathscr{G}, \mathscr{C})$ is given by a graph homomorphism $\mathscr{G} \to \mathscr{C}$. Since $\mathscr{G}$ has two nodes such a homomorphism is given by a pair of objects of $\mathscr{C}$. Since there are no arrows in $\mathscr{G}$, such a pair of arrows is exactly a graph homomorphism so the objects of $\text{Mod}(\mathscr{G}, \mathscr{C})$ are the pairs. An arrow from one pair to another is a pair of arrows as in $\mathscr{C} \times \mathscr{C}$, which are generally subject to commutativity conditions corresponding to arrows of $\mathscr{G}$. Since $\mathscr{G}$ has no arrows, there are no conditions in this case.

5. If $(h, k) : f \to g$ is an isomorphism with inverse (h', k'), then we have $(h \circ h', k \circ k') = (h, k) \circ (h', k') = \mathrm{id}_g = (\mathrm{id}_C, \mathrm{id}_D)$ so that $h \circ h' = \mathrm{id}_C$ and $k \circ k' = \mathrm{id}_D$. Similarly, $h' \circ h = \mathrm{id}_A$ and $k' \circ k = \mathrm{id}_B$. Conversely, suppose h and k are invertible. Then we must show that (h^{-1}, k^{-1}) is an arrow from $g \to f$. We have

$$f \circ h^{-1} = k^{-1} \circ k \circ f \circ h^{-1} = k^{-1} \circ g \circ h \circ h^{-1} = k^{-1} \circ g$$

It is evident that $(h^{-1}, k^{-1}) = (h, k)^{-1}$.

6. Let $f : D \to D'$ be an arrow of $\mathscr{D}$. There are objects C and C' of $\mathscr{C}$ and isomorphisms $h : D \to F(C)$ and $k : D' \to F(C')$. The arrow $k \circ f \circ h^{-1} : F(C) \to F(C')$ is $F(g)$ for a unique $g : C \to C'$ because an equivalence is full and faithful. Then $k \circ f = F(g) \circ h$, which means that (h, k) is an arrow from f to $F(g)$. Since h and k are isomorphisms, so is (h, k) (see previous exercise).

7. An object of $\mathscr{C}^{\to}$ is an arrow $A(u) : A(0) \to A(1)$ of $\mathscr{C}$ and an arrow $(f(0), f(1)) : A(u) \to B(u)$ is a commutative square

$$\begin{array}{ccc} A(0) & \xrightarrow{A(u)} & A(1) \\ {\scriptstyle f(0)}\downarrow & & \downarrow{\scriptstyle f(1)} \\ B(0) & \xrightarrow[B(u)]{} & B(1) \end{array}$$

A functor $A : \mathscr{G} \to \mathscr{C}^{\to}$ is thus required to produce, for each node a of $\mathscr{G}$ an arrow $A(a, u) : A(a, 0) \to A(a, 1)$ and to each arrow $s : a \to b$ a commutative diagram

$$\begin{array}{ccc} A(a, 0) & \xrightarrow{A(a, u)} & A(a, 1) \\ {\scriptstyle A(s, 0)}\downarrow & & \downarrow{\scriptstyle A(s, 1)} \\ A(b, 0) & \xrightarrow[A(b, u)]{} & A(b, 1) \end{array}$$

If we let $E(a) = A(a, 0)$, $E(s) = A(s, 0)$, one sees immediately that $E : \mathscr{G} \to \mathscr{C}$ is a model. Similarly, if we let $F(a) = A(a, 1)$ and $F(s) = A(s, 1)$ it is also a model. Finally, define $\alpha : E \to F$ by $\alpha(a) = A(a, u)$. The commutativity above implies that α is a natural transformation. Conversely, we can start with a natural transformation $\alpha : E \to F$ between two models and use the equations above to define A. The details are trivial.

8. We have to show that the naturality condition engendered by an arrow $f : a \to b$ in $\mathscr{G}$ is satisfied by β, given that it is satisfied by α. We have

$$D(f) \circ \beta a = \beta b \circ \alpha b \circ D(f) \circ \beta a = \beta b \circ E(f) \circ \alpha a \circ \beta a = \beta b \circ E(f)$$

(Compare this argument with that of Exercise 5 above.)

Section 4.3

1. Recall that if $\mathscr{G}$ is graph, ´$\mathscr{G}$ is the identity on objects and is de¯ned on arrows by ´$\mathscr{G}$[u] = (u), where we use square brackets to denote application to distinguish it from the parentheses used for lists. If $f : \mathscr{G}$ ¡! $\mathscr{H}$ is a graph homomorphism, we have to show that

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{´\mathscr{G}} & U(F(\mathscr{G})) \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle U(F(f))} \\ \mathscr{H} & \xrightarrow[´\mathscr{H}]{} & U(F(\mathscr{H})) \end{array}$$

commutes. Applied to objects, we get ´$\mathscr{H}$ ± f[a] = f[a], while

$$U(F(f)) \pm ´\mathscr{G}[a] = U(F(f))[a] = f[a]$$

If u : a ¡! b is an arrow, ´$\mathscr{H}$ ± f[u] = (f[u]), while

$$U(F(f)) \pm ´\mathscr{G}[u] = U(F(f))[(u)] = (f[u])$$

2. We must show that if $f : \mathscr{G}$ ¡! $\mathscr{H}$ is a graph homomorphism,

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{¯\mathscr{G}} & W(\mathscr{G}) \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle W(f)} \\ \mathscr{H} & \xrightarrow[¯\mathscr{H}]{} & W(\mathscr{H}) \end{array}$$

commutes. But for a node a of $\mathscr{G}$, W(f)(¯$\mathscr{G}$(a)) is calculated as the component of $\mathscr{H}$ containing f(a), which is exactly what ¯$\mathscr{H}$(f(a)) is. In other words, the naturality is the very de¯nition of W(f).

3. We must show that for each arrow f : C ¡! C^0 of $\mathscr{C}$,

$$\begin{array}{ccc} G(C) & \xrightarrow{i_C} & F(C) \\ {\scriptstyle G(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle F(f)} \\ G(C^0) & \xrightarrow[i_{C^0}]{} & F(C^0) \end{array}$$

commutes. We have, for x 2 G(C),

$$F(f)((i_C)(x)) = F(f)(x) = G(f)(x) = i_{C^0}(G(f)(x))$$

4. We must show that for any graph homomorphism $f : \mathscr{G}$ ¡! $\mathscr{H}$, the square

$$\begin{array}{ccc}
A(\mathscr{G}) & \xrightarrow{\ \mathrm{source}_{\mathscr{G}}\ } & N(\mathscr{G}) \\
{\scriptstyle A(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle N(f)} \\
A(\mathscr{H}) & \xrightarrow[\ \mathrm{source}_{\mathscr{H}}\ ]{} & N(\mathscr{H})
\end{array}$$

commutes. We have for $u : a \to b$ in $\mathscr{G}$, $N(f)(\mathrm{source}_{\mathscr{G}}(u)) = N(f)(a) = f(a)$, while $\mathrm{source}_{\mathscr{H}}(A(f)(u)) = \mathrm{source}(f(u)) = f(a)$. The argument for target is similar.

5. A functor $F : \mathscr{C} \to \mathbf{Set}$ is determined uniquely by giving, for each $C \in \mathscr{C}_0$, a set $F(C)$. The disjoint union of these sets can be modeled as $W(F) = \bigcup \{(x, C) \mid x \in F(C)\}$. The function $w(F) : W(F) \to \mathscr{C}_0$ defined by $w(F)(x, C) = C$ makes $w(F)$ an object of $\mathbf{Set}/\mathscr{C}_0$. Given a natural transformation $\alpha : F \to G$, define $W(\alpha) : W(F) \to W(G)$ by $W(\alpha)(x, C) = (\alpha C(x), C)$. This makes $W : \mathbf{Func}(\mathscr{C}, \mathbf{Set}) \to \mathbf{Set}/\mathscr{C}_0$ a functor.

Conversely, given an object $f : S \to \mathscr{C}_0$ of $\mathbf{Set}/\mathscr{C}_0$, define a functor $V(f) : \mathscr{C} \to \mathbf{Set}$ by $V(f)(C) = \{x \in S \mid f(x) = C\}$ for an object C of $\mathscr{C}$. If $u : f \to g : S' \to \mathscr{C}_0$ is an arrow of $\mathbf{Set}/\mathscr{C}_0$, define a natural transformation $V(u) : V(f) \to V(g)$ by $(V(u)C)(x) = u(x)$ for $x \in S'$. Then $V : \mathbf{Set}/\mathscr{C}_0 \to \mathbf{Func}(\mathscr{C}, \mathbf{Set})$ is a functor.

We have that $V(W(F))(C) = \{(x, C) \mid x \in F(C)\}$. Let $\beta C(x, C) = x$ for $x \in F(C)$. Then β is a natural isomorphism from $V \circ W$ to the identity functor on $\mathbf{Func}(\mathscr{C}, \mathbf{Set})$. In the other direction, for $f : S \to \mathscr{C}_0$,

$$W(V(f)) = \bigcup \{(x, C) \mid x \in V(f)(C)\} = \{(x, f(x)) \mid x \in S\}$$

which is isomorphic to S. The diagram

$$\begin{array}{ccc}
 & S & \\
{\scriptstyle \cong}\swarrow & & \searrow{\scriptstyle f} \\
W(V(f)) & \xrightarrow[\ w(V(f))\ ]{} & \mathscr{C}_0
\end{array}$$

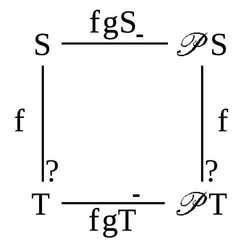
commutes and the naturality of the isomorphism is straightforward to verify.

6. a. We must show that if $f : S \to T$ is a function, then

$$\begin{array}{ccc}
S & \xrightarrow{\ \{\}_S\ } & \mathscr{P}S \\
{\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f_*} \\
T & \xrightarrow[\ \{\}_T\ ]{} & \mathscr{P}T
\end{array}$$

commutes. But each path applied to an $x \in S$ produces $\{f(x)\}$.

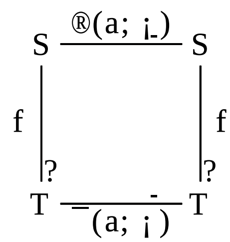
b. The diagram that would have to commute is



for an $f : S \to T$. The reader may want to verify that it does if and only if f is injective. When, for example, $f : 2 \to 1$ is the unique function, going around counter-clockwise gives, at 0, the element $\{0\}$. Going the other way, we get $f_!(\{0\})$ which is the set of $x \in 1$ whose every inverse image is included in $\{0\}$. Since no element of 1 has this property, $f_!\{0\} = \emptyset$.

c. It makes no sense to ask for a natural transformation between a contravariant functor and a covariant functor.

7. A natural transformation from $F \to G$ is a function f from $S = F(M)$ to $T = G(M)$ and it must satisfy the naturality condition that for any $a \in M$,



commutes. This is the definition of an equivariant function.

8. This refers to the answer to Exercise 7 of Section 4.2. In addition to that, in describing a model $A : \mathscr{D} \to \mathscr{C}^{\to}$ we must add, whenever we have arrows $s : a \to b$ and $t : b \to c$ of $\mathscr{D}$, the equations $A(t, 0) \circ A(s, 0) = A(t \circ s, 0)$ and $A(t, 1) \circ A(s, 1) = A(t \circ s, 1)$. Similar equations have to be added for E and F and will satisfied if and only if they are for A.

9. If $F, G : \mathscr{E} \to \mathscr{G}$ is such that $\alpha(F) = \alpha(G)$, then the sole vertex of $\alpha(F)$ is the same as that of $\alpha(G)$. But that is all there is to a homomorphism on $\mathscr{E}$. Thus α is injective. Similarly, every node of $\mathscr{G}$ does give graph homomorphism on $\mathscr{E}$ so α is surjective.

10. a. If $f : \mathscr{G} \to \mathscr{H}$ is a graph homomorphism, we define

$$P_k(f)((u_n, u_{n-1}, \ldots, u_1)) = (f(u_n), f(u_{n-1}), \ldots, f(u_1))$$

This makes sense since f preserves source and target. The functoriality is clear.

b. Since a path of length 0 is a node, the object functions are the same. The arrow functions are the same by de¯nition.

c. Since a path of length 1 is an arrow, the object functions are the same. The arrow functions are the same by de¯nition.

Section 4.4

1. By De¯nition 4.4.2, the component of $(G_1®)E$ at an object A of $\mathscr{A}$ is the arrow $(G_1®)E(A)$. This is $G_1(®E(A))$ by De¯nition 4.4.3. By De¯nition 4.4.3, the component of $G_1(®E)$ at A is $G_1((®E)A)$. This is $G_1(®E(A))$ by De¯nition 4.4.2.

2. The horizontal composites $¯ ¤ \mathrm{id}_F$ and $\mathrm{id}_H ¤ ®$ are de¯ned to be the two equal composites in each of these special cases of Diagram (4.30):

$$\begin{array}{ccc}
(H ± F)A & \xrightarrow{H(\mathrm{id}_F)A} & (H ± F)A \\
{\scriptstyle (¯F)A}\downarrow & & \downarrow{\scriptstyle (¯F)A} \\
(K ± F)A & \xrightarrow[K(\mathrm{id}_F)A]{} & (K ± F)A
\end{array}$$

$$\begin{array}{ccc}
(H ± F)A & \xrightarrow{(H®)A} & (H ± G)A \\
{\scriptstyle (\mathrm{id}_H)FA}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_H)GA} \\
(H ± F)A & \xrightarrow[(H®)A]{} & (H ± G)A
\end{array}$$

Since

$$H(\mathrm{id}_F)A = H(\mathrm{id}_{FA}) = \mathrm{id}_{(H±F)A}$$

(the ¯rst equality by the de¯nition of identity natural transformation in 4.2.13), we have that

$$(¯ ¤ \mathrm{id}_F)A = (¯F)A ± \mathrm{id}_{H±FA} = (¯F)A$$

as required. Similarly $(\mathrm{id}_H)GA = \mathrm{id}_{(H±G)A}$, so that

$$(\mathrm{id}_H ¤ ®)A = (\mathrm{id}_H)GA ± (H®)A = (H®)A$$

3. a. The ¯rst equation in Godement's ¯fth rule follows from this calculation, using the Interchange Law and Exercise 2.

$$\begin{aligned}
(°F_2) ± (G_1®) &= (° ¤ \mathrm{id}_{F_2}) ± (\mathrm{id}_{G_1} ¤ ®) \\
&= (° ± \mathrm{id}_{G_1}) ¤ (\mathrm{id}_{F_2} ± ®) = ° ¤ ®
\end{aligned}$$

A similar argument shows that °® = (G$_2$®) ± (°F$_1$)

b. This is shown by the following calculation, using (in order) Exercise 2, the Interchange Law, the de¯nition of the (vertical) composite of natural transformations and Exercise 2 again:

(G$_1$¯E) ± (G$_1$®E) = (id$_{G_1}$ ¤ ¯E) ± (id$_{G_1}$ ¤®E)
= (id$_{G_1}$ ± id$_{G_1}$) ¤ (¯E ± ®E)
= id$_{G_1}$ ¤(¯ ± ®)E = G$_1$(¯ ± ®)E

Section 4.5

1. Let H = Hom$_{Set}$(1; ¡) and I the identity functor. Let ® : H ¡! I assign to each function from 1 to I the element which is the image of that function. This is clearly bijective. If f : S ¡! T is a function, we have to show that the diagram

H(S) —®S— I(S)
H(f) |? I(f) |?
H(T) —®T— I(T)

commutes. If we take a function g : 1 ¡! S whose value is x 2 S, then

I(f)(®S(g)) = I(f)(x) = f(x) = ®T(f ± g) = ®T(H(f)(g))

2. A graph homomorphism **2** ¡! $\mathscr{G}$ takes e to an arrow u of $\mathscr{G}$ and takes 0 to the source of u and 1 to the target of u. In other words, it is completely determined by u. Thus A($\mathscr{G}$) » Hom(**2**;$\mathscr{G}$). It remains only to show that the isomorphism is natural in $\mathscr{G}$. If f : $\mathscr{G}$ ¡! $\mathscr{H}$, we must show that

A($\mathscr{G}$) —»— Hom(**2**;$\mathscr{G}$)
A(f) |? Hom(**2**; f) |?
A($\mathscr{H}$) —»— Hom(**2**;$\mathscr{H}$)

commutes. Then for an arrow u of $\mathscr{G}$, let h : **1** ¡! $\mathscr{G}$ be de¯ned by h(e) = u. Then the upper route in the diagram takes u to h and then to f ± h, whereas the lower route takes u to f(u) and then to the homomorphism h^0 de¯ned by h^0(e) = f(u). Since (f ± h)(e) = f(h(e)) = f(u), the two are the same.

3. Global elements are arrows from the terminal object. In the case of categories, the terminal object is the category **1** with one object and one arrow, the identity of that object. A functor from that object is determined by where it sends that object, which can be to any arbitrary object. The identity is sent to the identity of that chosen object. Thus $\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{1}, \mathscr{C})$ is isomorphic to the set of objects of $\mathscr{C}$. The set of objects is a functor $\mathscr{O} : \mathbf{Cat} \to \mathbf{Set}$: if $F : \mathscr{C} \to \mathscr{D}$ is a functor then $\mathscr{O}(F)$ is F_0, the object part of the functor. To verify naturality amounts to showing that if $G : \mathbf{1} \to \mathscr{C}$ is a functor, then $\mathrm{Hom}(\mathbf{1}, F)(G(*)) = F(G(*))$ which is immediate from the definition. (This says that the global elements of a category are its objects. Note that the global elements of a graph are its loops.)

4. Yes and yes. The representing object is the category **2** which is the graph **2** with the addition of two identity arrows. The argument is virtually identical to the argument for graphs.

5. Set $F^*(C) = \{(x, C) \mid x \in F(C)\}$ and for $f : C \to D$, set $F^*(f)(x, C) = (F(f)(x), D)$. The function $\beta C = x \mapsto (x, C)$ is clearly an isomorphism. To be natural requires that $F^*(f)(\beta C(x))$ be the same as $\beta D(F(f)(x))$, which is immediate from the definition.

6. Let $\mathscr{D} = \mathscr{C}^{\mathrm{op}}$. Then the ordinary Yoneda embedding

$$\mathscr{D}^{\mathrm{op}} \to \mathbf{Func}(\mathscr{D}, \mathbf{Set})$$

is full and faithful. But $\mathscr{C}^{\mathrm{op}} = \mathscr{D}$ means $\mathscr{D}^{\mathrm{op}} = \mathscr{C}$ so this is just 4.5.5.

7. For each $c \in F(C)$, define a natural transformation $\beta(C, F)(c) : \mathrm{Hom}(C, -) \to F$ as follows: for each $k : C \to A$, let $\beta(C, F)(c)(k) = F(f)(c)$. (This is the natural transformation defined by 4.5.6.) This is the value at c of a function we call $\beta(C, F) : F(C) \to \mathrm{NT}(\mathrm{Hom}(C, -), F)$ where $\mathrm{NT}(G, F)$ stands for the set of natural transformations between functors G and F. It is easy to see that $\mathrm{NT}(G, F)$ is contravariant in G and covariant in F. It is, in fact, the hom functor in the category $\mathbf{Func}(C, \mathbf{Set})$. Then the formulation of naturality is that if $f : C \to C'$ is an arrow and $\alpha : F \to F'$ is a natural transformation, the square

$$\begin{array}{ccc}
F(C) & \xrightarrow{\beta(C,F)} & \mathrm{NT}(\mathrm{Hom}(C,-),F) \\
{\scriptstyle \alpha f}\downarrow & & \downarrow{\scriptstyle \mathrm{NT}(\mathrm{Hom}(f,-),\alpha)} \\
F'(C') & \xrightarrow[\beta(C',F')]{} & \mathrm{NT}(\mathrm{Hom}(C',-),F')
\end{array}$$

commutes. Here αf is defined to be $F'f \circ \alpha C = \alpha C' \circ Ff$, equal by naturality. Note the double contravariance. From $f : C \to C'$, we get

$$\mathrm{Hom}(f, -) : \mathrm{Hom}(C', -) \to \mathrm{Hom}(C, -)$$

and then

$$NT(Hom(C; ¡); F) ¡! NT(Hom(C^0; ¡); F)$$

Now to prove this, we must apply it to a c 2 F(C). Going around clockwise gives us the natural transformation from Hom(C^0; ¡) ¡! F^0 whose value at an object A takes an arrow g : C^0 ¡! A to ®A(F(g ± f)(c)). Going the other way we get the natural transformation whose value at an object A is F^0g(®C^0(F f(c))). From the functoriality of F and naturality of ®, we have

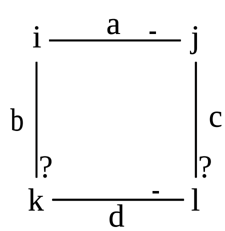
$$®A(F(g ± f)(c)) = ®A(F g(F f(c))) = F^0 g(®C^0(F f(c)))$$

8. If c 2 F(C) is a universal element of F, then there is a unique f : C ¡! C^0 such that F f(c) = c^0. Symmetrically, there is a unique g : C^0 ¡! C such that F g(c^0) = c. Then F g(F f(c)) = F g(c^0) = c. But the universality of c says that there is a unique arrow h : C ¡! C such that F h(c) = c. Clearly, h = id_C is one such; it follows that g ± f = id_C. Symmetrically, f ± g = id_{C^0}.

Section 4.6

1. Let $\mathscr{S}$ be the sketch whose graph has one node ¤ and no arrows, and (necessarily) no diagrams. If M is a model of $\mathscr{S}$, M determines and is determined by M(¤), which is a set. If N is a model and ® : M ¡! N a natural transformation, then ® has only one component, namely ®M, and the naturality condition of Diagram (4.20) is vacuous since the sketch has no arrows. The isomorphism takes M to M(¤) and ® to ®M.

2. The sketch has one node, call it s, and two arrows u; v : s ¡! s. There is one diagram D : $\mathscr{I}$ ¡! $\mathscr{S}$ based on the shape



de¯ned by D(i) = D(j) = D(k) = D(l) = s, D(a) = D(d) = u and D(b) = D(c) = v.

3. Since the sketch underlying a category has the same objects as the category and since the theory of a linear sketch also has the same objects as the sketch, the objects are same. For each arrow f : A ¡! B, there is an arrow in the sketch. Whenever (f_1; f_2; : : : ; f_n) is a path in the graph underlying $\mathscr{C}$ there is a diagram (f_1; f_2; : : : ; f_n) = (f_1 ± f_2 ± ¢¢¢ ± f_n) so that in the theory category every path is equal to a single arrow and the obvious functor is full. It is also faithful since there is no relation among paths in the theory that does not come from a commutative diagram in $\mathscr{C}$.

4. In general a homomorphism must commute in that way with every arrow in the sketch. However, an arrow that commutes in this way with an invertible arrow also commutes with the inverse:

$$\begin{aligned} f \circ M(v) &= N(v) \circ N(u) \circ f \circ M(v) \\ &= N(v) \circ f \circ M(u) \circ M(v) = N(v) \circ f \end{aligned}$$

5. The theory has nodes 0 and 1. The arrows are given by

(i) $\mathrm{Hom}(0,0) = \{\mathrm{id}_0, u \circ v\}$.
(ii) $\mathrm{Hom}(1,1) = \{\mathrm{id}_1, v \circ u\}$.
(iii) $\mathrm{Hom}(1,0) = \{v, v \circ u \circ v\}$.
(iv) $\mathrm{Hom}(0,1) = \{u\}$.

The nontrivial composites are given by $u \circ v \circ u = u$, $v \circ u \circ v \circ u = v \circ u$, and $u \circ v \circ u \circ v = u \circ v$.

Section 4.7

1. $I(0) = \{[x], [vux], [vy], [vuvy]\}$ and $I(1) = \{[y], [uvy], [ux]\}$. For any element $[z]$ of $I(0)$, $I(u)([z]) = [uz]$ and for any element $[w]$ of $I(1)$, $I(v)([w]) = [vw]$, all subject to the equation $uvu = u$.

2. If two terms are forced to be equal by the equivalence relation, then they are certainly equal in every model since the relations are valid in every model. On the other hand, the theory is a model and if the two terms are not forced to be equal by the equivalence relation, they are not equal in the theory.

3. Define the model $T : \mathscr{S} \to \mathbf{Set}$ this way: for any node a of the sketch, $T(a) = \{*\}$ (any one element set will do). For any arrow $f : a \to b$, $T(f)$ is the only possible function. If x is a constant of type a, then set $T(x) = *$ (the only possibility). If $\mathscr{S}$ has diagrams, T automatically takes them to commutative diagrams. If M is any model of $\mathscr{S}$, there is just one natural transformation from M to T whose value at a node a is the only possible map $M(a) \to T(a) = \{*\}$. The requisite naturality diagram commutes because there is only one possible map to $\{*\}$ from any set.

Section 4.8

1. Suppose the composites $\beta * \alpha$, $\delta * \gamma$, $\gamma \circ \alpha$ and $\delta \circ \beta$ are all defined. We must show that $\mathrm{cod}^v(\beta * \alpha) = \mathrm{dom}^v(\delta * \gamma)$.

$$\begin{aligned} \mathrm{id}^v\,\mathrm{cod}^v(\beta * \alpha) &= \mathrm{id}^v\,\mathrm{cod}^v\,\beta * \mathrm{id}^v\,\mathrm{cod}^v\,\alpha && \text{TC–5} \\ &= \mathrm{id}^v\,\mathrm{dom}^v\,\delta * \mathrm{id}^v\,\mathrm{dom}^v\,\gamma \\ &= \mathrm{id}^v\,\mathrm{dom}^v(\delta * \gamma) && \text{TC–5} \end{aligned}$$

where the second equality is because $\delta \circ \beta$ and $\gamma \circ \alpha$ are both defined.

2. We must show that for all posets A, B and C (with the ordering written $\leq$ in all of them), $\mathrm{comp} : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ is monotone. Suppose $f \leq g : A \to B$ and $h \leq k : B \to C$. We must show that $h \circ f \leq k \circ g$. For any $x \in A$, $h(f(x)) \leq h(g(x))$ because $f \leq g$ and h is monotone, and $h(g(x)) \leq k(g(x))$ because $h \leq k$. The result follows from transitivity.

3. Let α and α' be relations from A to B, γ and γ' relations from B to C, and $\alpha \subseteq \alpha'$ and $\gamma \subseteq \gamma'$. We must show that $\gamma \circ \alpha \subseteq \gamma' \circ \alpha'$. Suppose $(x, z) \in \gamma \circ \alpha$. Then there is $y \in B$ such that $(x, y) \in \alpha$ and $(y, z) \in \gamma$. But then $(x, y) \in \alpha'$ and $(y, z) \in \gamma'$, so that $(x, z) \in \gamma' \circ \alpha'$ as required.

4. Let $f, f' : A \to B$ be partial functions with f defined on A_0 and f' defined on A'_0, and $g, g' : B \to C$ partial functions with g defined on B_0 and g' defined on B'_0. We must show that if $f \leq f'$ and $g \leq g'$, then domain of definition of $g \circ f$ is included in the domain of definition of $g' \circ f'$ and for x such that $g(f(x))$ is defined, $g'(f'(x)) = g(f(x))$. So suppose $g(f(x))$ is defined for some $x \in A$. Then by definition of composition in 2.1.13, $x \in A_0$ and $f(x) \in B_0$. By the assumption that $f \leq f'$, $x \in A'_0$ and $f'(x) = f(x)$. Hence $f'(x) \in B_0$ and $g(f'(x))$ is defined and equal to $g(f(x))$. Now the assumption that $g \leq g'$ means that $f'(x) \in B'_0$ and $g(f'(x)) = g'(f'(x))$. Hence $g(f(x)) = g'(f'(x))$ as required.

Solutions for Chapter 5

Section 5.1

1. Let $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ be categories. The category $\mathscr{C} \times \mathscr{D}$ has functors $P_1 : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $P_2 : \mathscr{C} \times \mathscr{D} \to \mathscr{D}$ defined by $P_1(C, D) = C$, $P_2(C, D) = D$, $P_1(f, g) = f$ and $P_2(f, g) = g$ for C and D objects and f and g arrows of $\mathscr{C}$ and $\mathscr{D}$ respectively. That these are functors follows immediately from the fact that source, target and composition in the product category are defined coordinatewise (see 2.6.6). Now let $F : \mathscr{E} \to \mathscr{C}$ and $G : \mathscr{E} \to \mathscr{D}$ be functors. Define $\langle F, G\rangle : \mathscr{E} \to \mathscr{C} \times \mathscr{D}$ by $\langle F, G\rangle(E) = (F(E), G(E))$ and $\langle F, G\rangle(h) = (F(h), G(h))$ for E an object and h an arrow of $\mathscr{E}$. The proof that this is a functor is immediate. Then $P_1 \circ \langle F, G\rangle(E) = P_1(F(E), G(E)) = F(E)$ and similarly for arrows. And similarly, $P_2 \circ \langle F, G\rangle = G$. If $H : \mathscr{E} \to \mathscr{C} \times \mathscr{D}$ is any functor with $P_1 \circ H = F$ and $P_2 \circ H = G$, let $H(E) = (H_1(E), H_2(E))$. Then $F(E) = P_1 \circ H(E) = H_1(E)$ and similarly for arrows. Thus $F = H_1$ and similarly $G = H_2$, which proves uniqueness.

2. Let M and N be monoids. The product is the product of the underlying sets with multiplication $(m_1, n_1)(m_2, n_2) = (m_1m_2, n_1n_2)$. The identity element is $(1, 1)$.

3. If P and Q are posets, their product is the product of the underlying sets with $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq p_2$ and $q_1 \leq q_2$.

4. This is essentially the same as for categories.

5. Given any object B and arrows $f : B \to 1$ and $g : B \to A$, then $g : B \to A$ is evidently the unique arrow such that $\mathrm{id}_A\, g = g$. But also $\langle\rangle \circ g = f$ since that is the only arrow from B to 1.

6. In the category of sets the product of any set A with the empty set is the empty set. If A is nonempty, the projection onto A is not surjective, hence not an epimorphism.

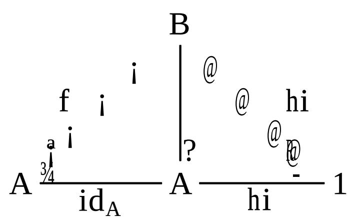
Section 5.2

1. The isomorphism is given by

$$f(x) = \begin{cases} (2,1) & \text{if } x = 1 \\ (1,1) & \text{if } x = 2 \\ (3,1) & \text{if } x = 3 \\ (2,2) & \text{if } x = 4 \\ (1,2) & \text{if } x = 5 \\ (3,2) & \text{if } x = 6 \end{cases}$$

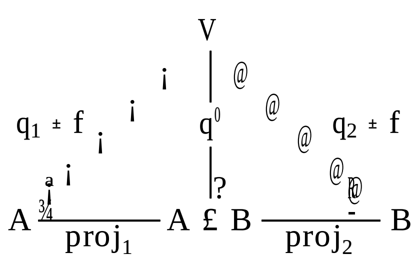
2. The isomorphism of the preceding exercise can be composed with any of the $6! = 720$ permutations of 6 to give another one.

3. A cone over A and 1 has to have this form, where $f : B \to A$ is any arrow.



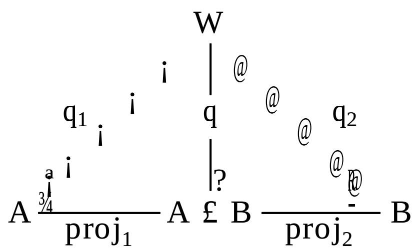
Clearly the only possible arrow in the middle is f.

4. By 5.2.13, the left vertical arrow in (5.12) takes the pair of arrows (q_1, q_2), where $q_1 : W \to A$ and $q_2 : W \to B$, to $(q_1 \circ f, q_2 \circ f)$, and the right vertical arrow takes $q : W \to A \times B$ to $q \circ f : V \to A \times B$. Therefore, by Definition 5.2.7, if you start at lower left with (q_1, q_2) and go north and then east, you get the unique arrow q' which makes



$(*)$

commute. If you go east and north, you get $q \circ f$, where q is the unique arrow which makes



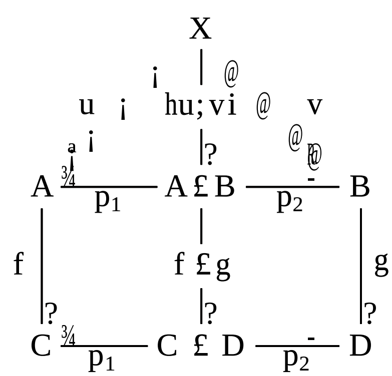
commute. Because $\mathrm{proj}_i \circ (q \circ f) = (\mathrm{proj}_i \circ q) \circ f = q_i \circ f$ for $i = 1, 2$, it follows that $q \circ f = q'$. This is a consequence of the fact that q' is the unique arrow making (*) commute. Hence (5.12) commutes.

5. Let $f : E \to A$ and $g : E \to B$. Then $i^{-1} \circ \langle f, g\rangle : C \to V$ is an arrow such that

$$\mathrm{proj}_1 \circ i \circ i^{-1} \circ \langle f, g\rangle = \mathrm{proj}_1 \circ \mathrm{id}_C \circ \langle f, g\rangle = \mathrm{proj}_1 \circ \langle f, g\rangle = f$$

and similarly $\mathrm{proj}_2 \circ i \circ i^{-1} \circ \langle f, g\rangle = g$. Moreover if $h : E \to V$ is such that $\mathrm{proj}_1 \circ i \circ h = f$ and $\mathrm{proj}_2 \circ i \circ h = g$, then we have $p_1 \circ i \circ h = p_1 \circ i \circ i^{-1} \circ \langle f, g\rangle$ and $p_2 \circ i \circ h = p_2 \circ i \circ i^{-1} \circ \langle f, g\rangle$. But since arrows to C are uniquely determined by their projections to A and B, we conclude that $i \circ h = i \circ i^{-1} \circ \langle f, g\rangle$ from which the isomorphism i can be cancelled to give $h = i^{-1} \circ \langle f, g\rangle$.

6. Chase the diagram



to show that

$$p_1 \circ (f \times g) \circ \langle u, v\rangle = f \circ p_1 \circ \langle u, v\rangle = f \circ u$$

and similarly $p_2 \circ (f \times g) \circ \langle u, v\rangle = g \circ v$, so that $(f \times g) \circ \langle u, v\rangle$ satisfies the condition that determines the map $\langle f \circ u, g \circ v\rangle$ uniquely.

Section 5.3

1. a. Define $q_1 = \mathrm{proj}_1 : A \times (B \times C) \to A$, $q_2 = \mathrm{proj}_1 \circ \mathrm{proj}_2 : A \times (B \times C) \to B$ and $q_3 = \mathrm{proj}_2 \circ \mathrm{proj}_2 : A \times (B \times C) \to C$. The meaning of the last, for example, is the second projection to $B \times C$, followed by the second projection from the latter to C. Now suppose D is an object and $f_1 : D \to A$, $f_2 : D \to B$ and $f_3 : D \to C$ are given. Then there is a unique arrow $\langle f_2, f_3 \rangle : D \to B \times C$ such that $\mathrm{proj}_1 \circ \langle f_2, f_3 \rangle = f_2$ and $\mathrm{proj}_2 \circ \langle f_2, f_3 \rangle = f_3$. It follows that there is a unique arrow $\langle f_1, \langle f_2, f_3 \rangle\rangle : D \to A \times (B \times C)$ such that $\mathrm{proj}_1 \circ \langle f_1, \langle f_2, f_3 \rangle\rangle = f_1$ and $\mathrm{proj}_2 \circ \langle f_1, \langle f_2, f_3 \rangle\rangle = \langle f_2, f_3 \rangle$ from which the required identities follow immediately. If $g : D \to A \times (B \times C)$ is another arrow with $q_i \circ g = f_i$ for $i = 1$, 2, 3, then $\mathrm{proj}_1 \circ \mathrm{proj}_2 \circ g = f_2$ and $\mathrm{proj}_2 \circ \mathrm{proj}_2 \circ g = f_3$, from which it follows from the uniqueness of arrows into a product, that $\mathrm{proj}_2 \circ g = \langle f_2, f_3 \rangle$. Also, $p_1 \circ g = f_1$ so that $g = \langle f_1, \langle f_2, f_3 \rangle\rangle$.

b. If $p : B \to A$ is a unary product diagram, then by definition there is for each object X a bijection

$$f \mapsto \langle f \rangle : \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$$

for which $p \circ \langle f \rangle = f$. This bijection is a natural isomorphism from $\mathrm{Hom}(-, A)$ to $\mathrm{Hom}(-, B)$: if $u : Y \to X$, then naturality follows from the fact that $p \circ \langle f \rangle \circ u = f \circ u$, so that $\langle f \rangle \circ u = \langle f \circ u \rangle$. It follows from Corollary 4.5.4 that p is an isomorphism.

c. The preceding part shows that every category has unary products, so such a category has n-ary products for $n = 0$, 1 and 2. The first part shows the same for $n = 3$ and also gives for that case the essential step for an obvious induction on n.

2. Given q_1 and q_2 as in (ii), we form $\langle q_1, q_2 \rangle$. From (iii), we see that $p_1 \circ \langle q_1, q_2 \rangle = q_1$ and $p_2 \circ \langle q_1, q_2 \rangle = q_2$. If h is another arrow satisfying the same identities, then (iv) tells us that $h = \langle p_1 \circ h, p_2 \circ h \rangle = \langle q_1, q_2 \rangle$ so that we have the uniqueness required by 5.1.3.

3. We saw in Exercises 1 and 4 of Section 5.1 that the product in each category took as objects of the product category the product of the objects and as arrows of the product the product of the arrows. Thus the product is constructed in the same way in both categories.

4. Let $\mathscr{C} = \mathscr{D}$ be the category of countably infinite sets and all functions between them. Let $F : \mathscr{C} \to \mathscr{D}$ be defined by $F(X) = \{X\} \cup X$ for such a set X (the idea is that $F(X)$ is X with a new element adjoined; the choice of $\{X\}$ to be the new element is convenient but not the only possible one) and if $f : X \to Y$, $F(f) = 1 + f$ is the function whose restriction to X is f and which takes the added element $\{X\}$ of $F(X)$ to the added element $\{Y\}$ of Y. Then although $F(X \times Y) \cong F(X) \times F(Y)$ since any two countable sets are isomorphic, the cone

$$\begin{array}{ccc} & F(X \times Y) & \\ {}^{F(\mathrm{proj}_1)} \swarrow & & \searrow {}^{F(\mathrm{proj}_2)} \\ F(X) & & F(Y) \end{array}$$

is not a product cone. In fact, $F(X \times Y)$ contains no point u with the property that $F(\mathrm{proj}_1)(u) = \{X\}$ but $F(\mathrm{proj}_2)(u) \neq \{Y\}$ or vice versa.

5. a. Since a unary product diagram is an isomorphism and every functor preserves isomorphisms, every functor preserves unary products. Now for $n \geq 3$, n-ary products are defined by induction. Assuming that a functor F preserves $n-1$-ary products, then the product

$$A_1 \times A_2 \times \cdots \times A_n = A_1 \times (A_2 \times \cdots \times A_n)$$

Then

$$\begin{array}{ccc} & F(A_1 \times \cdots \times A_n) & \\ {}^{F(\mathrm{proj}_1)} \swarrow & & \searrow {}^{F(\mathrm{proj}_2)} \\ F(A_1) & & F(A_2 \times \cdots \times A_n) \end{array}$$

is a product cone. By the inductive assumption so is

$$\begin{array}{ccccc} & & F(A_2 \times \cdots \times A_n) & & \\ & {}^{F(\mathrm{proj}_1)} \swarrow & \downarrow {}^{F(\mathrm{proj}_{i-1})} & \searrow {}^{F(\mathrm{proj}_{n-1})} & \\ F(A_2) & \cdots & F(A_i) & \cdots & F(A_n) \end{array}$$

from which the conclusion follows.

b. Let $\mathscr{C} = \mathbf{1}$, the category with one object called 0 and its identity arrow. Let $\mathscr{D} = \mathbf{2}$, the category with two objects, called 0 and 1, their identities and one arrow $0 \longrightarrow 1$. Then 0 is the terminal object of $\mathscr{C}$ and 1 is the terminal object of $\mathscr{D}$. Both categories have finite products, with $0^n = 0$ in both and $1^n = 1$ and $0 \times 1 = 0$ in $\mathscr{D}$. The products are canonical since there is only one possible choice. Then the functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ given by $F(0) = 0$ preserves n-ary products for $n \geq 1$, but not nullary products.

6. A terminal object in a category is an object that every other object has an arrow to (unique, of course). In a poset, that is an element that every element is less than or equal to, that is a top element. There is no largest integer so $\mathbf{N}$ has no terminal element. We saw in 5.1.8 that products in posets are just meets. Since the meet of two nonnegative integers is the smaller of the two, $\mathbf{N}$ has binary products.

Section 5.4

1. Using the version of the disjoint sum of 5.4.5, define $(f+g)(s,0) = (f(s),0)$ and $(f+g)(t,1) = (g(t),1)$.

2. Let P be a poset and $x, y \in P$. The sum $x + y$ is characterized by the fact that there is an arrow $x + y \to z$ corresponding to every pair consisting of an arrow $x \to z$ and an arrow $y \to z$. In a poset, there is an arrow $x \to z$ and only one if and only if $x \leq z$ and similarly for y. Thus $x + y \leq z$ if and only if $x \leq z$ and $y \leq z$. But this property characterizes the join $x \vee y$.

3. Let P and Q be two posets and $P + Q$ denote the disjoint sum of the sets P and Q as described in 5.4.5. Define $(x,i) \leq (y,i)$, $i = 0, 1$ if and only if $x \leq y$, while $(x,0) \not\leq (y,1)$ and $(x,1) \not\leq (y,0)$ for all $x, y \in P + Q$. The proof that this is the sum is essentially the same as the sum for the category of sets, augmented by the observation that an arrow from $P + Q \to R$ preserves the partial order just defined if and only if its restrictions to P and Q do.

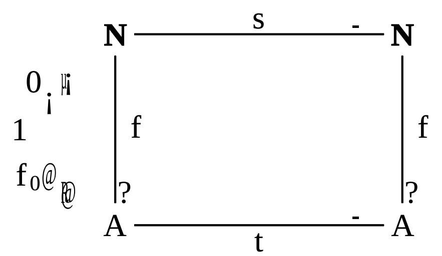
4. As noted in 5.4.7, we can take the canonical injections to be the same as in **Set**: $i_1(x) = (x,0)$ for $x \in S$ and $i_2(x) = (x,1)$ for $x \in T$. Then given $f : S \to X$, $g : T \to X$, define $\langle f|g\rangle : S + T \to X$ by $\langle f|g\rangle(s,0) = f(s)$ if and only if $f(s)$ is defined and $\langle f|g\rangle(t,1) = g(t)$ if and only if $g(t)$ is defined. Then, because i_1 is defined for all $x \in S$, $\langle f|g\rangle(i_1(x)) = \langle f|g\rangle(x,0) = f(x)$ if and only if $f(x)$ is defined, so that $\langle f|g\rangle \circ i_1 = f$, and similarly for g. It is clear that $\langle f|g\rangle$ is the only arrow such that $\langle f|g\rangle \circ i_1 = f$ and $\langle f|g\rangle \circ i_2 = g$.

5. Use the example given in the answer to Exercise 6 of Section 5.1 in the dual category.

Section 5.5

1. Given sets A and B, a function $f_0 : B \to A$ and $t : A \to A$, define a function $f : B \times \mathbf{N} \to A$ by letting $f(b,0) = f_0(b)$ and having defined $f(b,i)$ for $i \leq n$, define $f(b, n+1) = t(f(b,n))$.

2. To be a model M of that sketch is to be an object $A = M(n)$ together with arrows $f_0 = M(\text{zero}) : 1 \to A$ and $t = M(\text{succ}) : A \to A$, so that $(\mathbf{N}, 0, s)$ is certainly a model. If (A, f_0, t) is another model, then there is a unique arrow $f : \mathbf{N} \to A$ such that



commutes. But the commutation of the two parts of that diagram express the fact that f is an arrow in the category of models of the sketch. Since f is unique, this shows that there is exactly one such arrow from $(\mathbf{N}, 0, s)$ to any other model of the sketch, whence that is the initial model.

3. Define $t : B \times \mathbf{N} \times A \to B \times \mathbf{N} \times A$ by $t(b, n, a) = h(b, n, f(b, n))$. Define $k_0 : B \to B \times \mathbf{N} \times A$ by $k_0(b) = (b, 0, g(b))$. Then the induction property gives an arrow $k : B \times \mathbf{N} \to B \times \mathbf{N} \times A$ such that $k(b, 0) = k_0(b) = (b, 0, g(b))$ and $k(b, s(n)) = t(k(b, n))$. If we let $k_i = \mathrm{proj}_i \circ k$, $i = 1, 2, 3$, then $k(b, n) = (k_1(b, n), k_2(b, n), k_3(b, n))$. Next we claim that $k_1(b, n) = b$ and $k_2(b, n) = n$. For consider the diagram (in which we have abbreviated proj as p)

$$\begin{array}{ccccc}
 & & B \times \mathbf{N} & \xrightarrow{\ \mathrm{id}_B \times s\ } & B \times \mathbf{N} \\
 & \nearrow{\scriptstyle \langle \mathrm{id}_B, 0\rangle} & \Big\downarrow{\scriptstyle k} & & \Big\downarrow{\scriptstyle k} \\
B & \xrightarrow{\ k_0\ } & B \times \mathbf{N} \times A & \xrightarrow[\ t\ ]{} & B \times \mathbf{N} \times A \\
 & \searrow{\scriptstyle \langle \mathrm{id}_B, 0\rangle} & \Big\downarrow{\scriptstyle \langle p_1, p_2\rangle} & & \Big\downarrow{\scriptstyle \langle p_1, p_2\rangle} \\
 & & B \times \mathbf{N} & \xrightarrow[\ B \times s\ ]{} & B \times \mathbf{N}
\end{array}$$

Compare this to the diagram

$$\begin{array}{ccccc}
 & & B \times \mathbf{N} & \xrightarrow{\ \mathrm{id}_B \times \mathrm{succ}\ } & B \times \mathbf{N} \\
 & \nearrow{\scriptstyle \langle \mathrm{id}_B, 0\rangle} & & & \\
B & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\
 & \searrow{\scriptstyle \langle \mathrm{id}_B, 0\rangle} & & & \\
 & & B \times \mathbf{N} & \xrightarrow[\ \mathrm{id}_B \times \mathrm{succ}\ ]{} & B \times \mathbf{N}
\end{array}$$

and the uniqueness of recursively defined arrows implies that

$$\langle p_1, p_2\rangle \circ k = \langle \mathrm{id}_B, \mathrm{id}_{\mathbf{N}}\rangle$$

Then $f = k_3$ has the required properties.

Section 5.6

1. There is, by CC–1 and CC–2, a proof of $A \Rightarrow \mathit{true}$ and only one for each object A of $\mathscr{C}$ so that true is the terminal object. If $q_1 : X \to A$ and $q_2 : X \to B$ are arrows in $\mathscr{C}$, then from CC–3, there is an arrow $\langle q_1, q_2 \rangle : X \to A \wedge B$. According to CC–4, $p_1 \circ \langle q_1, q_2 \rangle = q_1$ and $p_2 \circ \langle q_1, q_2 \rangle = q_2$ and by CC–5, the arrow $\langle q_1, q_2 \rangle$ is unique with this property, so that $A \wedge B$ together with p_1 and p_2 is a product of A and B in $\mathscr{C}$.

2. a. Either projection is a proof.
   b. $\langle \mathrm{id}_A, \mathrm{id}_A \rangle$ is a proof.
   c. $\langle \mathrm{proj}_2, \mathrm{proj}_1 \rangle$ is a proof.
   d. $\langle \mathrm{proj}_1 \circ \mathrm{proj}_1, \langle \mathrm{proj}_2 \circ \mathrm{proj}_1, \mathrm{proj}_2 \rangle\rangle$ is a proof.

Section 5.7

1. We have, for any objects X, Y and Z, that $X + Y + X \cong (X + Y) + Z$ by the dual of the argument in 5.3.3. Then $D \times (A + B + C) \cong D \times ((A + B) + C) \cong D \times (A + B) + D \times C \cong (D \times A + D \times B) + D \times C \cong D \times A + D \times B + D \times C$.

2. The arrow h is the composite

$$
\begin{aligned}
C \times T &\xrightarrow{\mathrm{id}_C \times c} C \times (1 + 1 + 1) \xrightarrow{\cong} C \times 1 + C \times 1 + C \times 1 \\
&\xrightarrow{p_1 + p_1 + p_1} C + C + C \xrightarrow{f + g + h} D + D + D \\
&\xrightarrow{\langle \mathrm{id}_D \mid \mathrm{id}_D \mid \mathrm{id}_D \rangle} D
\end{aligned}
$$

Solutions for Chapter 6

Section 6.1

1. Define $\mathrm{eval} : [\mathscr{C} \to \mathscr{D}] \times \mathscr{C} \to \mathscr{D}$ on objects as follows. Let $F : \mathscr{C} \to \mathscr{D}$ and let C be an object of $\mathscr{C}$. Then $\mathrm{eval}(F, C) = F(C)$. On arrows, suppose $\alpha : F \to F'$ is a natural transformation and $f : C \to C'$ is an arrow of $\mathscr{C}$. Then set $\mathrm{eval}(\alpha, f) = F'f \circ \alpha C = \alpha C' \circ Ff : F(C) \to F'(C')$ (they are the same by naturality).

Since (α, f) is an arrow from (F, C) to (F', C'), eval preserves source and target because $\mathrm{eval}(F, C) = F(C)$ and $\mathrm{eval}(F', C') = F'(C')$. Preservation of identities is easy to verify. As for composition, let $\beta : F' \to F''$ and $g : C' \to C''$. Then

$$\begin{aligned} \mathrm{eval}((\beta, g) \circ (\alpha, f)) &= \mathrm{eval}(\beta \circ \alpha, g \circ f) \\ &= F''(g \circ f) \circ (\beta \circ \alpha)C \\ &= F''(g) \circ F''(f) \circ \beta C \circ \alpha C \\ &= F''(g) \circ \beta C' \circ F'(f) \circ \alpha C \\ &= \mathrm{eval}(\beta, g) \circ \mathrm{eval}(\alpha, f) \end{aligned}$$

The third line follows from the functoriality of F'' and the definition of composition of natural transformations; the fourth line is by naturality of β.

For a functor $F : \mathscr{B} \times \mathscr{C} \to \mathscr{D}$ and an object (B, C) of $\mathscr{B} \times \mathscr{C}$,

$$\begin{aligned} \mathrm{eval} \circ (\lambda F \times \mathscr{C})(B, C) &= \mathrm{eval} \circ (\lambda F(B), C) \\ &= \lambda F(B)(C) = F(B, C) \end{aligned}$$

Thus $\mathrm{eval} \circ (\lambda F \times \mathscr{C}) = F$, as required. Similarly, for an arrow $(f, g) : (B, C) \to (B', C')$ of $\mathscr{B} \times \mathscr{C}$,

$$\begin{aligned} \mathrm{eval} \circ (\lambda F \times \mathscr{C})(f, g) &= \mathrm{eval} \circ (\lambda F(f), g) \\ &= \lambda F(B')(g) \circ \lambda F(f)(C) \\ &= F(B', g) \circ F(f, C) = F(f, g) \end{aligned}$$

as required.

2. Let $\mathscr{G}_0$ be the set of nodes of $\mathscr{G}$ regarded as a graph with no arrows. We must show that

$$\mathscr{G} \xleftarrow{\;i\;} \mathscr{G}_0 \xrightarrow{\;()\;} \mathrm{No}$$

is a product diagram, where $()$ is the unique function and $i : \mathscr{G}_0 \to \mathscr{G}$ is the identity function on nodes and (necessarily) the empty function on arrows. Let $\mathscr{T}$ be a graph and $f : \mathscr{T} \to \mathscr{G}$, $g : \mathscr{T} \to \mathrm{No}$ be graph homomorphisms. We must find $u : \mathscr{T} \to \mathscr{G}_0$ for which $i \circ u = f$ and $() \circ u = ()$. The second equation is automatic. The first equation requires that u be the same as f on nodes. The fact that there is a homomorphism from $\mathscr{T}$ to No (which has no arrows) means that $\mathscr{T}$ has no arrows. Thus u has to be the empty function on arrows; since f must also be the empty function on arrows it follows that $i \circ u = f$.

A node of $\mathscr{G} \times \mathrm{No}$ as constructed in Exercise 4 of Section 5.1 is a pair (g, n) where g is a node of $\mathscr{G}$ and n is the unique node of No. (The construction in that exercise shows that $\mathscr{G} \times \mathrm{No}$ has no arrows since No has none.) Thus the function $(g, n) \mapsto g$ is clearly a bijection between $\mathscr{G} \times \mathrm{No}$ and the set of nodes of $\mathscr{G}$. What we have constructed here is more than that: it is a natural isomorphism in the category of graphs.

3. Given $f : \mathscr{C} \times \mathscr{G} \to \mathscr{H}$, we must show that $\text{eval} \circ (\lambda f \times \mathscr{G}) = f$. First, suppose that (c, n) is a node of $\mathscr{C} \times \mathscr{G}$. Then

$$\begin{aligned} \text{eval}((\lambda f \times \mathscr{G})(c, n)) &= \text{eval}(\lambda f(c), n) \\ &= \lambda f(c)(n) = f(c, n) \end{aligned}$$

For an arrow $(a : c \to d, w)$ of $\mathscr{C} \times \mathscr{G}$,

$$\begin{aligned} \text{eval}((\lambda f \times \mathscr{G})(a, w)) &= \text{eval}(\lambda f(a), w) \\ &= f_a(w) = f(a, w) \end{aligned}$$

as required.

4. Corollary 4.5.13 says that two representing objects for the same functor are isomorphic by a unique isomorphism that preserves the universal element. Let $\phi(A, B) : [A \to B]' \to [A \to B]$ be the isomorphism. In this case the functor is $\text{Hom}(- \times A, B)$ and preserving the universal element means that

$$\text{Hom}(\phi(A, B) \times A, B)(\text{eval}) = \text{eval} \circ (\phi(A, B) \times A) = \text{eval}'$$

which is the left diagram of the proposition. The right diagram follows from this calculation: $\text{eval} \circ (\phi(A, B) \times A) \circ \lambda' f = \text{eval}' \circ \lambda' f = f$, so by the uniqueness requirement of CCC–3, $(\phi(A, B) \times A) \circ \lambda' f = \lambda f$.

Section 6.2

1. By CCC–3, $\text{eval} \circ ((\lambda(\text{eval})) \times A) = \text{eval} : [A \to B] \times A \to B$. By the uniqueness requirement of CCC-3, $(\lambda(\text{eval})) \times A = [A \to B] \times A$ (the identity arrow) so the first component of $(\lambda(\text{eval}) \times A)$ must be the identity.

2. By CCC–3, $\text{eval} \circ (\lambda f \times A) \circ (g \times A) = f \circ (g \times A)$. But by 5.2.19, $\text{eval} \circ (\lambda f \times A) \circ (g \times A) = \text{eval} \circ ((\lambda f \circ g) \times A)$. By the uniqueness property of eval, it follows that $\lambda(f \circ (g \times A)) = (\lambda f) \circ g$.

3. Since $f \mapsto \lambda f$ is a bijection on each hom set by CCC-3, we only need to show that the following diagrams is commutative, where $g : B \to B'$.

$$\begin{array}{ccc} \text{Hom}(C \times A, B) & \xrightarrow{f \mapsto \lambda f} & \text{Hom}(C, [A \to B]) \\ {\scriptstyle \text{Hom}(C \times A, g)} \downarrow & & \downarrow {\scriptstyle \text{Hom}(C, [A \to g])} \\ \text{Hom}(C \times A, B') & \xrightarrow[f \mapsto \lambda f]{} & \text{Hom}(C, [A \to B']) \end{array}$$

This requires that $\lambda(g \circ f) = [A \to g] \circ \lambda f$. This follows from CCC–3 and Exercise 2 by the following calculation:

$$\begin{aligned} [A \to g] \circ \lambda f &= \lambda(g \circ \mathrm{eval}) \circ \lambda f \\ &= \lambda(g \circ \mathrm{eval} \circ (\lambda f \times A)) \\ &= \lambda(g \circ f) \end{aligned}$$

4. Let $h : 1 \to [A \to B]$ be a global element. Let $u : A \to 1 \times A$ be an isomorphism (see Section 5.3). The bijection from $\mathrm{Hom}(1, [A \to B])$ to $\mathrm{Hom}(A, B)$ takes h to

$$\mathrm{eval} \circ (h \times A) \circ u : A \to 1 \times A \to [A \to B] \times A \to B$$

Its inverse takes $f : A \to B$ to $\lambda(f \circ u^{-1}) : 1 \to [A \to B]$, which fits since $f \circ u^{-1} : 1 \times A \to A \to B$. That this is indeed the inverse follows from CCC–3. (In categorical writing, these manipulations with u are nearly always suppressed by assuming that $A = A \times 1$ and $u = \mathrm{id}_A$.)

5. In this answer, we regard λ as binding more tightly than composition or product, so that for example $\lambda f \circ g$ means $(\lambda f) \circ g$ and $\lambda f \times g$ means $(\lambda f) \times g$.

a. Define $\alpha_B : \mathrm{Hom}(C, [A \to B]) \to \mathrm{Hom}(C \times A, B)$ by $\alpha_B(g) = \mathrm{eval} \circ (g \times A)$. Then by CCC–3, $\alpha_B \circ \beta_B(f) = \mathrm{eval} \circ (\lambda f \times A) = f$ for $f : C \times A \to B$, and by the uniqueness requirement for eval,

$$\beta_B \circ \alpha_B(g) = \lambda(\mathrm{eval} \circ (g \times A)) = g$$

for $g : C \to [A \to B]$, so α_B is the inverse of β_B which is therefore bijective.

b. Let $g : C' \to C$ and $h : B \to B'$. Naturality requires that this diagram commute:

$$\begin{array}{ccc} \mathrm{Hom}(C, [A \to B]) & \xrightarrow{\hat{h}_C} & \mathrm{Hom}(C, [A \to B']) \\ {\scriptstyle \mathrm{Hom}(g, [A \to B])}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(g, [A \to B'])} \\ \mathrm{Hom}(C', [A \to B]) & \xrightarrow[\hat{h}_{C'}]{} & \mathrm{Hom}(C', [A \to B']) \end{array}$$

Because β_B is a bijection, an arbitrary arrow of $\mathrm{Hom}(C, [A \to B])$ can be taken to be λf for some $f : C \times A \to B$. The upper route around the diagram takes λf to

$$\lambda(h \circ \mathrm{eval} \circ (\lambda f \times A)) \circ g = \lambda(h \circ f) \circ g$$

whereas the lower route takes it to

$$\begin{aligned} \lambda(h \circ \mathrm{eval} \circ ((\lambda f \circ g) \times A)) &= \lambda(h \circ \mathrm{eval} \circ (\lambda f \times A) \circ (g \times A)) \\ &= \lambda(h \circ f \circ (g \times A)) \end{aligned}$$

which is the same thing by Exercise 2.

c. Let $h : B \to B'$. The required diagram for naturality, namely

$$\begin{array}{ccc}
\mathrm{Hom}(C\times A,B) & \xrightarrow{\mathrm{Hom}(C\times A,h)} & \mathrm{Hom}(C\times A,B')\\
\phi B\Big\downarrow & & \Big\downarrow\phi B'\\
\mathrm{Hom}(C,[A\to B]) & \xrightarrow[\mathrm{Hom}(C,[A\to h])]{} & \mathrm{Hom}(C,[A\to B'])
\end{array}$$

commutes by definition of $[A\to h]$. It is a natural isomorphism because each component is a bijection.

d. Let $C=[A\to B]$ in the preceding diagram and start with eval in the upper right corner. The upper route gives $\lambda(h\circ \mathrm{eval})$ and the lower route gives $[A\to h]\circ\lambda(\mathrm{eval})$, which is $[A\to h]$ by Exercise 1.

6. We use the formulation of $[\mathscr{G}\to\mathscr{H}]$ in Exercise 3. Let $h:1\to[\mathscr{G}\to\mathscr{H}]$ be a global element. Remember that in the category of graphs and graph homomorphisms, 1 is the graph with one node n and one arrow e. The node n goes to a node of $[\mathscr{G}\to\mathscr{H}]$, which is a function $f_0:G_0\to H_0$. The arrow e goes to an arrow of $[\mathscr{G}\to\mathscr{H}]$, which is an ordered triple $(f_1,f_2,f_3):f_1\to f_2$ as described in Exercise 3. The fact that 1 has only one node means that, since h is a graph homomorphism, necessarily $f_1=f_2=f_0$. Then the conditions in the description in Exercise 3 say that for any arrow g of $\mathscr{G}$, $\mathrm{source}(f_3(g))=f_0(\mathrm{source}(g))$ and $\mathrm{target}(f_3(g))=f_0(\mathrm{target}(g))$. Thus f_0 must be the node map and f_3 the arrow map of a graph homomorphism from $\mathscr{G}$ to $\mathscr{H}$. It follows from Exercise 4 that the loops of $[\mathscr{G}\to\mathscr{H}]$ are in one to one correspondence with the graph homomorphisms from $\mathscr{G}$ to $\mathscr{H}$: to recover the homomorphism from the loop $(f_1,f_1,f_3):f_1\to f_1$, take the node map to be f_1 and the arrow map to be f_3.

Section 6.3

1. For $i=1,2$,

$$\begin{array}{rlll}
\mathrm{proj}_i(c) & =_x & \lambda_{x\in A\times B}(\mathrm{proj}_i(x))`c & (\text{TL–17})\\
& =_x & \lambda_{x\in A\times B}(\mathrm{proj}_i(x))`c' & (\text{TL–12})\\
& =_x & \mathrm{proj}_i(c') & (\text{TL–17})
\end{array}$$

where in the applications of TL–17, we judiciously choose a variable x of type $A\times B$ that does not occur freely in c or c'.

2.

$$\begin{array}{rlll}
(a,b) & =_x & \lambda_{x\in A}(x,b)`a & (\text{TL–17})\\
& =_x & \lambda_{x\in A}(x,b)`a' & (\text{TL–12})\\
& =_x & (a',b) & (\text{TL–17})
\end{array}$$

and similarly $(a',b)=_x(a',b')$. The result follows from TL–9.

Section 6.4

1. We must show that for $f : C \times A \to B$, $\mathrm{eval} \circ (\lambda f \times A) = f$. First note that $\lambda f \times A = \langle \lambda f \circ p_1, p_2 \rangle : C \times A \to [A \to B] \times A$. (See 5.2.17.) Now let z be a variable of type C, y a variable of type A which is not in X, and suppose f is determined by a term $\phi(z, y)$ of type B. Then $\lambda f \times A$ is represented by $(\lambda_y \phi(z, y) \circ z, y) =_X (\lambda_y \phi(z, y), y)$ by definition of composition in $C(\mathscr{L})$. Then $\mathrm{eval} \circ (\lambda f \times A)$ is

$$(p_1(\lambda_y \phi(z, y), y))\,\text{`}\,p_2(\lambda_y \phi(z, y), y) =_X \lambda_y \phi(z, y)\,\text{`}\,y$$

by TL–15 and Exercise 2 of Section 6.3. Since $y \notin X$, this is $\lambda_y \phi(z, y)$ by TL–18.

2.

$$\begin{aligned}
\Gamma(\Lambda(\phi(u))) &= \Gamma(\lambda_x \phi((z, x))) \\
&= \lambda_x \phi(\mathrm{proj}_1 u, x)\,\text{`}\,\mathrm{proj}_2 u \\
&=_X \phi((\mathrm{proj}_1 u, \mathrm{proj}_2 u)) \\
&=_X \phi(u)
\end{aligned}$$

Section 6.5

1. a. $\mathbf{N} \times \mathbf{N} \xrightarrow{\langle p_1, p_1, p_2, p_2 \rangle} \mathbf{N} \times \mathbf{N} \times \mathbf{N} \times \mathbf{N} \xrightarrow{\ast \times \ast} \mathbf{N} \times \mathbf{N} \xrightarrow{+} \mathbf{N}$

   b. $\mathbf{N} \times \mathbf{N} \times \mathbf{N} \xrightarrow{\langle p_1, p_1, p_2, p_2 \rangle} \mathbf{N} \times \mathbf{N} \times \mathbf{N} \times \mathbf{N} \xrightarrow{\ast \times \ast} \mathbf{N} \times \mathbf{N} \xrightarrow{+} \mathbf{N}$

   c. $\mathbf{N} \times \mathbf{N} \xrightarrow{()} 1 \xrightarrow{5} \mathbf{N}$

Solutions for Chapter 7

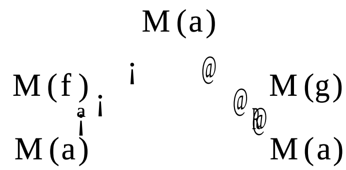
Section 7.1

1. Let us temporarily denote the usual addition by $\oplus$. The fact that $0 + m = m$ implies that $k + m = k \oplus m$ when $k = 0$. Assuming that equation for some k, we have that

$$\mathrm{succ}(k) + m = \mathrm{succ}(k + m) = \mathrm{succ}(k \oplus m) = \mathrm{succ}(k) \oplus m$$

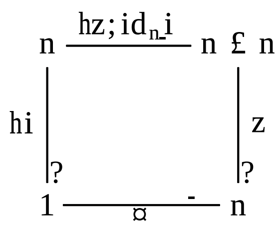
2. The cone requires that $M(a) = M(a) \times M(a)$. If $M(a)$ is finite then the number of elements of $M(a) \times M(a)$ is the square of the number of elements of $M(a)$. Hence $M(a)$ has either no elements or one element or an infinite number.

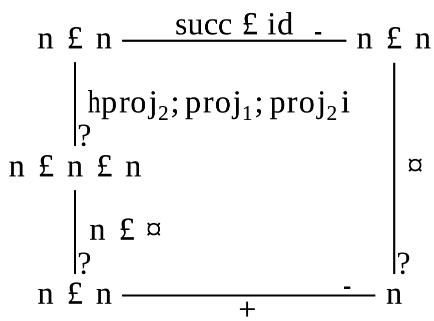
To see that $\mathscr{S}$ has an infinite model, define $M(a) = \mathbf{N}$, $M(f)(n) = r$ and $M(g)(n) = s$, where (r, s) is the unique pair of nonnegative integers for which $n - 1 = 2^r(2s + 1)$. It follows from the unique factorization of integers that $n \mapsto (r, s)$ is a bijection from $\mathbf{N}$ to $\mathbf{N} \times \mathbf{N}$. By Proposition 5.2.3, this means that



is a product diagram.

3. What we must do is to add an operation and equations to implement the standard inductive definition of multiplication: $0 \ast m = 0$ and $\mathrm{succ}(k) \ast m = m + k \ast m$. We do this by adding one operation $\ast : n \times n \to n$ and diagrams



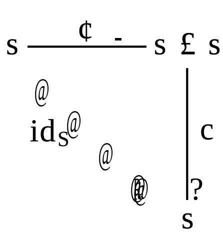


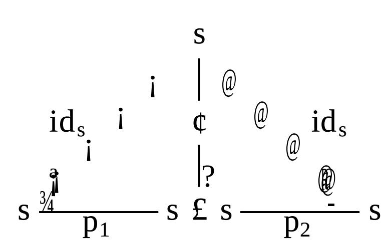
In order to make these diagrams, we also have to add arrows $n \to n \times n$, $n \times n \to n \times n$, $n \times n \to n \times n \times n$ and $n \times n \times n \to n \times n$ and, following the pattern of similar constructions in 7.1.7, diagrams forcing them to be $\langle z, \mathrm{id}_n \rangle$, $\mathrm{succ} \times \mathrm{id}$, $\langle \mathrm{proj}_2, \mathrm{proj}_1, \mathrm{proj}_2 \rangle$ and $n \times \ast$ respectively.

Section 7.2

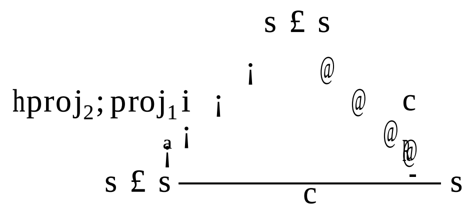
1. If we write xy for $M(c)(x, y)$ for $x, y \in M(s)$, then the diagram requires that $xy = x$ for all $x, y \in M(s)$. Then $x(yz) = x$ and $(xy)z = xz = x$. Of course, most semigroups do not have such a multiplication, so this sketch is not a sketch for semigroups.

2. One arrow $\delta : s \to s \times s$ is needed, with the following diagrams:

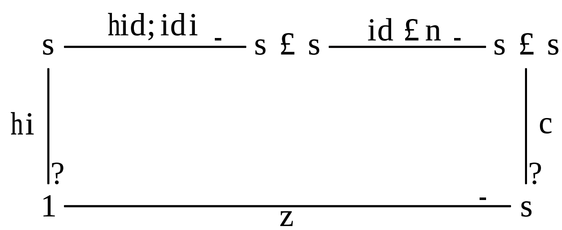




3. We give the answer for the case of real vector spaces; the other is similar. We assume the real number field **R** as a given structure. We suppose, for each $r \in R$, a unary operation we will denote $r^*: s \to s$. We require a unit element $z: 1 \to s$ for the operation c and a diagram similar to the previous exercise to say that z is the unit element. The following diagram says that c is commutative:



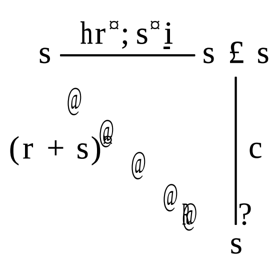
We have to add a unary negation operator $n: s \to s$ together with a diagram to say it is the negation operator:



In addition, we need diagrams that express the following identities:

$$
\begin{aligned}
0^*(x) &= z \\
r^*(c(x, y)) &= c(r^*(x), r^*(y)) \\
(r + s)^*(x) &= c(r^*(x), s^*(x)) \\
1^*(x) &= x \\
r^*(s^*(x)) &= (rs)^*(x)
\end{aligned}
$$

We give, for example, the diagram required to express the third of the equations above:



Section 7.4

1. We define a functor $F : \mathbf{Mod}(\mathscr{S}, \mathbf{Set}) \to \mathbf{Set}$: Given a model M of $\mathscr{S}$, let $F(M) = M(s)$. Given a homomorphism $\alpha : M \to M'$ of models, let $F(\alpha) = \alpha s : M(s) \to M(s)$. Define the inverse $G : \mathbf{Set} \to \mathbf{Mod}(\mathscr{S}, \mathbf{Set})$ this way: For a set A, $G(A)$ is the model M with $M(s) = A$ and multiplication $M(c) : A \times A \to A$ defined by $M(c)(a, a') = a$. $M(p)$ and $M(q)$ are necessarily the projections and clearly $M(c) = M(p)$ as required by the lone diagram of $\mathscr{S}$. If $\phi : A \to B$ is a set function, define the homomorphism $G(\phi) : G(A) \to G(B)$ by $G(\phi)s = \phi$ and $G(\phi)(s \times s) = \phi \times \phi$. It is easy to see that this is a homomorphism of models and that F and G are inverse functors.

2. Let the functor $F : \mathbf{Mod}(\mathscr{S}, \mathscr{C}) \to \mathbf{Mod}(\mathscr{T}, \mathscr{C})$ take a model M of $\mathscr{S}$ to the model $F(M)$ of $\mathscr{T}$ that is the same as M on the nodes and arrows with the same labels, and such that $F(M)(c) = M(a) = F(M)(a)$, $F(M)(h) = F(M)(j) = \mathrm{id}_{F(M)(a)}$ and $F(M)(k) = M(f)$. Given a natural transformation $\alpha : M \to M'$, let $F(\alpha)$ have the same components as α on the nodes of $\mathscr{S}$, and set $F(\alpha)c = \alpha a$.

Define $G : \mathbf{Mod}(\mathscr{S}, \mathscr{C}) \to \mathbf{Mod}(\mathscr{T}, \mathscr{C})$ so that $G(K)$ is K restricted to the nodes and arrows of $\mathscr{S}$. For $\beta : K \to K'$ define $G(\beta)$ to have the same component as β on a and b.

Then F is an equivalence of categories with pseudoinverse G. $G \circ F$ is actually the identity functor on $\mathscr{S}$. The required natural isomorphism η from $F \circ G$ to the identity functor on $\mathscr{T}$ has $\eta a = \mathrm{id}_a$, $\eta_b = \mathrm{id}_b$ and $\eta c = h$.

Section 7.6

1. It is the model I with $I(d) = \{a, b\}$ and $I(l) = \emptyset$.

2. Let I be the initial model. $I(l)$ contains u, so it contains elements obtained by repeatedly applying tail to u. Let us write tu for tail(u), t^2u for tail(tail(u)), and so on. The operation head can be applied to each of these, producing a countably infinite list of elements ht^nu of $I(d)$, one for each $n \in \mathbf{N}$. By the product property, we then get all possible pairs (ht^mu, t^nu) for $m, n \in \mathbf{N}$. These pairs include the elements t^nu because $t^nu = (ht^nu, t^{n+1}u)$. Again by the product property we get elements $(ht^{m_1}u, (ht^{m_2}u, t^nu))$, $(ht^{m_1}u, (ht^{m_2}u, (ht^{m_3}u, t^nu)))$ and so on.

This suggests simplifying the notation further as follows: We define $I(l)$ to be the set of equivalence classes of finite sequences $s = (s_1, s_2, s_3, \ldots, s_m)$ of all possible nonzero lengths m, with all $s_i \in \mathbf{N}$, using the equivalence relation $\sim$ generated by requiring that $s \sim s'$, where s' is obtained from s by adjoining $s_m + 1$ as the $m + 1$st entry. (Thus $(3, 1) \sim (3, 1, 2) \sim (3, 1, 2, 3)$, etc.) We denote the equivalence class of s by $[s]$. Note that all elements of an equivalence class have the same first entry and that every equivalence class contains entries of length greater than 1. Now define $I(d) = \mathbf{N}$, head($[s]$) $= s_1$ and tail($[s]$) the the equivalence class of the sequence obtained by dropping the first entry of a representative of $[s]$ of length greater than 1.

The only problematical thing to verify is that $I(C)$ is a product cone. Suppose $f : X \to \mathbf{N}$ and $g : X \to I(l)$ are given. By definition of head and tail there is only one possible map $h : X \to I(l)$ that makes $\text{head} \circ h = f$ and $\text{tail} \circ h = g$, and that is for $h(x)$ to be the equivalence class of the infinite sequence s with $s_1 = f(x)$ and $s_k = g(x)_{k-1}$ for $i > 1$, and that definition does indeed work.

To see that I is an initial model, let M be any model. Define a natural transformation $\alpha : I \to M$ as follows. Let $I(u) = M(u)$. If $s \in [s]$ has length greater than 1, let s' denote the sequence obtained by deleting the first entry of s, and define

$$\alpha l([s]) = (M(\text{head})(M(\text{tail})^{s_1}(M(u))); \alpha l([s']))$$

This is the only possible definition for a natural transformation from I to M and it is easy to prove inductively that it is well-defined and a natural transformation.

3. Let the elements of X be x and y. Form the set $\mathbf{N}_x = \{(n; x) \mid n \in \mathbf{N}\}$ and similarly $\mathbf{N}_y = \{(n; y) \mid n \in \mathbf{N}\}$. Then $S = \mathbf{N} \cup \mathbf{N}_x \cup \mathbf{N}_y$ is certainly the disjoint union of three copies of $\mathbf{N}$. We can identify x with $(0; x)$ and y with $(0; y)$ so that, up to isomorphism, $x \in S$ and $y \in S$. We define succ on S by $\text{succ}(n) = n + 1$, $\text{succ}(n; x) = (n + 1; x)$ and $\text{succ}(n; y) = (n + 1; y)$. With these definitions and no relations, this is the initial term model.

Section 7.7

1. Let the sort on which the operations are defined be σ. Thus $\cdot : \sigma \times \sigma \to \sigma$ and $+ : \sigma \times \sigma \to \sigma$. The diagram is then

$$\begin{array}{ccccc}
\sigma \times \sigma \times \sigma & \xrightarrow{v = \mathrm{id}_{\sigma\times\sigma\times\sigma}} & \sigma \times \sigma \times \sigma & \xrightarrow{\langle \mathrm{id}_\sigma; + \rangle} & \sigma \times \sigma \\
{\scriptstyle w = \langle p_1; p_2; p_1; p_3 \rangle}\Big\downarrow & & & & \Big\downarrow{\scriptstyle \cdot} \\
\sigma \times \sigma \times \sigma \times \sigma & \xrightarrow[\cdot \times \cdot]{} & \sigma \times \sigma & \xrightarrow[+]{} & \sigma
\end{array}$$

Solutions for Chapter 8

Section 8.1

1. Suppose by induction that this is true for every list of $n - 1$ parallel arrows and $f_1; \ldots; f_n : A \to B$ is a list of n parallel arrows. Let $j : E \to A$ be an equalizer of $f_1; \ldots; f_{n-1}$. We may clearly suppose that $n \geq 3$. Then the parallel pair $f_1 \circ j; f_n \circ j : E \to B$ has an equalizer $k : F \to E$. I claim that $j \circ k : F \to A$ is an equalizer of the list. In fact, for $i < n$, $f_i \circ j \circ k = f_1 \circ j \circ k$, because

j simultaneously equalizes all those arrows, while $f_n \circ j \circ k = f_1 \circ j \circ k$ because k equalizes $f_1 \circ j$ and $f_n \circ j$. If $h : C \to A$ is simultaneously equalized by $f_1, \ldots, f_n$, then there is a unique $m : C \to E$ such that $j \circ m = h$. Since $f_1 \circ j \circ m = f_1 \circ h = f_n \circ h = f_n \circ j \circ m$, there is a unique $g : C \to F$ such that $k \circ g = m$. Then $j \circ k \circ g = j \circ m = h$. If $j \circ k \circ g' = h$, then $g' = g$ because $j \circ k$ is monic; hence g has the required uniqueness property.

2. Let A and B be monoids and $f, g : A \to B$ be monoid homomorphisms. Let $E = \{x \in A \mid f(a) = g(a)\}$. Then $f(1) = 1 = g(1)$ so that $1 \in E$. Also if $x, y \in E$, then $f(xy) = f(x)f(y) = g(x)g(y) = g(xy)$ so that $xy \in E$ and E is a submonoid of A. Let $j : E \to A$ be the inclusion homomorphism. Now let $h : C \to A$ be a monoid homomorphism with $f \circ h = g \circ h$. Then for all $x \in C$, $f(h(x)) = g(h(x))$ so that $h(x) \in E$. Thus $h(C) \subseteq E$ so there is a function $k : C \to E$ with $j \circ k = h$. It has to be shown that k is a homomorphism, but $k(x) = h(x)$ for all $x \in C$, so that is trivial. The fact that j is injective makes the uniqueness of k evident.

3. a. For x and y to have an equalizer, we would need, at least, an element z with $xz = yz$ and this never happens in a free monoid.

b. Suppose the element x is the equalizer of y and z. Then x is monic, which means, according to the cited exercise, that x is invertible. But then $yx = zx$ implies that $y = yxx^{-1} = zxx^{-1} = z$.

4. A monomorphism in **Set** is an injective function (see Theorem 2.8.3), so let $f : A \to B$ be an injective function. Let C be the set of all pairs

$$\{(b, i) \mid b \in B,\ i = 0, 1\}$$

and impose an equivalence relation on these pairs forcing $(b, 0) = (b, 1)$ if and only if there is an $a \in A$ with $f(a) = b$ (and not forcing $(b, i) = (c, j)$ if b and c are distinct). Since f is injective, if such an a exists, there is only one. Let $g : B \to C$ by $g(b) = (b, 0)$ and $h : B \to C$ by $h(b) = (b, 1)$. Then clearly $g(b) = h(b)$ if and only if there is an $a \in A$ with $f(a) = b$. Now let $k : D \to B$ with $g \circ k = h \circ k$. It must be that for all $x \in D$, there is an $a \in A$, and only one, such that $k(x) = f(a)$. If we let $l(x) = a$, then $l : D \to A$ is the unique arrow with $f \circ l = k$.

5. The reason v and w are inverse to each other is that there is no other arrow for $v \circ u$ to be but id_C and similarly $u \circ v = \mathrm{id}_D$. Let $F : \mathscr{A} \to \mathscr{B}$ be the inclusion. For any functors $G, H : \mathscr{B} \to \mathscr{C}$, if $G \circ F = H \circ F$, then $G(u) = H(u)$. But then

$$G(v) = G(u^{-1}) = G(u)^{-1} = H(u)^{-1} = H(u^{-1}) = H(v)$$

Since the objects of $\mathscr{B}$ and the other arrows are in the image of F, it follows that $G = H$. Since F is not an isomorphism, it cannot be a regular monomorphism by Proposition 8.1.8.

Section 8.2

1. If D is the base of Diagram (8.1), then $\mathrm{cone}(E, D)$ is the set of all commutative cones of that shape. The equalizer of f and g is the universal element of the functor F of the proof of Proposition 8.1.4. These two functors are naturally isomorphic by an isomorphism $\phi : F \to \mathrm{cone}(-, D)$: if X is any object and $u : X \to A$ is in $F(X)$ (so $f \circ u = g \circ u$), then $\phi A(u)$ is the cone with components $u : X \to A$ and $f \circ u : X \to B$. By its very definition this is a commutative cone, so an element of $\mathrm{cone}(X, D)$. Since u determines $f \circ u$ uniquely, this function ϕA is a bijection, so ϕ is a natural isomorphism. Since the two functors are isomorphic, so are the objects that represent them, by the uniqueness of universal elements.

2. The limits are as indicated:

$$\begin{array}{ccc}
S & & \\
{\scriptstyle id_S}\downarrow & \searrow^{f} & \\
S & \xrightarrow[f]{} & T
\end{array}
\qquad
\begin{array}{ccccc}
 & & S \times_T S & & \\
 & \swarrow^{proj_1} & \downarrow & \searrow^{proj_2} & \\
S & \xrightarrow[f]{} & T & \xleftarrow[f]{} & S
\end{array}
\qquad
\begin{array}{ccccc}
 & & S \times S & & \\
 & \swarrow^{proj_1} & \downarrow & \searrow^{proj_2} & \\
S & \longrightarrow & 1 & \longleftarrow & S
\end{array}$$

(a) (b) (c)

where $S \times_T S$ is the subset $\{(s, s') \mid f(s) = f(s')\}$ of $S \times S$. (This is standard notation to be introduced in Section 8.3.)

3. a. An arrow $f : C \to \prod_{i \in \mathscr{I}} Di$ is simply a collection of arrows $f_i : C \to Di$, one for each object of $\mathscr{I}$. In order that it be a commutative cone on the diagram D it is necessary and sufficient that it satisfy the additional condition that $Da \circ f_j = f_k$. This is equivalent to

$$proj_k \circ f = f_k = Da \circ f_j = Da \circ proj_j \circ f$$

which is a necessary and sufficient condition that f factor through $E \to \prod Di$. Thus a cone over D is equivalent to an arrow to E, which means that E is a limit of D.

b. An arrow $f : C \to \prod Di$ is a collection of arrows $f_i : C \to Di$. In order that it be a commutative cone on D it must simultaneously satisfy the conditions $Da \circ f_j = f_k$ and $Db \circ f_l = f_m$. This is equivalent to $\langle f_i, f_l \rangle : C \to Di \times Dl$ and $\langle f_k, f_m \rangle : C \to Dk \times Dm$ satisfying $(Da \times Db) \circ \langle f_j, f_l \rangle = \langle f_k, f_m \rangle$, in turn equivalent to $r \circ f = s \circ f$, that is to factoring through $E \to \prod Di$.

c. Again an arrow $f : C \to A$ is a family of arrows $f_i : C \to Di$. In order to be a commutative cone on D it must satisfy $Da \circ f_{\mathrm{source}(a)} = f_{\mathrm{target}(a)}$ for every arrow $a \in \mathscr{I}$. This is exactly the condition that $r \circ f = s \circ f$, which means that f is a cone over D if and only if it factors through $E \to A$. Hence a cone over D is equivalent to an arrow to E, which means that E is the limit.

Section 8.3

1. Let $f : R \to S$ and $k : R \to A$ be functions such that $g \circ f = i \circ k$. Then for each $x \in R$, $g(f(x)) \in A$ which means that $f(x) \in g^{-1}(A)$. Thus f factors through $g^{-1}(A)$ by the corestriction $m : R \to g^{-1}(A)$ of f to $g^{-1}(A)$, so $f = j \circ m$. Also $i \circ h \circ m = i \circ k$, so $h \circ m = k$ because i is monic.

2. Suppose that $f : A \rightarrowtail B$ is monic. Then the only way you can have $g, h : C \to A$ such that $f \circ g = f \circ h$ is if $g = h$. In that case $g : C \to A$ is the unique arrow such that $\mathrm{id} \circ g = g$ and $\mathrm{id} \circ g = h$ so that

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{\mathrm{id}} & \downarrow f & \searrow^{\mathrm{id}} & \\ A & \xrightarrow{f} & B & \xleftarrow{f} & A \end{array}$$

satisfies the condition of being a pullback cone. Thus (a) implies (b). It is obvious that (b) implies (c). Now suppose that the diagram in (c) is a pullback. Let h, $k : C \to A$ be arrows such that $f \circ h = f \circ k$. Then we have a cone

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{h} & \downarrow f \circ h & \searrow^{k} & \\ A & \xrightarrow{f} & B & \xleftarrow{f} & A \end{array}$$

so that there is an arrow $l : C \to P$ such that $h = g \circ l = k$. Thus (c) implies (a).

3. Suppose $h, k : D \to P$ with $p_2 \circ h = p_2 \circ k$. We have

$$f \circ p_1 \circ h = g \circ p_2 \circ h = g \circ p_2 \circ k = f \circ p_1 \circ k$$

and f is monic by assumption so that $p_1 \circ h = p_1 \circ k$. Thus $x = h$ and $x = k$ are solutions to the equation $p_1 \circ x = p_1 \circ h$ and $p_2 \circ x = p_2 \circ h$. But the definition of pullback requires that solution to be unique so that $h = k$.

4. Let $q_1 : S \times U \to S$ and $q_2 : S \times U \to U$ be the product projections. Suppose there are arrows $u, v : X \to P$ such that $\langle p_1, p_2 \rangle \circ u = \langle p_1, p_2 \rangle \circ v$. Then

$$\begin{aligned} f \circ p_1 \circ u &= f \circ q_1 \circ \langle p_1, p_2 \rangle \circ u \\ &= f \circ q_1 \circ \langle p_1, p_2 \rangle \circ v \\ &= f \circ p_1 \circ v \end{aligned}$$

and similarly $g \circ p_2 \circ u = g \circ p_2 \circ v$ so $u = v$ by the uniqueness part of the universal property of pullbacks.

5. If $f, g : A \to B$, then an equalizer is the arrow $j : E \to A$ in a pullback

$$\begin{array}{ccc} E & \xrightarrow{\quad j \quad} & A \\ \Big\downarrow & & \Big\downarrow \langle f, g\rangle \\ B & \xrightarrow[\langle \mathrm{id}, \mathrm{id}\rangle]{} & B \times B \end{array}$$

The proof of this fact comes down to showing that any pullback of this square and an equalizer of f and g represent equivalent functors: the functor G representing a pullback of the square has value

$$\begin{aligned} G(X) &= \{(u : X \to A, v : X \to B) \mid \langle \mathrm{id}, \mathrm{id}\rangle \circ v = \langle f, g\rangle \circ u\} \\ &= \{(u, v) \mid \langle v, v\rangle = \langle f \circ u, g \circ u\rangle\} \\ &= \{(u, v) \mid f \circ u = v = g \circ u\} \\ &\cong \{u \mid f \circ u = g \circ u\} \end{aligned}$$

which is the value of the functor representing an equalizer of f and g. (Compare the proof of Exercise 1 of Section 8.2.)

6. A pullback of a diagram $A \to 1 \leftarrow B$ is just a product $A \times B$. We know from the preceding exercise that a category with products and pullbacks has equalizers. Hence such a category has finite products and equalizers. The conclusion now follows from Corollary 8.2.11.

7. In **Set**, a pullback is

$$P = \{(a, b) \mid f(a) = g(b)\}$$

Since f is surjective, for any $b \in B$, there is a $a \in A$ such that $f(a) = g(b)$. Then $p_2(a, b) = b$. Thus p_2 is surjective. The condition now follows from Proposition 2.9.2.

8. In the category of monoids, we have seen in 2.9.3 that the inclusion of the monoid **N** of nonnegative integers into the monoid **Z** of integers is an epimorphism. Let $A \subseteq \mathbf{Z}$ denote the submonoid of nonpositive integers. Both inclusions are epic, but their pullback, which is the intersection, is the submonoid $\{0\}$ and the maps to both **N** and A are not epic. In fact all pairs of arrows on **N** (or on A) agree on $\{0\}$.

9. Suppose the arrow $g : A \to C$ in Diagram (8.2) is a split epi, so that there is an arrow $h : C \to A$ so that $g \circ h = \mathrm{id}_C$. Then we have $\mathrm{id} : B \to B$ and $h \circ f : B \to A$ satisfy $f \circ \mathrm{id} = g \circ h \circ f$ so there is an arrow $k : B \to P$ such that $p_1 \circ k = h \circ f$ and $p_2 \circ k = \mathrm{id}_B$. This latter identity is what we want.

10. Suppose we have arrows A ¡! B and A ¡! C giving a commutative cone. By composing the latter with C ¡! D, we get a pair of arrows A ¡! B and A ¡! D giving a commutative cone and this leads to a unique arrow A ¡! Q. We now have A ¡! C and A ¡! Q giving a commutative cone and this leads to a unique arrow A ¡! P making the left hand cone commute. Clearly, the outer rectangle commutes as well.

It says that the weakest precondition of the composite of two procedures can be calculated as the weakest precondition under the second procedure of the weakest precondition under the ¯rst.

Section 8.4

1. In Exercise 8.1.5 of Section 8.1, we showed that any equalizer is a monomorphism. Interpreted in the dual category, it is the result of this exercise.

2. By Proposition 2.9.2, every epimorphism in Set is surjective. We claim that every surjective function is a regular epimorphism. In fact, let f : S ¡! T be surjective and suppose E = f(x; y) 2 S £ S j f(x) = f(y)g. Let p; q : E ¡! S be the ¯rst and second projections. We claim that f is the coequalizer of p and q. Clearly f ± p = f ± q. Let h : S ¡! R such that h ± p = h ± q. De¯ne k : T ¡! R as follows. For t 2 T, there is an s 2 S with f(s) = t. We would like to de¯ne k(t) = h(s). If s^0 is another element of S with f(s^0) = t, then (s; s^0) 2 E and so h(s) = (h ± p)(s; s^0) = (h ± q)(s; s^0) = h(s^0). Thus k is well de¯ned and clearly k ± f = h. The uniqueness of k follows from the fact that f is epic.

3. In Proposition 8.1.8, it is shown that a regular mono that is an epimorphism is an isomorphism. The dual says that a monomorphism that is a regular epimorphism is an isomorphism. In 2.9.3 the inclusion was shown to be epic. Since the inclusion is evidently a monomorphism and not an isomorphism, it cannot be a regular epimorphism.

4. We use the terminology of Exercise 2, just assuming that S and T are monoids and f a monoid homomorphism. First we have to say that if f is not surjective, then it still may be an epimorphism, as the inclusion of **N** into **Z** shows, but it cannot be regular. The reason is the easily veri¯ed fact that the image of a monoid homomorphism f : S ¡! T is a submonoid T_0 µ T and if it is not all of T, the properties of regular epimorphism and the fact that the inclusion is a monomorphism combine to provide an arrow T ¡! T_0 such that the composite T ¡! T_0 ¡! T is the identity of T. This is possible if and only if T_0 = T.

To go the other way, we need add to the construction of Exercise 2 only the facts that E µ S £ S is a submonoid and that all the arrows constructed are monoid homomorphism. Of all of these, only the fact that k is a monoid homomorphism is interesting. In fact, if t and t^0 are elements of T and if s, s^0 2 S are such that f(s) = t and f(s^0) = t^0, then since f is a monoid homomorphism,

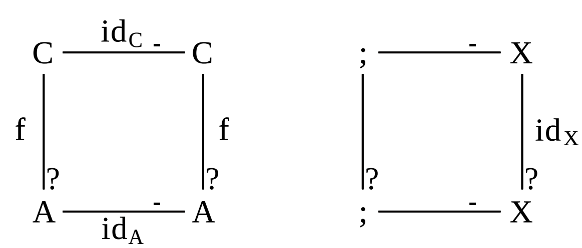
$f(ss') = f(s)f(s') = tt'$ so that $k(tt') = h(ss') = h(s)h(s') = k(t)k(t')$. Similarly, since f is a monoid homomorphism, $f(1) = 1$, so $k(1) = h(1) = 1$.

5. A kernel pair of f is characterized by the following mapping property: $f \circ d^0 = f \circ d^1$ and if $e^0, e^1 : C \to A$ satisfies $f \circ e^0 = f \circ e^1$, then there is a unique $g : C \to K$ such that $d^i \circ g = e^i$, $i = 0, 1$. Now if f is the coequalizer of e^0 and e^1 and the kernel pair exists, let $g : C \to K$ as described. Suppose $h : A \to D$ is an arrow such that $h \circ d^0 = h \circ d^1$. Then $h \circ e^0 = h \circ d^0 \circ g = h \circ d^1 \circ g = h \circ e^1$ so that there is a unique $k : B \to D$ with $k \circ f = h$. Thus f is the coequalizer of d^0 and d^1.

Section 8.5

1. Suppose E is some object and $\langle l | l' \rangle : C + C' \to E$ and $\langle m | m' \rangle : B + B' \to E$ satisfy $\langle l | l' \rangle \circ (g + g') = \langle m | m' \rangle \circ (f + f')$. If i and j represent the inclusions of the components, then by definition, $\langle l | l' \rangle \circ (g + g') \circ i = \langle l | l' \rangle \circ i \circ g = l \circ g$ and similarly $\langle m | m' \rangle \circ (f + f') \circ i = m \circ f$ so these equations imply that $l \circ g = m \circ f$. By using j we similarly conclude that $l' \circ g' = m' \circ f'$. The mapping properties of the pushout imply the existence of $n : D \to E$ and $n' : D' \to E$ such that $n \circ k = l$, $n \circ h = m$ $n' \circ k' = l'$ and $n' \circ h' = m'$. Then $\langle n | n' \rangle$ is the required arrow. Uniqueness follows from the uniqueness of the components, together with the uniqueness of an arrow from a sum, given its components.

2. a. Since e is injective, $e(C) \cong C$. Up to isomorphism, the diagram is the sum of the following two:



and the preceding exercise completes the argument.

b. e is an arbitrary monomorphism and the function i is injective.

3. a. As suggested by the hint, we take pairs of natural numbers with coordinatewise addition and subject to the relation that for any $a \in \mathbf{N}$, $(b, c) = (a + b, a + c)$. This relation is not an equivalence relation (it is not symmetric because a cannot be negative), but the symmetric closure is an equivalence relation. We will show that the quotient is isomorphic to $\mathbf{Z}$. To do this we define a function $f : \mathbf{N} \times \mathbf{N} \to \mathbf{Z}$ by $f(b, c) = b - c$. Clearly, $f(a + b, a + c) = f(b, c)$ so that $f \circ g = f \circ h$. On the other hand, if $f(b, c) = f(b', c')$, then $c - b = c' - b'$ or $c - c' = b - b'$. If $c \le c'$, then $(b, c) = g(c' - c, b, c)$ while $(b', c') = h(c' - c, b, c)$. If $c' \le c$, their roles are reversed. In either case, f is the quotient by the generated equivalence relation. The function f is also surjective, since every $n \ge 0$ is $f(n, 0)$, while every $n < 0$ is

$f(0; -n)$. Since $f(0;0) = 0$ and $f(b+b'; c+c') = b+b' - (c+c') = b - c + b' - c' = f(b;c) + f(b';c')$ so that f is a monoid homomorphism and the coequalizer is the coequalizer in the category of monoids.

b. Let us denote this subset by A. The function $f : A \to \mathbf{Z}$ defined by

$$f(n;m) = \begin{cases} n & \text{if } m = 0 \\ -m & \text{if } n = 0 \end{cases}$$

is obviously bijective. It is a matter of consideration of cases to see that it is additive if addition is defined in A as suggested above.

4. Define $h : \mathbf{Z} \times \mathbf{N}^+ \to \mathbf{Q}$ by $h(b;c) = b/c$. Since $b/c = (ab)/(ac)$, $h \circ f = h \circ g$. Moreover, if $h(b;c) = h(b';c')$, then $bc' = b'c$. We have the equations

$$\begin{aligned} f(c'; b; c) &= (b; c) \\ g(c'; b; c) &= (c'b; c'c) \\ f(c; b'; c') &= (b'; c') \\ g(c; b'; c') &= (cb'; cc') \end{aligned}$$

Thus the coequalizer of f and g must render $(b;c)$ and $(b';c')$ equal. Thus h is injective. It is clearly surjective.

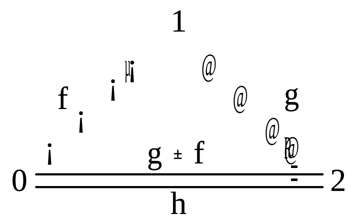
5. Let X be a set, $r : X \to B + B'$ and $s : X \to C + C'$ functions such that $(h + h') \circ r = (k + k') \circ s = v$. If we let $Y = v^{-1}(D)$ and $Y' = v^{-1}(D')$, then $X = Y + Y'$. Moreover, it is clear that $r(Y) \subseteq B$, $r(Y') \subseteq B'$, $s(Y) \subseteq C$ and $s(Y') \subseteq C'$. Let $t : Y \to B$ and $t' : Y' \to B'$ be the restrictions of r to Y and Y', respectively. Then $r = \langle t|t' \rangle$. Similarly, we have $s = \langle u|u' \rangle$ for $u : Y \to C$ and $u' : Y' \to C'$. Moreover, $(h + h') \circ r = (k + k') \circ s$ is equivalent to $h \circ t = k \circ u$ and $h' \circ t' = k' \circ u'$. Since the original two squares were pullbacks, it follows that there are arrows $w : Y \to A$ and $w' : Y' \to A'$ such that $f \circ w = t$, $g \circ w = u$, $f' \circ w' = t'$ and $g' \circ w' = u'$. This implies that $(f + f') \circ \langle w|w' \rangle = \langle t|t' \rangle = r$ and $(g + g') \circ \langle w|w' \rangle = \langle u|u' \rangle = s$. We also have to show uniqueness, but the arguments are similar.

6. a. Let $e^0 : \mathbf{1} \to \mathbf{2}$ take the single object of $\mathbf{1}$ to one object of $\mathbf{2}$ and let e^1 take it to the other. Then to say of a functor $u : \mathbf{2} \to \mathscr{A}$ that $u \circ e^0 = u \circ e^1$ is simply to say that u takes the two objects to the same one. If a is the single arrow in $\mathbf{2}$, then $\text{source}(u(a)) = \text{target}(u(a))$ so that not only will there be $u(a)$, but also $u(a) \circ u(a)$ and $u(a) \circ u(a) \circ u(a)$ and so on. Now let $\mathbf{N}$ also denote the category with one object and the natural numbers as arrows as defined in the exercise. Then $q(a) \circ q(a) = 2$, $q(a) \circ q(a) \circ q(a) = 3$ and so on. Given a functor u as above, we can define a functor $v : \mathbf{N} \to \mathscr{A}$ by $v(0) = \mathrm{id}_{\mathrm{source}(u(a))}$, $v(1) = u(a)$, $v(2) = u(a) \circ u(a)$ and so on. Clearly v is unique such that $v \circ q = u$. Thus q is the coequalizer of u and v.

b. In the diagram, the arrow from $\mathbf{1}+\mathbf{1}$ to $\mathbf{2}$ is inclusion and the arrow from $\mathbf{1}+\mathbf{1}$ to $\mathbf{N}$ is the only possible one. Let $v:\mathscr{A}\to\mathbf{2}$ and $w:\mathscr{A}\to\mathbf{N}$ be such that $q\circ v = t\circ w$. If f is an arrow in $\mathscr{A}$ then $q(v(f)) = t(w(f))$. But the only integer that is in the image of $q\circ v$ and $t\circ w$ is 0. Thus $q(v(f)) = 0$, which means that $v(f)$ is an identity. Thus v takes every arrow to an identity arrow, that is it factors through $\mathbf{1}+\mathbf{1}$, and the factorization is clearly unique.

The arrow $s:\mathbf{1}+\mathbf{1}\to\mathbf{N}$ is not even epic, let alone regular. In fact, $t\circ s = s = \mathrm{id}_{\mathbf{N}}\circ s$ without $t = \mathrm{id}_{\mathbf{N}}$.

c. Let $\mathscr{D}$ be the category whose nonidentity arrows can be pictured as:



There is a functor $G:\mathscr{C}\to\mathscr{D}$ that is like F except that $G(h) = h$. This functor cannot factor through F because under any $H:\mathbf{3}\to\mathscr{D}$, $H(F(h)) = H(g\circ f) = H(g)\circ H(f) \neq h$.

d. A complete answer is too long, but we give enough details that the reader should have no difficulty filling in the rest. The first thing to observe is that being surjective on composable pairs of arrows includes, as special cases, being surjective on objects and on arrows. Suppose $T:\mathscr{A}\to\mathscr{B}$ is a functor which fails to be surjective on composable pairs, but is surjective on arrows and objects. Let f_1 and g_1 be a composable pair of arrows such that whenever $T(f_2) = f_1$ and $T(g_2) = g_1$, then f_2 and g_2 are not composable. Then let $S:\mathbf{3}\to\mathscr{B}$ be the unique functor such that $S(f) = f_1$ and $S(g) = g_1$. The pullback of S along T will be a category that includes subcategories like $\mathscr{C}$ of part (c) of this problem and other pieces. It will have a functor to the category $\mathscr{D}$ of the preceding part that takes every arrow lying above f to f, every arrow lying above g to g and every arrow lying above $g\circ f$ to h, which is not the composite $g\circ f$. Then just as in the preceding part, this functor makes all the identifications made by the functor to $\mathbf{3}$, but does not factor through that functor. Thus that functor is not a regular epi. If T fails to be epi on arrows or on objects, even easier arguments suffice.

For the other direction, in fact a functor that is surjective on composable pairs of arrows is a stable regular epi (which is more than the problem asked for). It is immediate that the condition of being surjective on composable pairs is stable under pullback, so it is sufficient to show that such an arrow is a regular epi. The argument is similar to that for monoids and we omit it.

Section 8.6

1. The functor that assigns to each object the set of commutative cocones to that object on the base $A \leftarrow 0 \rightarrow B$ is naturally isomorphic to $\mathrm{Hom}(A, -) \times \mathrm{Hom}(B, -)$, which is the functor which assigns the set of cocones on the discrete base consisting of A and B. The isomorphism forgets the arrow $0 \rightarrow A + B$ and its inverse puts back the only arrow that can go from 0 to $A + B$. Thus the universal elements are isomorphic.

2. Use exactly the same argument as for the preceding exercise.

3. We have to show that if

$$\begin{array}{ccc} P_1 & \xrightarrow{p_1} & A \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle f} \\ C_1 & \xrightarrow[g_1]{} & B \end{array} \qquad \begin{array}{ccc} P_2 & \xrightarrow{p_2} & A \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle f} \\ C_2 & \xrightarrow[g_2]{} & B \end{array}$$

are both pullbacks, then so is

$$\begin{array}{ccc} P_1 + P_2 & \xrightarrow{\langle p_1 | p_2 \rangle} & A \\ {\scriptstyle q_1 + q_2}\downarrow & & \downarrow{\scriptstyle f} \\ C_1 + C_2 & \xrightarrow[\langle g_1 | g_2 \rangle]{} & B \end{array}$$

Let $i_1 : C_1 \rightarrow C_1 + C_2$, $i_2 : C_2 \rightarrow C_1 + C_2$, $j_1 : P_1 \rightarrow P_1 + P_2$ and $j_2 : P_2 \rightarrow P_1 + P_2$ be the coproduct injections. If $a \in A$ and $c \in C_1 + C_2$ such that $f(a) = \langle g_1 | g_2 \rangle(c)$, then either $c = i_1(c_1)$ for some $c_1 \in C_1$ and $f(a) = g_1(c)$ or $c = i_2(c_2)$ for some $c_2 \in C_2$ and $f(a) = g_2(c)$. In the first case, there is a unique $x_1 \in P_1$ such that $p_1(x_1) = a$ and $q_1(x_1) = c_1$. Then also $(q_1 + q_2)(j_1(x_1)) = j_1(q_1(x_1)) = j_1(c_1) = c$ and $\langle p_1 | p_2 \rangle(j_1(x_1)) = p_1(x_1) = a$. We get a similar conclusion if $c = j_2(c_2)$. This shows that $P_1 + P_2$ satisfies the existence condition of pullback. The uniqueness condition is similar.

4. Let us deal first with the monoid case. If M is a monoid and $f, g : E \rightarrow M$ are monoid homomorphisms, then $f = g$ is the function taking the unique element of $e \in E$ to the identity of M. Hence $f : E \rightarrow M$ is the unique monoid homomorphism such that $f \circ \mathrm{id}_E = f$ and $f \circ \mathrm{id}_E = g$. Thus the cocone

$$\begin{array}{ccccc} E & & & & E \\ & {\scriptstyle \mathrm{id}_E}\searrow & & \swarrow{\scriptstyle \mathrm{id}_E} & \\ & & E & & \end{array}$$

is a colimit.

In the category of semigroups, the situation is quite different. If $f : E \to S$ is a semigroup homomorphism, $f(e)$ need not be an identity arrow, if any exists. Since $e^2 = e$, it must be that $f(e)^2 = f(e)$ (such an element of S is called an **idempotent**). Other than that, there is no restriction on what $f(e)$ can be. Now let S be the semigroup of endofunctions on $\mathbf{N}$. The elements u and v defined in the problem are idempotents such that for all n, $(u \circ v)^n$ are distinct. In fact, $(u \circ v)^n$ adds $2n$ to odd integers and $2n - 2$ to even ones (and $(v \circ u)^n$ does the opposite). Now let $i : E \to F$ and $j : E \to F$ give a sum $F = E + E$. Let $i(e) = a$ and $j(e) = b$. Since there are arrows $f, g : E \to S$ defined by $f(e) = u$ and $g(e) = v$, there is an $h : F \to S$ such that $h \circ i = f$ and $h \circ j = g$. In F there are all the elements $(ab)^n$ and since $h((ab)^n) = (u \circ v)^n$ in order to be a semigroup homomorphism, all the $(ab)^n$ must also be distinct. Thus F is infinite.

5. This follows because they are each isomorphic to E in the previous example.

6. a. The recursive functions are defined as the smallest set of functions including successor and projections and closed under certain operations, of which the simplest is composition. (Details are in [Lewis and Papadimitriou, 1981], where recursive functions are called μ-recursive.) Since the identity is also recursive (it is, for one thing, the projection from the unary product), it follows that this defines a category.

b. The first thing we do is to choose a bijective pairing with the additional property that it reflects. This means we choose a pair of functions $l, r : \mathbf{N} \to \mathbf{N}$ such that the function $\langle l, r\rangle : \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ is bijective and such that if $l(m) \le l(m')$ and $r(m) \le r(m')$, then $m \le m'$. It follows that if $l(m) = a$, then the number of $m' \le m$ for which $l(m') = a$ is $r(m)$. A pairing that works is given by $l(m) = \exp_2(m)$, the largest integer l for which 2^l divides m, and

$$r(m) = \frac{m + 1 - 2^l}{2^{l+1}}$$

where $l = l(m)$. Let M_i be the ith Turing machine in some enumeration of them. Let M be the process that at step n runs $M_{l(n)}$ one step. Because of our assumption on the bijective pairing, this is actually the $r(n)$th step that this machine has executed. Define a recursive function $f : \mathbf{N} \to \mathbf{N}$ by letting $f(n) = r(n)$ if machine $M_{l(n)}$ halts at the nth step and 0 otherwise. This is not injective (takes the value 0 more than once) but the composite $\langle r, l\rangle^{-1} \circ \langle \mathrm{id}, f\rangle$ is and defines an injective recursive function $\mathbf{N} \to \mathbf{N}$ whose image is not decidable (else the halting problem would be). But then there cannot be any recursive function whose image is the complement. Thus the subobject defined by that injection has no complement in the category.

7. Let N denote the sum of countably many copies of 1 and $i_n : 1 \to N$ be the nth injection to the sum. Let $z = n_0 : 1 \to N$ and let $s : N \to N$ be defined by

$s \circ i_n = i_{n+1}$. Given an object A, an arrow $f_0 : 1 \to A$ and an arrow $t : A \to A$, we use ordinary induction to define a sequence of arrows $f_n : 1 \to A$, for $n \geq 1$ by $f_{n+1} = t \circ f_n$. Then the arrow $f = \langle f_0; f_1; \ldots; \rangle : N \to A$ clearly satisfies the recursion. The uniqueness is shown similarly. If the countable sums are stable under pullbacks, then this construction is clearly stable.

Section 8.7

1. The definition of epimorphism implies that for any $h : B \to X$, the condition $h \circ f = h \circ g$ is equivalent to the condition $h \circ f \circ e = h \circ g \circ e$ so that the condition to be satisfied by a coequalizer of f and g is identical to that to be satisfied by a coequalizer of $f \circ e$ and $g \circ e$. Thus the cocone functors are equivalent and are therefore the universal elements.

2. If $h = \langle h_1 | h_2 \rangle : B_1 + B_2 \to D$, then the condition $\langle h_1 | h_2 \rangle \circ (f_1 + f_2) = \langle h_1 | h_2 \rangle \circ (g_1 + g_2)$ is equivalent to the two conditions $h_1 \circ f_1 = h_1 \circ g_1$ and $h_2 \circ f_2 = h_2 \circ g_2$. There results unique arrows $k_1 : C_1 \to D$ and $k_2 : C_2 \to D$ such that $k_1 \circ c_1 = h_1$ and $k_2 \circ c_2 = h_2$. This gives $\langle k_1 | k_2 \rangle \circ (c_1 + c_2) = \langle h_1 | h_2 \rangle$.

3. By definition of coequalizer, $e \circ d^0 \circ u_1 = e \circ d^1 \circ u_1$, so $c \circ e \circ d^0 \circ u_1 = c \circ e \circ d^1 \circ u_1$, and also $c \circ e \circ d^0 \circ u_2 = c \circ e \circ d^1 \circ u_2$, so by the universal property of sums, $c \circ e \circ d^0 = c \circ e \circ d^1$. Suppose $f \circ d^0 = f \circ d^1$. Then $f \circ d^0 \circ u_1 = f \circ d^1 \circ u_1$, so there is a unique arrow $x : C \to X$ for which $x \circ e = f$. Since $x \circ e \circ d^0 \circ u_2 = f \circ d^0 \circ u_2 = f \circ d^1 \circ u_2 = x \circ e \circ d^1 \circ u_2$, there is a unique arrow $y : D \to X$ for which $y \circ c = x$, hence $y \circ c \circ e = f$. Thus $c \circ e$ satisfies the existence part of the definition of coequalizer of d^0 and d^1. If $y' \circ c \circ e = f$, then $y \circ c = y' \circ c$ since e is a coequalizer, and so $y = y'$ because c is a coequalizer. Thus $c \circ e$ also satisfies the uniqueness requirement.

Solutions for Chapter 9

Section 9.1

1. By definition, g takes the identity of $F(X)$ to that of M. If $a = (x_1; \ldots; x_n)$ and $b = (y_1; \ldots; y_m)$, then $ab = (x_1; \ldots; x_n; y_1; \ldots; y_m)$ and

$$\begin{aligned} g(ab) &= g(x_1; \ldots; x_n; y_1; \ldots; y_m) \\ &= u(x_1) \cdot \cdots \cdot u(x_n) \cdot u(y_1) \cdot \cdots \cdot u(y_m) \\ &= g(x_1; \ldots; x_n) \cdot g(y_1; \ldots; y_m) \\ &= g(a) \cdot g(b) \end{aligned}$$

Section 9.2

1. If $S_0 \subseteq f^{-1}(T_0)$, let $y \in f(S_0)$. Then there is some $x \in S_0$ such that $f(x) = y$ and since $x \in S_0 \subseteq f^{-1}(T_0)$, $x \in f^{-1}(T_0)$ so that $y = f(x) \in T_0$. Therefore $f_*(S_0) \subseteq T_0$. Conversely, suppose $f_*(S_0) \subseteq T_0$. For $x \in S_0$, $f(x) \in F(S_0)$ so that $f(x) \in T_0$, whence $x \in f^{-1}(T_0)$. Therefore $S_0 \subseteq f^{-1}(T_0)$.

2. Let $\Sigma \dashv \Delta$. Then for every object C,

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}}(\Sigma(A, B), C) &\cong \mathrm{Hom}_{\mathscr{C} \times \mathscr{C}}((A, B), \Delta(C)) \\
&\cong \mathrm{Hom}_{\mathscr{C} \times \mathscr{C}}((A, B), (C, C)) \\
&\cong \mathrm{Hom}_{\mathscr{C}}(A, C) \times \mathrm{Hom}_{\mathscr{C}}(B, C)
\end{aligned}$$

which is the mapping condition that determines $A + B$ uniquely.

3. The unit of the adjunction is the map from $\Delta\Pi(A, B) \to (A, B)$ that corresponds under the adjunction to the identity arrow $\Pi(A, B) \to \Pi(A, B)$. Since $\Pi(A, B) = A \times B$, we are looking for an arrow $(A \times B, A \times B) \to (A, B)$. The arrow is $(\mathrm{proj}_1, \mathrm{proj}_2)$ since that is the pair that gives the universal mapping property.

4. If n is an integer, let us (temporarily) denote by $\mathrm{coe}(n)$ (for coercion), the real number n. If there is a left adjoint, say ci, then it is characterized by the property that for all real $n \in \mathbf{N}$ and $r \in \mathbf{R}$, $\mathrm{ci}(r) \leq n$ if and only if $r \leq \mathrm{coe}(n)$, the property that obviously characterizes the ceiling function. In a similar way the right adjoint fl is characterized by the property that $n \leq \mathrm{fl}(r)$ if and only if $\mathrm{coe}(n) \leq r$, which is the defining property of the floor function.

5. There is a sketch for monoids gotten by augmenting that of semigroups given in Section 7.2 by adding a constant of type s and diagrams that force it to be a left and right identity. Call this sketch $\mathscr{M}$. For a set X, let $\mathscr{M}(X)$ denote the sketch gotten from $\mathscr{M}$ by adding to $\mathscr{M}$ the set X of constants. Then the initial algebra for $\mathscr{M}(X)$ is just the free monoid $F(X)$. The reason is that a model of $\mathscr{M}(X)$ is just a model M of $\mathscr{M}$ together with a chosen element $u(x) \in M$ for each $x \in X$, which is just a function $u : X \to U(M)$. Thus an initial model of the sketch is the free monoid generated by X. In 4.7.16, exactly the same construction of adding a set of constants to a sketch and forming the initial model yields the free model on the original sketch generated by that set.

6. This is the content of Proposition 3.1.15.

7. If the functor defined in 9.2.5 has a left adjoint F, then by definition of left adjoint there is for any set X an arrow $\eta X : X \to FX \times A$ with the property that for any function $f : X \to Y \times A$ there is a unique function $g : FX \to Y$ for which $(g \times A) \circ \eta X = f$. Now take $Y = 1$, the terminal object (any one element set). There is only one function $g : FX \to 1$, so there can be only one function $f : X \to 1 \times A \cong A$. If A has more than one element and X is non-empty, this is a contradiction.

8. Since it is the arrow that corresponds under adjunction to the identity, its value at B is the unique arrow $\eta B : B \to R_A(B \times A)$ such that the composite

$$B \times A \xrightarrow{\eta B \times \mathrm{id}_A} R_A(B \times A) \times A \xrightarrow{e} B \times A$$

is the identity. In **Set**, $R_A(B \times A)$ is the set of functions from A to $A \times B$, and ηB takes $b \in B$ to the function $a \mapsto (b, a)$.

Section 9.3

1. We use Theorem 9.3.2. We have

$$\mathrm{Hom}_{\mathscr{A}}(LTA, A') \cong \mathrm{Hom}_{\mathscr{B}}(TA, TA') \cong \mathrm{Hom}_{\mathscr{A}}(A, RTA')$$

and

$$\mathrm{Hom}_{\mathscr{B}}(TLB, B') \cong \mathrm{Hom}_{\mathscr{A}}(LB, RB') \cong \mathrm{Hom}_{\mathscr{B}}(B, TRB')$$

2. a. A limit of the inclusion $X \subseteq R$ is a real number r such that $r \le x$ for all $x \in X$ (that is the cone with vertex r) and that if $r' \le x$ for all $x \in X$, then $r' \le r$. This is precisely the definition of the infimum. The second half is dual.

b. Since the floor function is a right adjoint, it preserves all limits that exist, in particular, all infimums and dually the ceiling function preserves colimits, which includes supremums. On the other hand, if we take, say a sequence that converges downwards on an integer, say the sequence $X = 1, 1/2, \ldots, 1/n, \ldots$ the infimum of the ceilings is 1, while the infimum is 0, whose ceiling is 0. Similarly, floor does not necessarily preserve supremum.

3. This translates to showing for any $g : A' \to A$ and $h : B \to B'$, that

$$\begin{array}{ccc}
\mathrm{Hom}(FA, B) & \xrightarrow{\beta(A,B)} & \mathrm{Hom}(A, UB) \\
\Big\downarrow{\scriptstyle \mathrm{Hom}(Fg, h)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}(g, Uh)} \\
\mathrm{Hom}(FA', B') & \xrightarrow[\beta(A', B')]{} & \mathrm{Hom}(A', UB')
\end{array}$$

Applied to an $f \in \mathrm{Hom}(FA, B)$, this requires that we show that

$$U(h \circ f \circ Fg) \circ \eta A' = Uh \circ (Uf \circ \eta A) \circ g$$

But using the functoriality of U and naturality of η, we have

$$U(h \circ f\,) \circ \eta A' = Uh \circ Uf \circ UFg \circ \eta A' = Uh \circ Uf \circ \eta A \circ g$$

4. The adjointness $F \dashv U$, for $F : \mathscr{C} \to \mathscr{D}$ and $U : \mathscr{D} \to \mathscr{C}$ is equivalent to the natural isomorphism

$$\mathrm{Hom}_{\mathscr{C}}(-, U-) \cong \mathrm{Hom}_{\mathscr{D}}(F-, -) : \mathscr{C}^{op} \times \mathscr{D} \to \mathbf{Set}$$

But that is the same as the natural isomorphism

$$\mathrm{Hom}_{\mathscr{C}^{op}}(U^{op}-,-) \cong \mathrm{Hom}_{\mathscr{D}^{op}}(-,F^{op}-) : \mathscr{D} \times \mathscr{C}^{op} \to \mathbf{Set}$$

and since $(\mathscr{D}^{op})^{op} = \mathscr{D}$, the conclusion follows.

5. a. Here and below, we use Hom for $\mathrm{Hom}_{\mathscr{C}}$ and Hom_A for $\mathrm{Hom}_{\mathscr{C}/A}$. If $f : B \to A$ is an object of $\mathscr{C}/A$ and C is an object of $\mathscr{C}$, an arrow $g : f \to P_A(C) = p_2 : C \times A \to A$ has two coordinates, say $g_1 = p_1 \circ g : B \to C$ and $g_2 = p_2 \circ g : B \to A$. It is an arrow in $\mathscr{C}/A$ if and only if $g_2 = f$. Thus there are no conditions on g_1, so $\mathrm{Hom}_A(f, P_A(C)) \cong \mathrm{Hom}(B, C) = \mathrm{Hom}(L_A(f), C)$.

b. This is an application of Exercise 1.

c.

$$\begin{aligned}\mathrm{Hom}(B, \Phi(P_A(C))) &\cong \mathrm{Hom}(B, [A \to C]) \cong \mathrm{Hom}(B \times A, C)\\ &\cong \mathrm{Hom}_A(B \times A \to A, C \times A \to A) \cong \mathrm{Hom}_A(P_A(B), P_A(C))\end{aligned}$$

d. The hint suggests that we find a natural transformation from $\mathrm{Hom}(-, \Phi(C))$ to $\mathrm{Hom}(-, \Phi(D))$. It is simple to prove that the isomorphism of part (c) is natural in B. Thus we have for any B, the arrow

$$\begin{aligned}&\mathrm{Hom}(B, \Phi(P_A(C))) \cong \mathrm{Hom}(P_A(B), P_A(C))\\ &\xrightarrow{\mathrm{Hom}(P_A(B), f)} \mathrm{Hom}(P_A(B), P_A(D)) \cong \mathrm{Hom}(B, \Phi(P_A(D)))\end{aligned}$$

The fact that these isomorphisms are natural in B implies the existence of the required arrow $\Phi(f)$. The desired commutation is essentially the definition. This part justifies the use of the notation $\Phi(P_A(C))$ for it implies that if $P_A(C) \cong P_A(D)$, then $\Phi(P_A(C)) \cong \Phi(P_A(D))$.

e. The required diagram is given by letting $d = \langle \mathrm{id}_C, f\rangle : C \to C \times A$, $d^0 = \langle \mathrm{proj}_1, \mathrm{proj}_2, \mathrm{proj}_2\rangle$ and $d^1 = \langle \mathrm{proj}_1, f \circ \mathrm{proj}_1, \mathrm{proj}_2\rangle : C \times A \to C \times A \times A$. These are readily seen to be arrows in $\mathscr{C}/A$. The equalizer of d^0 and d^1 is

$$\begin{aligned}&\{(c, a) \in C \times A \mid (c, a, a) = (c, f(c), a)\}\\ &= \{(c, a) \in C \times A \mid a = f(c)\} = \{(c, f(c)) \mid c \in C\}\end{aligned}$$

which is the image of d.

f. In the diagram

$$\begin{array}{ccccc}\mathrm{Hom}(B, \Phi(f)) & \longrightarrow & \mathrm{Hom}(B, \Phi(P_A(C))) & \rightrightarrows & \mathrm{Hom}(B, \Phi(P_A L_A P_A(C)))\\ \Big\downarrow & & \Big\downarrow \cong & & \Big\downarrow \cong\\ \mathrm{Hom}_A(P_A(B), f) & \longrightarrow & \mathrm{Hom}_A(P_A(B), P_A(C)) & \rightrightarrows & \mathrm{Hom}_A(P_A(B), P_A L_A P_A(C))\end{array}$$

both lines are equalizers since both $\mathrm{Hom}(B, -)$ and $\mathrm{Hom}_A(P_A(B), -)$ preserve equalizers. Moreover the middle and right hand vertical arrows are isomorphisms by part (c) and by definition. Hence by uniqueness of equalizers, the left hand vertical arrow is an isomorphism.

g. This is an immediate consequence of the Pointwise Adjointness Theorem 9.3.5.

Solutions for Chapter 10

Section 10.1

1. In order that h be a homomorphism of algebras, it is required that the diagram

$$\begin{array}{ccc} 1 + \mathbf{N} & \xrightarrow{\langle 0, s\rangle} & \mathbf{N} \\ {\scriptstyle 1+h}\downarrow & & \downarrow{\scriptstyle h} \\ 1 + S & \xrightarrow[f]{} & S \end{array}$$

commute. Applied to the element $* \in 1$, this means that $h(0) = f(*)$ and applied to $h(n)$ that $f(h(n)) = h(\mathrm{succ}(n))$. This shows the uniqueness of h and since this is the only condition to be satisfied, it shows that h works.

2. The category associated with S has the elements of S as objects and pairs (x, y) as the unique arrow $y \longrightarrow x$ when $y \le x$. Then f takes elements to elements, that is objects to objects and we define f on arrows by $f(x, y) = (f(x), f(y))$ which is well defined since $y \le x$ implies that $f(y) \le f(x)$. Since (x, x) is the identity and $f(x, x) = (f(x), f(x))$, we see that f preserves identities. Since $(x, y) \circ (y, z) = (x, z)$ is the composite, it is immediate that f preserves composition as well.

Now an object of $(f : S)$ is an x together with an arrow $f(x) \longrightarrow x$. Such an arrow exists if and only if $f(x) \le x$. So the objects of $(f : S)$ can be identified as those elements. If also $f(y) \le y$, then $y \le x$ implies that $f(y) \le f(x)$ and that

$$\begin{array}{ccc} f(y) & \longrightarrow & y \\ \downarrow & & \downarrow \\ f(x) & \longrightarrow & x \end{array}$$

commutes. Thus $(f : S)$ can be identified as the set of all x with $f(x) \le x$ with the restricted order. If x_0 is initial in $(f : S)$, then $f(y) \le y$ implies that $x_0 \le y$. But then $f(x_0) \le x_0$ implies $f(f(x_0)) \le f(x_0)$ so that $y = f(x_0)$ is such an element and so $x_0 \le f(x_0)$. Thus x_0 is fixed and it is clearly the least fixed point.

3. a. Since $S \neq \emptyset$, let s_0 be an arbitrary element of S. Define

$$g(t) = \begin{cases} s & \text{if } f(s) = t \\ s_0 & \text{otherwise} \end{cases}$$

b. Choose $g : T \to S$ as above. From $F(g) \circ F(f) = \mathrm{id}_{F(S)}$ it follows easily that $F(f)$ is monic.

c. If $S = \emptyset$, there is nothing to prove since in that case the inclusions $S_0 \to S$ and $S_1 \to S$ are isomorphisms. Otherwise let $S_2 = S_0 \cup S_1$ unless $S_0 = S_1 = \emptyset$ and in that case let S_2 be any singleton set. Then if $i_2 : S_2 \to S$ is inclusion, $R(i_2)$ is monic. Hence from $i_2 \circ j_0 = i_0$, $i_2 \circ j_1 = i_1$, $Ri_0(x_0) = Ri_1(x_1)$ we see that $Ri_2 \circ Rj_0(x_0) = Ri_2 \circ Rj_1(x_1)$ from which Ri_2 can be canceled to give $Rj_0(x_0) = Rj_1(x_1)$.

Section 10.2

1. In effect we claim that if A^* is the set of lists of elements of A (including the empty list ()), then $\mathrm{rec}(A; B) = A^* \times B$. Let $\mathrm{cons} : A \times A^* \to A^*$ denote the function that adjoins an element of A to the head of a list. Then $r_0(A; B) : B \to A^* \times B$ takes $b \in B$ to $(\langle\rangle; b)$ and $r(A; B) : A \times A^* \to B$ is just $\mathrm{cons} \times \mathrm{id}_B$. Now let $t_0 : B \to X$ and $t : A \times X \to X$ be given. In order that $f : A^* \times B \to X$ make the diagram from 10.2.5 commute, it is necessary that $f(\langle\rangle; b) = t_0(b)$ and that $f(\mathrm{cons}(a; l); b) = t(a; f(l; b))$ for $a \in A$, $l \in A^*$ and $b \in B$. But those conditions define, by induction on the length of a word in A^*, a unique function $f : A^* \times B \to X$ that makes the necessary diagram commute.

2. From the definition of recursive in 10.2.5, we see that when $A = B = 1$, we get $r_0(1; 1) : 1 \to \mathrm{rec}(1; 1)$ and $r(1; 1) : \mathrm{rec}(1; 1) \to \mathrm{rec}(1; 1)$ (taking $\mathrm{rec}(1; 1)$ as the product $1 \times \mathrm{rec}(1; 1)$) and the diagram of that section describes exactly the universal mapping property of a natural numbers object.

Section 10.3

1. Since T must be a functor, it is a monotone function so that for any x and y with $x \leq y$, $T(x) \leq T(y)$. Then the existence of the natural transformations $\eta : \mathrm{id} \to T$ and $\mu : T^2 \to T$ imply that for any x, $x \leq T(x)$ and $T(T(x)) \leq T(x)$. Since $x \leq T(x)$, it follows from these inequalities and monotonicity that $T(x) \leq T(T(x)) \leq T(x)$, so that $T(x) = T(T(x))$.

2. We freely use the Godement rules of Section 4.4 together with the identities satisfied by an adjoint pair given in Section 9.2 to compute

$$\mu \circ T\mu = U\epsilon F \circ UF\eta = U(\epsilon F \circ F\eta) = U(\mathrm{id}_F) = \mathrm{id}_{UF} = \mathrm{id}_T$$

$$\mu \circ \eta T = U\epsilon F \circ \eta UF = (U\epsilon \circ \eta U)F = \mathrm{id}_U F = \mathrm{id}_{UF} = \mathrm{id}_T$$

Now the naturality of ϵ implies that for any $\alpha : R \to S$,

$$\alpha \circ \epsilon R = \epsilon S \circ FU\alpha$$

Applied when $R = FU$, $S = I$ (the identity functor) and $\alpha = \epsilon$, this gives that $\epsilon \circ FU\epsilon = \epsilon \circ \epsilon FU$. We then compute

$$\begin{aligned}\mu \circ \mu T &= U\epsilon F \circ U\epsilon FUF = U(\epsilon \circ \epsilon FU)F = U(\epsilon \circ FU\epsilon)F \\ &= U\epsilon F \circ UFU\epsilon F = \mu \circ T\mu\end{aligned}$$

3. If we can identify the η and μ with those determined by the triple, then the conclusion follows from the preceding exercise. Let us temporarily call the natural transformations determined by the triple $\hat{\eta}$ and $\hat{\mu}$. Then $\hat{\eta}A$ is defined as the unique $A \to A^*$ such that the extension to a monoid homomorphism $F(A) \to F(A)$ is the identity. But $\eta(a) = (a)$ clearly extends to the identity on $F(A)$ since the extension takes $(a_1, \ldots, a_n)$ to $(a_1)\cdots(a_n) = (a_1, \ldots, a_n)$ by definition. Hence $\hat{\eta} = \eta$.

As for $\hat{\mu} = U\epsilon F$, that is the function underlying the unique homomorphism from $FUF(A)$ to $F(A)$ that is gotten by extending the identity function from $UF(A)$ to $UF(A)$. This function takes $(a_1, \ldots, a_n)$ to itself and the extension to a homomorphism takes, for example, the two letter string

$$((a_1, \ldots, a_n), (a'_1, \ldots, a'_{n'})) \in FUF(A)$$

to the product (concatenate)

$$(a_1, \ldots, a_n, a'_1, \ldots, a'_{n'}) \in FU(A)$$

Here we see explicitly that μ and $\hat{\mu}$ agree on two letter strings and it is clear they agree on strings of arbitrary length.

Section 10.4

1. Let us write $f : A \Rightarrow B$ for an arrow in the Kleisli category and $f \bullet g$ for the Kleisli composition. Then for $f : A \Rightarrow B$, we have $f \bullet \eta A = \mu B \circ Tf \circ \eta A = \mu B \circ \eta TB \circ f = \mathrm{id}_B \circ f = f$. For $g : C \Rightarrow A$, we have $\eta A \bullet g = \mu A \circ T\eta A \circ g = \mathrm{id}_A \circ g = g$. In this exercise and many of the later ones we use naturality without comment. Here, for example, the fact that $Tf \circ \eta A = \eta TB \circ f$ is a consequence of the naturality of η.

2. Using the same notation as in the previous problem, we have, for $f : A \Rightarrow B$, $g : B \Rightarrow C$ and $h : C \Rightarrow D$ that $(h \bullet g) \bullet f = \mu D \circ T(\mu C \circ Th \circ g) \circ f = \mu D \circ T\mu D \circ T^2h \circ Tg \circ f = \mu D \circ \mu TD \circ T^2h \circ Tg \circ f = \mu D \circ Th \circ \mu C \circ Tg \circ f = h \bullet (g \bullet f)$.

3. a. Let us take the last point first. Given a monoid structure $\cdot$ and 1 on A, define $\alpha : A^* \to A$ by $\alpha() = 1$, $\alpha(a) = a$ and $\alpha(a_1; a_2; \cdots; a_n) = a_1 \cdot a_2 \cdot \cdots \cdot a_n$. Because of the associativity, it is not necessary to parenthesize that expression. Since $\alpha(a) = a$, the identity $\alpha \circ \eta A = \mathrm{id}_A$ is satisfied. As for the identity $\alpha \circ T\alpha = \alpha \circ \mu A$, let us do this for a list of length two of A^*; the general case follows by an easy induction. So let $l = (a_1; a_2; \cdots; a_n)$ and $l' = (a'_1; a'_2; \cdots a'_{n'})$. If $n = 0$, we have

$$\begin{aligned} \alpha \circ T\alpha(()l') &= \alpha(1; a'_1 \cdot a'_2 \cdot \cdots \cdot a'_{n'}) \\ &= 1 \cdot a'_1 \cdot a'_2 \cdot \cdots \cdot a'_{n'} \\ &= \alpha(l') = \alpha \circ \mu A(()l') \end{aligned}$$

and similarly if $n' = 0$. For $n > 0$ and $n' > 0$,

$$\begin{aligned} \alpha \circ T\alpha(ll') &= \alpha(a_1 \cdot a_2 \cdot \cdots \cdot a_n; a'_1 \cdot a'_2 \cdot \cdots \cdot a'_{n'}) \\ &= a_1 \cdot a_2 \cdot \cdots \cdot a_n \cdot a'_1 \cdot a'_2 \cdot \cdots \cdot a'_{n'} \\ &= \alpha(ll') = \alpha \circ \mu A(ll') \end{aligned}$$

This shows that each monoid gives an algebra structure. An algebra structure on A assigns to each list $(a_1 a_2 \cdots a_n) \in A^*$ an element $\alpha(a_1 a_2 \cdots a_n) \in A$. In particular, there is a multiplication given by $a \cdot b = \alpha(ab)$. Also let 1 denote $\alpha()$. We must show that $\cdot$ and 1 constitute a monoid structure and that the associated algebra structure is the one we started with. It is already evident that if α is the algebra structure just constructed, then this monoid structure is the original one. We have $\alpha \circ T\alpha(()(a)) = \alpha(1a) = 1 \cdot a$ and $\alpha \circ \mu A(()(a)) = \alpha(a) = a$ so that $1 \cdot a = a$ and similarly $a \cdot 1 = a$. Next $\alpha \circ T\alpha((a); (b; c)) = a \cdot (b \cdot c)$ while $\alpha \circ \mu A((a); (b; c)) = \alpha(abc)$. Similarly, using $((a; b); (c))$ we can show that $(a \cdot b) \cdot c = \alpha(abc)$ and so $\cdot$ is associative. The proof that $a \cdot b \cdot c = \alpha(abc)$ extends by an obvious induction to show that $\alpha(a_1; a_2; \cdots; a_n) = a_1 \cdot a_2 \cdot \cdots \cdot a_n$, which means that the monoid structure determines uniquely the algebra structure.

b. Let $(A; \alpha)$ and $(B; \beta)$ be algebra structures with corresponding monoid structures that we denote $\cdot$. Let $f : A \to B$. We show that f is a homomorphism of monoids if and only if it is a homomorphism of algebra structures. Suppose f is a homomorphism of algebra structures. Then from $\beta \circ Tf() = \beta() = 1$ and $f \circ \alpha() = f(1)$, we conclude that $f(1) = 1$. From $\beta \circ Tf(a; a') = \beta(f(a)f(a')) = f(a) \cdot f(a')$ and $f \circ \alpha(a; a') = f(a \cdot a')$ we see that f is a monoid homomorphism.

If f is a monoid homomorphism, then

$$\beta \circ Tf() = \beta() = 1 = f(1) = f \circ \alpha()$$

and

$$\beta \circ Tf(a) = \beta(f(a)) = f(\alpha(a)) = f \circ \alpha(a)$$

If $l = (a_1; a_2; \cdots; a_n)$, then

$$
\begin{aligned}
\beta \circ Tf(a_1 a_2 \cdots a_l) &= \beta(f(a_1)f(a_2)\cdots f(a_n)) \\
&= f(a_1)\cdot f(a_2)\cdot \cdots \cdot f(a_n) \\
&= f(a_1 \cdot a_2 \cdot \cdots \cdot a_n) \\
&= f \circ \alpha(a_1; a_2; \cdots; a_n)
\end{aligned}
$$

4. The first thing to be checked is that $(TA; \mu A)$ is actually a **T**-algebra. The relevant diagrams that have to be checked to be commutative are

$$
\begin{array}{ccc}
T^3A & \xrightarrow{\mu TA} & T^2A \\
{\scriptstyle T\mu A}\downarrow & & \downarrow{\scriptstyle \mu A} \\
T^2A & \xrightarrow[\mu A]{} & TA
\end{array}
\qquad\qquad
\begin{array}{ccc}
TA & \xrightarrow{\eta TA} & T^2A \\
 & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \mu A} \\
 & & TA
\end{array}
$$

which are two of the three commutative diagrams required for a triple. Next we have to show that for $f : A \to B$, $Ff : FA \to FB$ is an homomorphism in the category of **T**-algebras. To do this we must show that the diagram

$$
\begin{array}{ccc}
T^2A & \xrightarrow{T^2f} & T^2B \\
{\scriptstyle \mu A}\downarrow & & \downarrow{\scriptstyle \mu B} \\
TA & \xrightarrow[Tf]{} & TB
\end{array}
$$

commutes. But the commutation of this diagram for all f is exactly what is meant by the statement that μ is a natural transformation. The fact that F is a functor is left to the end.

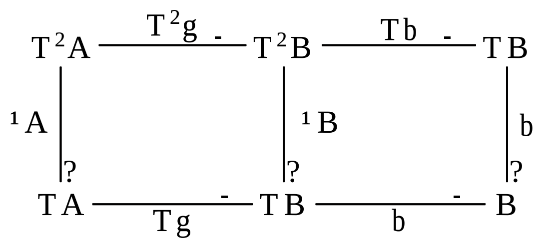
Next we use the result of Theorem 9.3.5 to show that F is left adjoint to U. To do so we must give an isomorphism

$$\mathrm{Hom}(FA; (B; b)) \to \mathrm{Hom}(A; B)$$

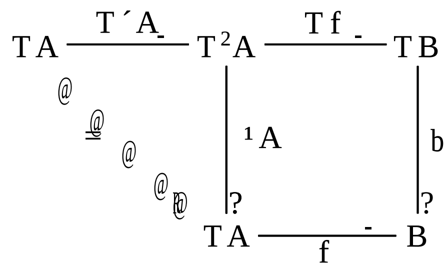
which is natural in B. The function associates to each arrow

$$f : (TA; \mu A) \to (B; b)$$

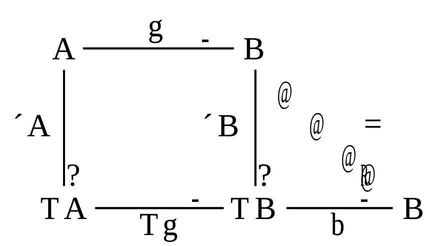
the arrow $f \circ \eta A : A \to B$. The naturality of this function follows from that of η. It must be shown to be an isomorphism. To do this, we define a function in the other direction that sends $g : A \to B$ to $b \circ Tg : TA \to B$. First we claim that that arrow is an arrow of **T**-algebras. To see that, we must show that the diagram



commutes. But the left hand square does by the naturality of μ and the right hand one does because the commutativity of that square is one of the hypotheses on b. Next we observe that if $f : TA \longrightarrow B$ is an algebra homomorphism, then the square in the diagram



commutes and the triangle does by one of the identities that define a triple. Thus the whole square commutes which shows that one of the composites is the identity. As for the other, that follows from the commutativity of the diagram



whose square commutes by naturality of η and triangle by one of the hypotheses on b.

Finally we show that F and U are functors, that $F \dashv U$ and that the triple gotten from the adjoint pair is **T**. For an object A, $U\eta A = \mu A \circ T\eta A = \mathrm{id}_{TA} = \mathrm{id}_{UA}$ so U preserves identities. If $f : A \Rightarrow B$ and $g : B \Rightarrow C$ are arrows, then

$$\begin{aligned} Ug \circ Uf &= \mu C \circ Tg \circ \mu B \circ Tf = \mu C \circ \mu TC \circ T^2 g \circ Tf \\ &= \mu C \circ T\mu C \circ T^2 g \circ Tf = \mu C \circ T(\mu C \circ Tg \circ f) \\ &= \mu C \circ T(g \bullet f) = U(g \bullet f) \end{aligned}$$

so that U preserves composition. Thus U is a functor.

As for F, $F\,\mathrm{id}_A = \eta A \circ \mathrm{id}_A = \eta_A$ so F preserves identities. If $f : A \longrightarrow B$ and $g : B \longrightarrow C$, then

$$\begin{aligned} Fg \bullet Ff &= \mu C \circ TFg \circ Ff = \mu C \circ T(\eta C \circ g) \circ \eta B \circ f \\ &= \mu C \circ T\eta C \circ Tg \circ \eta B \circ f = \eta C \circ g \circ f = F(g \circ f) \end{aligned}$$

and so F is also a functor.

It is evident that $T = UF$. Also, $\eta A : A \to UFA = TA$ is natural and we let $\epsilon A = \mathrm{id}_{TA} : UFA \Rightarrow A$ in $\mathscr{K}(\mathbf{T})$. Then

$$\epsilon FA \bullet F\eta A = \mu A \circ T(\mathrm{id}_{TA}) \circ \eta TA \circ \eta A = \mu A \circ \eta TA \circ \eta A$$

which is the identity of FA. For the other adjointness, we have for $B = FA$ in $\mathscr{K}(\mathbf{T})$,

$$U\epsilon FA \circ \eta UFA = \mu A \circ T(\mathrm{id}_A) \circ \eta TA = \mu A \circ \eta TA = \mathrm{id}_{TA}$$

This completes the proof.

Section 10.5

1. If A and B are ω-CPOs, let $[A \to B]$ denote the set of monotone functions from A to B that preserve joins of countable chains. Make it a poset by the pointwise ordering, that is $f \le g$ if $f(a) \le g(a)$ for all $a \in A$. We claim it is an ω-CPO. In fact, if $f_0 \le f_1 \le \cdots$ is a countable increasing chain of such functions, then for each $a \in A$, let $f(a) = \bigvee f_i(a)$. Let us show that f is countably chain complete (it will obviously be the join in that case). If $a_0 \le a_1 \le \cdots$ is a countable increasing sequence with join a we have a double sequence in which every row and every column except perhaps the bottom row consists of a countable increasing sequence and its join.

$$\begin{array}{ccccc}
f_0(a_0) & f_0(a_1) & f_0(a_2) & \cdots & f_0(a) \\
f_1(a_0) & f_1(a_1) & f_1(a_2) & \cdots & f_1(a) \\
f_2(a_0) & f_2(a_1) & f_2(a_2) & \cdots & f_2(a) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f(a_0) & f(a_1) & f(a_2) & \cdots & f(a)
\end{array}$$

For each i, $f(a_i) = \bigvee_j f_j(a_i) \le \bigvee_j f_j(a) = f(a)$ so $f(a)$ is an upper bound for the bottom row. If $b \in B$ were another bound, we would have for each i, $f_i(a) = \bigvee_j f_i(a_j) \le \bigvee_j f(a_j) \le b$ so that $f(a) \le b$. Thus $f(a) = \bigvee f(a_i)$.

Now we must show that if C is any other ω-CPO,

$$\mathrm{Hom}(C, [A \to B]) \cong \mathrm{Hom}(C \times A, B)$$

and this isomorphism is natural in B. If $f : C \times A \to B$ is a function, let $\phi(f) : C \to [A \to B]$ be the function defined by $\phi(f)(c)(a) = f(c, a)$. So far this is just the cartesian closed structure on sets. Now we claim that f is countably chain complete if and only if for all $c \in C$, $\phi(f)(c)$ is countably chain complete and $\phi(f)$ is countably chain complete.

Suppose f is countably chain complete. This means that if $c = \bigvee c_i$ and $a = \bigvee a_i$ are sups along increasing chains, then $f(c, a) = \bigvee f(c_i, a_i)$. This can be specialized to the case that all $c_i = c$ to conclude that $f(c, a) = \bigvee f(c, a_i)$ so that

$\varphi(f)(c)$ is countably chain complete. Similarly, if $c = \bigvee c_i$ is the sup of a countable increasing chain, then the fact that $f(c; a) = \bigvee f(c_i; a)$ implies that $\varphi(f)(c) = \bigvee \varphi(f)(c_i)$. This shows one direction. By reversing all these implications, one easily shows that if $\varphi(f)(c)$ preserves joins along countable chains and $\varphi(f)$ also preserves them, then f preserves joins in its variables separately. To prove that it preserves them jointly, let $c = \bigvee c_i$ and $a = \bigvee a_i$ and consider the double sequence

$$\begin{array}{ccccc}
f(c_0; a_0) & f(c_0; a_1) & f(c_0; a_2) & \cdots & f(c_0; a) \\
f(c_1; a_0) & f(c_1; a_1) & f(c_1; a_2) & \cdots & f(c_1; a) \\
f(c_2; a_0) & f(c_2; a_1) & f(c_2; a_2) & \cdots & f(c_2; a) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f(c; a_0) & f(c; a_1) & f(c; a_2) & \cdots & f(c; a)
\end{array}$$

in which all rows and all columns are joins of increasing sequences. Clearly, $f(c_i; a_i) \le f(c; a)$ for all i and if b is an upper bound for the $f(c_i; a_i)$, then for all $j \ge i$, $f(c_i; a_j) \le f(c_j; a_j) \le b$, whence $f(c_i; a) = \bigvee_j (c_i; a_j) \le b$. Since this is true for all i, it follows that $f(c; a) \le b$. Hence f preserves joins of countable increasing chains.

2. This is clearly a poset. If $c^0 \le c^1 \le c^2 \le \cdots$ is an increasing chain in $\hat{P}$, let c^n be the chain $c^n_0 \le c^n_1 \le c^n_2 \le \cdots$. Construct a chain d as follows. Let $d_0 = c^0_0$ and having chosen $d_0 \le d_1 \le \cdots \le d_n$ such that $d_i = c^i_{j(i)}$, choose an integer $j(n+1) > j(n)$ so that $c^n_{j(n)} \le c^{n+1}_{j(n+1)}$. This is always possible since $c^n \le c^{n+1}$. Let $d_{n+1} = c^{n+1}_{j(n+1)}$. An obvious induction shows that $j(n) \ge n$ and so $c^i_n \le d_n$ for $n \ge i$. For a fixed i, the finite set of $n \le i$ does not matter and so $c^i \le d$ for all i. Thus d is an upper bound on the c^i. Suppose $e = e_0 \le e_1 \le e_2 \le \cdots$ is an upper bound on the c^i. Then since $d_n = c^n_{j(n)}$, and $c_n \le e$, it follows that d_n is less than or equal to some term of e, so that $d \le e$, whence $d = e$. Thus $\hat{P}$ is an ω-CPO. Now suppose that A is an ω-CPO and $f : P \to A$ is an order-preserving map. Define $\hat{f} : \hat{P} \to A$ by $\hat{f}(c) = \bigvee f(c_i)$ for $c = c_0 \le c_1 \le c_2 \le \cdots$. It is easy to see that this preserves the order and therefore the equality. Since the join of a constant sequence is itself, this extends f. Finally, if d is the join of the c^i as constructed above, it easily follows from the fact that the terms of d are a selection of those of the c^i that $\hat{f}(d) = \bigvee \hat{f}(c^i)$.

3. Let $\mathscr{C}$ be the category of ω-CPOs and functions that preserve joins along countable increasing chains and let $D : \mathscr{I} \to \mathscr{C}$ be a diagram. Let A be the set that is the limit of the diagram in sets. Then A comes equipped with projections $p_I : A \to DI$ for I an object of $\mathscr{I}$. Say that $a \le a'$ in A if for every object I of $\mathscr{I}$, $p_I(a) \le p_I(a')$. Then A is a poset and the projections preserve order. If $a_0 \le a_1 \le a_2 \le \cdots$ is an increasing chain in A, then for each I, $p_I(a_0) \le p_I(a_1) \le p_I(a_2) \le \cdots$ is an countable increasing chain in DI and has a join a_I. If $\alpha : I$

$\to I'$ is an arrow in $\mathscr{I}$, then

$$D(\alpha)(a_I) = D(\alpha)\left(\bigvee_n (p_I(a_n))\right) = \bigvee_n D(\alpha)(p_I(a_n)) = \bigvee_n p_{I'}(a_n) = a_{I'}$$

so there is a unique $a \in A$ such that $p_I(a) = a_I$. Evidently $a_n \le a$ for each integer n. Also if $a' \in A$ is another upper bound for the $\{a_n\}$ then for each I, $p_I(a_n) \le p_I(a')$ so that $a_I = p_I(a) \le p_I(a')$ and then $a \le a'$. This shows that A is an ω-CPO and it is clear that the projections preserve the joins along countable chains. If B is any poset and $\{f_I\} : B \to D$ is a natural transformation from the constant diagram to D, let $f : B \to A$ be the induced function. Let $b_0 \le b_1 \le b_2 \le \cdots$ be an increasing chain in B with join b. Then for each I, $f_I(b) = \bigvee_n f_I(b_n)$ so that $p_I(f(b)) = \bigvee_n p_I(f(b_n))$. An argument similar to the above shows that this implies $f(b) = \bigvee_n f(b_n)$. The uniqueness of f follows because of the uniqueness of functions into a limit. This completes the proof for limits.

Assuming that the category of posets has colimits, one way to get colimits in the category of ω-CPOs is to form the colimit in the category of posets and apply the completion process of the preceding problem. If A is the colimit and $\hat{A}$ its completion then any map from A to an ω-CPO extends to a unique map on $\hat{A}$ that preserves joins along ω-chains. Putting the two universal mapping conditions together gives the result.

4. Since a chain is a directed set, one direction is immediate. For the other direction, let P be an ω-CPO and $D \subseteq P$ be a countable directed set in P. Since D is countable, we can name the elements of D as $d_0, d_1, d_2, \ldots$. Let $c_0 = d_0$ and having chosen $c_0 \le c_1 \le c_2 \le \cdots \le c_n$ an increasing sequence of elements of D, let $c_{n+1} \in D$ be a common upper bound of d_{n+1} and c_n. A join of the $\{c_n\}$ is clearly an upper bound of D.

Solutions for Chapter 11

Section 11.2

1. Suppose $f : A' \to A$ is given. In the diagram below, the upper and lower trapezoids commute by the naturality of the isomorphism, the left and right hand trapezoids commute from the definitions of $B \multimap (f \multimap C)$ and $(f \otimes B) \multimap C$, resp. and the inner square commutes by definition of $f \multimap C$. Therefore the outer

square commutes.

$$\begin{array}{ccccccc}
\mathrm{Hom}(D, B \multimap (A \multimap C)) & & \longrightarrow & & \mathrm{Hom}(D, (A \otimes B) \multimap C) \\
\downarrow & \searrow \cong & & \cong \nearrow & \downarrow \\
& \mathrm{Hom}(B \otimes D, A \multimap C) & \longrightarrow & \mathrm{Hom}(A \otimes B \otimes D, C) & \\
& \downarrow & & \downarrow & \\
& \mathrm{Hom}(B \otimes D, A' \multimap C) & \longrightarrow & \mathrm{Hom}(A' \otimes B \otimes D, C) & \\
\downarrow & \cong \swarrow & & \nwarrow \cong & \downarrow \\
\mathrm{Hom}(D, B \multimap (A' \multimap C)) & & \longrightarrow & & \mathrm{Hom}(D, (A' \otimes B) \multimap C)
\end{array}$$

According to the Yoneda Lemma, it follows that

$$\begin{array}{ccc}
B \multimap (A \multimap C) & \xrightarrow{\ \cong\ } & (A \otimes B) \multimap C \\
{\scriptstyle B \multimap (f \multimap C)} \downarrow & & \downarrow {\scriptstyle (f \otimes B) \multimap C} \\
B \multimap (A' \multimap C) & \xrightarrow[\ \cong\ ]{} & (A' \otimes B) \multimap C
\end{array}$$

commutes. This is just the internalized version (that is, with Hom sets replaced by $\multimap$) of the main diagram used to prove the naturality in the preceding exercise. By replacing Hom sets everywhere in that argument with these 'internal hom objects', the above proof can be adapted to give the solution to this exercise.

2. We begin by showing that M is the unit. We must give, for each M-action S an isomorphism $l = l_S : M \otimes_M S \to S$. Since $S \otimes M$ is defined only up to isomorphism as a coequalizer, one way of doing this is to show how S itself can be made into the relevant coequalizer, that is that there is a coequalizer diagram of the form

$$M \times M \times S \underset{d^1}{\overset{d^0}{\rightrightarrows}} M \times S \xrightarrow{\ d\ } S$$

where $d^0(m, n, s) = (mn, s)$ and $d^1(m, n, s) = (m, ns)$. Define $d(m, s) = ms$. The fact that $d \circ d^0 = d \circ d^1$ is just condition A–2 of 3.3.2.1. Now suppose that $f : M \times S \to T$ is an equivariant map of M-actions such that $f \circ d^0 = f \circ d^1$. Define $g : S \to T$ by $g(s) = f(1, s)$. Then $(g \circ d)(m, s) = g(ms) = f(1, ms) = (f \circ d^1)(1, m, s) = (f \circ d^0)(1, m, s) = f(m, s)$ so that $f = g \circ d$. If $h : S \to T$ is another map such that $h \circ d = f$, then for all $s \in S$, $h(s) = (h \circ d)(1, s) =$

$f(1;s) = g(s)$. This shows that S is the coequalizer, so that $M \otimes S$ can be taken to be S.

To show associativity, suppose that R, S and T are M-actions. Define $a = a(R;S;T) : R \otimes (S \otimes T) \longrightarrow (R \otimes S) \otimes T$ by $a(r \otimes (s \otimes t)) = (r \otimes s) \otimes t$. We have that

$$\begin{aligned} a(mr \otimes (s \otimes t)) &= (mr \otimes s) \otimes t = (r \otimes ms) \otimes t \\ &= a(r \otimes (ms \otimes t)) = a(r \otimes m(s \otimes t)) \end{aligned}$$

so that a is well defined on the tensor product. It is just as easy to show it is M-equivariant and the similarly constructed inverse map shows it is an isomorphism. We similarly define $c = c(S;T) : S \otimes T \longrightarrow T \otimes S$ by $c(s \otimes t) = t \otimes s$ which can be similarly shown to be well-defined and equivariant.

Now suppose $f : R \otimes S \longrightarrow T$ is an equivariant map. Define $\varphi(f) : S \longrightarrow R \multimap T$ by $\varphi(f)(s) \in S \multimap T$ is the map for which $\varphi(f)(s)(r) = f(r;s)$. We have to show for each f and s, that $\varphi(f)(s)$ is equivariant; for each f, that $\varphi(f)$ is equivariant and, finally, that φ is equivariant. Each of the three is a different assertion. For the first, we have that

$$\varphi(f)(s)(mr) = f(mr \otimes s) = f(m(r \otimes s)) = mf(r;s) = m\varphi(f)(s)(r)$$

The second follows from

$$\begin{aligned} \varphi(f)(ms)(r) &= f(r \otimes ms) = f(m(r \otimes s)) = mf(r \otimes s) \\ &= m\varphi(f)(s)(r) = (m\varphi(f)(s))(r) \end{aligned}$$

The second computation looks quite similar to the first, but really is different since it depends on the fact that the action of M on a map $g : R \longrightarrow T$ is by $(mg)(r) = mg(r)$. As for the third, we have that

$$\begin{aligned} \varphi(mf)(s)(r) &= (mf)(r;s) = mf(r;s) = m\varphi(f)(s)(r) \\ &= (m\varphi(f)(s))(r) = (m\varphi(f))(s)(r) \end{aligned}$$

so that $\varphi(mf) = m\varphi(f)$ where we have twice used the definition of the action of M on a homomorphism. To go the other way, given an equivariant map $g : S \longrightarrow R \multimap T$, define $\psi(g) : R \otimes S \longrightarrow T$ by $\psi(g)(r;s) = g(s)(r)$. The crucial fact, that $\psi(g)(mr;s) = \psi(g)(r;ms)$, is shown by

$$\begin{aligned} \psi(g)(mr;s) &= g(s)(mr) = m(g(s)(r)) \\ &= (mg(s))(r) = g(ms)(r) = \psi(g)(r;ms) \end{aligned}$$

The rest, such as that ψ is equivariant, is similar. It is easily seen that $\psi = \varphi^{-1}$.

3. a. If F is a functor, then let $F(A,-)$ be defined on arrows $g : B \to B'$ by $F(A,g) = F(\mathrm{id}_A, g)$ and $F(-,B)$ similarly defined on arrows $f : A \to A'$ by $F(f,B) = F(f,\mathrm{id}_B)$. The commutation follows from the equations $(f,g) = (f,\mathrm{id}_{B'}) \circ (\mathrm{id}_A, g) = (\mathrm{id}_{A'}, g) \circ (f, \mathrm{id}_B)$.

Conversely, if we have the condition satisfied, define

$$F(f,g) = F(A,g) \circ F(f,B) = F(f,b) \circ F(A,g)$$

Since $\mathrm{id}_{(A,B)} = (\mathrm{id}_A, \mathrm{id}_B)$, we have

$$\begin{aligned} F(\mathrm{id}_{(A,B)}) &= F(\mathrm{id}_A, \mathrm{id}_B) = F(\mathrm{id}_A, B) \circ F(A, \mathrm{id}_B) \\ &= \mathrm{id}_{F(A,B)} \circ \mathrm{id}_{F(A,B)} = \mathrm{id}_{F(A,B)} \end{aligned}$$

If $(f,g) : (A,B) \to (C,D)$ and $(f',g') : (A',B') \to (A'',B'')$ then

$$\begin{aligned} F(f',g') \circ F(f,g) &= F(f',B'') \circ F(A',g') \circ F(A',g) \circ F(f,B) \\ &= F(f',B'') \circ F(A',g' \circ g) \circ F(f,B) \\ &= F(f',B'') \circ F(f,B'') \circ F(A,g \circ g') \\ &= F(f' \circ f, B'') \circ F(A, g \circ g') = F(f' \circ f, g' \circ g) \end{aligned}$$

b. The definition of naturality of $F(f,-)$ is that for all $g : B \to B'$, the square

$$\begin{array}{ccc} F(A,B) & \xrightarrow{F(A,g)} & F(A,B') \\ {\scriptstyle F(f,B)}\downarrow & & \downarrow{\scriptstyle F(f,B')} \\ F(A',B) & \xrightarrow[F(A',g)]{} & F(A',B') \end{array}$$

commutes, which is exactly the third condition and also exactly the naturality of $F(-,g)$.

c. Naturality of α means that for each arrow $(f,g) : (A,B) \to (A',B')$ the diagram

$$\begin{array}{ccc} F(A,B) & \xrightarrow{F(f,g)} & F(A',B') \\ {\scriptstyle \alpha(A,B)}\downarrow & & \downarrow{\scriptstyle \alpha(A',B')} \\ G(A,B) & \xrightarrow[G(f,g)]{} & G(A',B') \end{array}$$

Specializing to the case that $f : A \to A$ is the identity, we see that if α is natural, then so is $\alpha(A,-)$ and similarly for $\alpha(-,B)$. Conversely, suppose that all $\alpha(A,-)$

and $\alpha(-,B)$ are natural. First we observe that $(f,g) = (\mathrm{id},g) \circ (f,\mathrm{id})$ (this is the crucial observation, actually) and hence $F(f,g) = F(\mathrm{id},g) \circ F(f,\mathrm{id})$ and analogously for G. This gives us the following commutative diagram

$$\begin{array}{ccccc}
 & & \overset{F(f,g)}{\frown} & & \\
F(A,B) & \xrightarrow{F(f,B)} & F(A',B) & \xrightarrow{F(A,g)} & F(A',B') \\
\Big\downarrow{\scriptstyle \alpha(A,B)} & & \Big\downarrow{\scriptstyle \alpha(A',B)} & & \Big\downarrow{\scriptstyle \alpha(A',B')} \\
G(A,B) & \xrightarrow[G(f,B)]{} & G(A',B) & \xrightarrow[G(A',g)]{} & G(A',B') \\
 & & \underset{G(f,g)}{\smile} & &
\end{array}$$

Since the two squares and the two triangles commute, so does the outer diagram, which is just the naturality of α.

d. Suppose we have $f : A' \to A$. Since the bottom map in the diagram below is an isomorphism, there is, for each B and C a unique arrow $\alpha(B,C) : \mathrm{Hom}(B, A \multimap C) \to \mathrm{Hom}(B, A' \multimap C)$ such that

$$\begin{array}{ccc}
\mathrm{Hom}(B, A \multimap C & \xrightarrow{\cong} & \mathrm{Hom}(A \otimes B, C) \\
\Big\downarrow{\scriptstyle \alpha(B,C)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}(f \otimes B, C)} \\
\mathrm{Hom}(B, A' \multimap C) & \xrightarrow[\cong]{} & \mathrm{Hom}(A' \otimes B, C)
\end{array}$$

commutes. I claim that this is natural in B. In fact, if $B' \to B$ is a map, we have a diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}(B, A \multimap C) & & \longrightarrow & & \mathrm{Hom}(B', A \multimap C) \\
\Big\downarrow & \searrow{\scriptstyle \cong} & & {\scriptstyle \cong}\swarrow & \Big\downarrow \\
 & \mathrm{Hom}(A \otimes B, C) & \longrightarrow & \mathrm{Hom}(A \otimes B', C) & \\
 & \Big\downarrow & & \Big\downarrow & \\
 & \mathrm{Hom}(A' \otimes B, C) & \longrightarrow & \mathrm{Hom}(A' \otimes B', C) & \\
\Big\downarrow & \nearrow{\scriptstyle \cong} & & {\scriptstyle \cong}\nwarrow & \Big\downarrow \\
\mathrm{Hom}(B, A' \multimap C) & & \longrightarrow & & \mathrm{Hom}(B', A' \multimap C)
\end{array}$$

The left and right hand trapezoids commute by definition of α; the top and bottom ones because of the naturality of isomorphisms in the adjunction. The middle square commutes because $\otimes$ is a functor. It follows from a simple diagram chase that the outer square commutes, which is just naturality of $\alpha(B, C)$ with respect to B. Thus we have, for each C, a natural transformation $\mathrm{Hom}(-, A \multimap C) \to \mathrm{Hom}(-, A' \otimes C)$. The Yoneda lemma implies that any such natural transformation is induced by a unique map we call $f \multimap C : A \multimap C \to A' \multimap C$. If f is the identity, then the identity map of $A \multimap C$ has all the properties required of $f \multimap C$. It then follows from uniqueness that $\mathrm{id} \multimap C$ is the identity. If $f' : A'' \to A'$ is a morphism, then one easily sees that $(f \multimap C) \circ (f' \multimap C)$ has all the properties required of $(f \circ f') \multimap C$ and the uniqueness implies they are equal. Thus the object function $A \mapsto A \multimap C$ extends to a functor.

If $h : C \to C'$ is given, then the same argument with appropriate changes gives maps $A \multimap h : A \multimap C \to A \multimap C'$ for each A. For example, for each A and B we have a unique function $\gamma(A, B) : \mathrm{Hom}(B, A \multimap C) \to \mathrm{Hom}(B, A \multimap C')$ for which

$$\begin{array}{ccc}
\mathrm{Hom}(B, A \multimap C & \xrightarrow{\cong} & \mathrm{Hom}(A \otimes B, C) \\
{\scriptstyle \gamma(A,B)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(A \otimes B, h)} \\
\mathrm{Hom}(B, A \multimap C') & \xrightarrow[\cong]{} & \mathrm{Hom}(A \otimes B, C')
\end{array}$$

commutes. This is natural in B and thus there is a unique map $A \multimap h : A \multimap C \to A \multimap C'$ as above. To complete the argument, we must show that for $f : A' \to A$ and $h : C \to C'$, $(A' \multimap h) \circ (f \circ C) = (f \circ C') \circ (A \multimap h)$. This follows from the commutativity of

$$\begin{array}{ccccccc}
\mathrm{Hom}(B, A \multimap C) & & \longrightarrow & & \mathrm{Hom}(B, A' \multimap C) \\
 & \searrow^{\cong} & & \swarrow_{\cong} & \\
\big\downarrow & \mathrm{Hom}(A \otimes B, C) & \longrightarrow & \mathrm{Hom}(A' \otimes B, C) & \big\downarrow \\
 & \big\downarrow & & \big\downarrow & \\
 & \mathrm{Hom}(A \otimes B, C') & \longrightarrow & \mathrm{Hom}(A' \otimes B, C') & \\
 & \nearrow^{\cong} & & \nwarrow^{\cong} & \\
\mathrm{Hom}(B, A \multimap C') & & \longrightarrow & & \mathrm{Hom}(B, A' \multimap C')
\end{array}$$

The four trapezoids commute from the definitions of $f \multimap -$ and $- \multimap h$, while the inner square does because $\otimes$ is a functor and hence the outer square does too. One easily sees that for three variables it is su±cient that all the restrictions to two variables be functorial, since all the necessary commutations can be performed by permuting two variables at a time.

Section 11.3

1. a. Since there is are only ¯nitely many elements, there is a sup of all them, clearly the greatest element. Given any two elements x and y the set of elements z such that z · x and z · y is ¯nite and therefore has a sup, which is evidently the inf of x and y.

b. This is actually an instance of the fact that right adjoints preserve limits, since if L and M are treated as categories, f and g are functors and f is left adjoint to g, since from the de¯nition of g, x · g(y) if and only if f(x) · y. In particular, f(x) · y and f(x) · y^0 if and only if f(x) · y ^ y^0. Thus x · g(y) and x · g(y^0) if an only if x · g(y ^ y^0) from which it follows that x · g(y) ^ g(y^0) if and only if x · g(y ^ y^0) which implies that g(y ^ y^0) = g(y) ^ g(y^0).

c. The previous part shows that when f : L ¡! M is sup preserving, then so is $f^{¤}$ = g : $M^{¤}$ ¡! $L^{¤}$. The uniqueness of adjoints implies that $f^{¤¤}$ = f since $f^{¤¤}$ is another left adjoint to f (note that although g : M ¡! L is a right adjoint to f, g : M^{op} ¡! L^{op} is left adjoint.) Thus ¤ looks like a duality. We must also show that L ¡±M ≌ $M^{¤}$ ¡±$L^{¤}$. But in fact the correspondence f 7 $f^{¤}$ is a one-one correspondence between the two hom sets. If f · g : L ¡! M are sup preserving, then I claim that $f^{¤}$ · $g^{¤}$: $M^{¤}$ ¡! $L^{¤}$, which is equivalent to the statement that $g^{¤}$ · $f^{¤}$: M ¡! L. In fact, this is a general fact about adjoints. If F, G : $\mathscr{A}$ ¡! $\mathscr{B}$ are functors with right adjoints $F^{¤}$, $G^{¤}$, resp., then any natural transformation ® : F ¡! G induces a natural transformation $G^{¤}$ ¡! $F^{¤}$ by the composite $G^{¤}$ ¡! $G^{¤}FF^{¤}$ ¡! $G^{¤}GF^{¤}$ ¡! $F^{¤}$, where the ¯rst and third map are induced by the adjunctions and the middle one by ®. Between posets, there is one natural transformation f ¡! g if and only if f · g and none otherwise. Thus f · g if and only if $g^{¤}$ · $f^{¤}$, just what is required for this to be a duality in the closed monoidal category.