The volume of separable states is super-doubly-exponentially small

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Abstract

In this note we give sharp estimates on the volume of the set of separable states on N qubits. In particular, the magnitude of the "effective radius" of that set in the sense of volume is determined up to a factor which is a (small) power of N, and thus precisely on the scale of powers of its dimension. Similar but less precise results are obtained for qudits. Additionally, one of the appendices contains new sharp estimates (by known methods) for the expected values of norms of the GUE random matrices. We employ standard tools of classical convexity, high-dimensional probability and geometry of Banach spaces.

Let $\mathcal{H} = \mathcal{H}_N := (\mathbb{C}^2)^{\otimes N}$ be the N-fold tensor power of $\mathbb{C}^2$ and denote by $d = 2^N$ its dimension. In this note we investigate the structure of the set $\mathcal{D} = \mathcal{D}_N = \mathcal{D}(\mathcal{H}_N)$ of states on $\mathcal{B}(\mathcal{H}_N)$ and, in particular, of its subset $\mathcal{S} = \mathcal{S}_N$ consisting of (mixtures of) *separable* states. We recall that when $\mathcal{D}$ is identified with the set of density matrices $\{\rho \in \mathcal{B}(\mathcal{H}) : \rho$ is positive semi-definite and $\mathrm{tr}\rho = 1\}$, then

$$\mathcal{S} = \mathrm{conv}\{\rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_N : \rho_j \in \mathcal{D}(\mathbb{C}^2), j = 1, 2, \ldots, N\}.$$

[Above and in what follows, we skip the subscript N whenever its value is clear from the context.]

The question of the relative size of $\mathcal{S}$ as a subset of $\mathcal{D}$ was first considered in [23] and further investigated, among others, in [3], [22] and [5]. In particular, it was shown in [3] that $\mathcal{S}$ contains homothetic images of $\mathcal{D}$, more precisely

$$\epsilon\mathcal{D} + (1 - \epsilon)I_d/d \subset \mathcal{S}, \tag{1}$$

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provided $\epsilon \leq 2/(d^2+2)$ (I_d stands here for the identity matrix in d dimensions; in the present context I_d/d is referred to as "the maximally mixed state") and that the above inclusion does not hold if $\epsilon \geq 2/(\sqrt{d}+1)$. Very recently, analogous inclusions with $\mathcal{D}$ replaced by the Euclidean ball were studied (and somewhat more precise estimates obtained) in [5]. However, all these results left open the question of the precise asymptotic order of the "in radii" of $\mathcal{S}$ on the power scale in $\dim \mathcal{S} = \dim \mathcal{D} = d^2 - 1$ and the issue of the "size" of $\mathcal{S}$ when measured by global invariants such as volume. In the latter direction we obtain here the following bounds

$$ \frac{c}{d^{1/2+\alpha}} \ \leq \ \left(\frac{\operatorname{vol} \mathcal{S}}{\operatorname{vol} \mathcal{D}} \right)^{1/\dim \mathcal{S}} \ \leq \ \frac{C(\log d \log\log d)^{1/2}}{d^{1/2+\alpha}} \ , \qquad (2) $$

where $c, C > 0$ are universal effectively computable numerical constants and $\alpha = \log_2 (27/16)/8 \approx 0.094361$ (in other words, $d^{1/2+\alpha} = 3^{3N/8}$). The "effective radius" of $\mathcal{S}$ in the sense of volume is thus precisely determined on the scale of powers of d. Since complexity of a set can be estimated using volumetric methods (see [11] for a modern exposition of this circle of ideas), this goes a long way towards the ability to compare complexities of $\mathcal{S}$ and of $\mathcal{D}$ – even though the so-called Bures metric and the related volume may be more appropriate measures of size in the present context (see [2]). In what follows we shall present the main line of the argument leading to (2), relegating to Appendices the discussion of some peripheral issues as well as the description of results and concepts from convexity and geometry of Banach spaces that are being used. We refer to [3], [8] and [5] for a more professional exposition of the relevance of separability to quantum computation in general and to NMR computing in particular.

For comparison, we note that [3] contained, in addition to the results mentioned above, a hint at an argument which implies, by known methods, a *lower* estimate on $(\operatorname{vol} \mathcal{S}/\operatorname{vol} \mathcal{D})^{1/\dim \mathcal{S}}$ which is of order $d^{-\beta}$, where $\beta = \log 10/\log 4 \approx 1.660964$. An improvement to $\beta = 1$ follows similarly from [5]. [The latter results are more closely related to our estimates (9) and so we comment on them again in that context and later in Appendix H.] By contrast, no non-trivial *upper* estimate on the volume of $\mathcal{S}$ was apparently available. In the other direction, the expression on the right hand side of (2) provides an upper estimate on the ϵ that may work in (1) which is stronger, at least for sufficiently large N, than the $2/(\sqrt{d}+1)$ mentioned above. However, our method does not give – at least without any additional work –

any explicit state that constitutes an obstruction to the inclusion in (1) for $\epsilon < 2/(\sqrt{d}+1)$. Since it is conceivable that the inequalities (2) may be of interest not just asymptotically, but also for some specific "moderately large" values of N, we put some effort into obtaining reasonable (but certainly not optimal) values of the numerical constants. Our main argument gives $c = 1/4$ and shows that (2) holds with $4(N\log_2(4N))^{1/2} = 4(\log_2 d \, \log_2(4\log_2 d))^{1/2}$ in the numerator of its third member. A slightly more precise (and more tedious) calculation yields $c = \sqrt{e/8\pi} \approx 0.32887$; see the comments following (10) and Appendix E. The smallest N for which the upper bound $4(N\log_2(4N))^{1/2}/d^{1/2+\alpha}$ is nontrivial (i.e., < 1) is 8. It is easy to follow the argument and to obtain somewhat sharper estimates for specific values of N, which may be of interest, e.g., in the context of a threshold of 23 mentioned in [5]. It is also possible to show a better asymptotic (as $N \to \infty$) behavior of the constants than the bounds given above; for example one may have $c = c_N \to e^{3/4}/\sqrt{2\pi} \approx 0.84456$ and $C = C_N \to e^{1/4}\sqrt{2/\log 2} \approx 2.1811$. We comment more on these issues in Appendices E and G. Likewise, our methods allow analyzing separable states on tensor products involving spaces $\mathbb{C}^k$ with $k > 2$, leading to non-trivial but not definitive results; some remarks to that effect are presented in Appendix I.

Instead of working directly with $\mathcal{D}$ and $\mathcal{S}$, we shall consider their respective symmetrizations

$$\Delta := \operatorname{conv}(\mathcal{D} \cup (-\mathcal{D})), \quad \Sigma := \operatorname{conv}(\mathcal{S} \cup (-\mathcal{S})), \tag{3}$$

where all sets are thought of as being contained in the real d^2-dimensional vector space of self-adjoint elements of $\mathcal{B}(\mathcal{H})$ (further identifiable with $\mathcal{M}_d^{sa}$, the space of $d \times d$ complex Hermitian matrices). We do that because, firstly, the geometry of *symmetric* convex sets is much better understood than that of the general ones and, secondly, the specific symmetric sets Δ and Σ are familiar objects in geometry of Banach spaces, which allows us to refer to known concepts and results. In Appendix D we indicate how one can treat directly $\mathcal{D}$ and $\mathcal{S}$ without passing to symmetrizations; however, this yields only a very small improvement in the estimates in (2) at the price of obscuring somewhat the argument.

It is readily verified that Δ consists exactly of those (Hermitian) elements of $\mathcal{B}(\mathcal{H})$ whose *trace class norm* is ≤ 1. Equivalently, Δ is the unit ball of the space $\mathcal{C}_1^d := (\mathcal{M}_d^{sa}, \|\cdot\|_1)$, where, for $p \in [1,\infty)$, $\|A\|_p := (\operatorname{tr}(A^*A)^{p/2})^{1/p}$ is the Schatten-von Neumann p-norm . A similar argument shows that Σ is the

unit ball of the Nth projective tensor power of $\mathcal{C}_1^2$ (in the sense of the Banach space theory, see Appendix B). We shall denote the corresponding norm on $\mathcal{M}_d^{sa}$ by $\|\cdot\|_\pi$. For future reference, we note that in the above notation $\|\cdot\|_\infty$ corresponds to $\|\cdot\|_{op}$, the usual norm of a matrix as an operator on the Euclidean space. We also point out that while in this note we focus on spaces of Hermitian matrices and operators, the Schatten-von Neumann classes $\mathcal{C}_p^d$ are most often defined in the literature to include all scalar matrices and not just the self-adjoint ones.

The plan of the rest of the argument is as follows. First, using classical general results from convexity, we relate the volumes of Δ and Σ to those of $\mathcal{D}$ and $\mathcal{S}$. Next, we obtain two-sided estimates for volΔ and volΣ, which are most conveniently described using the following concept: if K is a subset of an n-dimensional Euclidean space with the unit ball B, we call $(\text{vol}\,K/\text{vol}\,B)^{1/n}$ the *volume radius* of K. [In the present context the Euclidean structure is determined by the 2-norm defined above, also often called the Hilbert-Schmidt norm or the Frobenius norm, and the inner product is $\langle u, v \rangle = \text{tr}\,uv$.] Our arguments will determine the volume radius of Σ up to a factor which is a power of $\log d$, in particular precisely on the scale of powers of d; this is the principal result of this note. The corresponding problem for Δ, the unit ball in the trace class norm, is much better understood. Indeed, two-sided estimates for the volume radius of Δ involving a rather large (but universal, i.e., independent of N) constant follow from an early paper [19]. Moreover, explicit formulae for the volume of $\mathcal{D}$ involving multiple integrals can be produced; see [14] for an analysis of a closely related problem, which can be routinely modified to yield similar expressions for $\mathcal{D}$. After a preliminary version of this note has been circulated, the author has learned that this circle of ideas has led to a closed formula for the volume of $\mathcal{D}$ in a very recent work [24]; see Appendix E for more details and [18] for related results concerning the Bures volume. Finally, the unified argument for estimating the volume radii that is presented in this note allows to deduce (from known facts) the value of the volume radius of $\mathcal{D}$ up to a factor of 4.

For the first point, i.e., comparing the volumes of convex sets and their symmetrizations, we use a 1958 result of Rogers and Shephard (see Appendix C for more details and background) to deduce that

$$\frac{2}{\sqrt{d}}\,\text{vol}\mathcal{D} \leq \text{vol}\Delta \leq \frac{2}{\sqrt{d}}\,\frac{2^n}{n+1}\text{vol}\mathcal{D}, \tag{4}$$

where $n = \dim \mathcal{D} = d^2 - 1$. [The factor $2/\sqrt{d}$ appears because it is the distance between the hyperplanes containing $\mathcal{D}$ and $-\mathcal{D}$; note that strictly speaking we should be writing $\mathrm{vol}_n \mathcal{D}$ and $\mathrm{vol}_{n+1} \Delta$ to refer to n and $n+1$-dimensional volume respectively.] Similarly

$$\frac{2}{\sqrt{d}} \, \mathrm{vol}\, \mathcal{S} \leq \mathrm{vol}\, \Sigma \leq \frac{2}{\sqrt{d}} \, \frac{2^n}{n+1} \mathrm{vol}\, \mathcal{S}. \tag{5}$$

Combining (4) and (5) we obtain

$$\left(\frac{2^n}{n+1} \right)^{-1} \frac{\mathrm{vol}\, \Sigma}{\mathrm{vol}\, \Delta} \; \leq \; \frac{\mathrm{vol}\, \mathcal{S}}{\mathrm{vol}\, \mathcal{D}} \; \leq \; \frac{2^n}{n+1} \frac{\mathrm{vol}\, \Sigma}{\mathrm{vol}\, \Delta}. \tag{6}$$

Given that the proper homogeneity is achieved by raising the volume ratios to the power $1/n$, we see that one may replace $\mathcal{D}$ and $\mathcal{S}$ in (2) by Δ and Σ with the accuracy of the estimates affected at most by a factor of 2.

It remains to estimate $\mathrm{vol}\, \Delta$ and $\mathrm{vol}\, \Sigma$; this will be accomplished by comparing each of these bodies with the d^2-dimensional Euclidean ball B_{HS} (the unit ball with respect to the Hilbert-Schmidt norm; we shall also denote by S_{HS} the corresponding $d^2 - 1$-dimensional sphere).

Concerning Δ, we claim that its volume radius satisfies

$$1/\sqrt{d} \leq (\mathrm{vol}\Delta / \mathrm{vol}B_{HS})^{1/d^2} \leq 2/\sqrt{d} \tag{7}$$

To show this, we note first the "trivial" inclusions $B_{HS}/\sqrt{d} \subset \Delta \subset B_{HS}$, which just reflect the inequalities $\| \cdot \|_2 \leq \| \cdot \|_1 \leq \sqrt{d} \, \| \cdot \|_2$ between the trace class and the Hilbert-Schmidt norms. The first inclusion implies the lower estimate on the volume radius in (7). The upper bound is less obvious, but it may be shown by the following rather general argument. The first step is the classical Urysohn inequality, which in our context asserts that

$$\left(\frac{\mathrm{vol}\, \Delta}{\mathrm{vol} B_{HS}} \right)^{1/d^2} \leq \int_{S_{HS}} \|A\|_{op} dA \; =: \mu_d, \tag{8}$$

where the integration is performed with respect to the normalized Lebesgue measure on the Hilbert-Schmidt sphere. (For clarity and to indicate flexibility of the approach we shall present a general statement and a short proof in Appendix A.) The quantity μ_d is most easily handled by passing to an integral with respect to the standard Gaussian measure, which reduces the problem to finding expected value of the norm of a random $d \times d$ Gaussian matrix

$G = G(\omega)$ usually called the Gaussian Unitary Ensemble, or GUE. It is well known that $\mathbb{E}\|G\|_{op} = \gamma_{d^2}\mu_d$, where $\gamma_k := \sqrt{2}\Gamma(\frac{k+1}{2})/\Gamma(\frac{k}{2})$ for $k \in \mathbb{N}$, and it is easy to check that $\sqrt{k-1} < \gamma_k < \sqrt{k}$ for all k. In other words, $\mu_d \sim \mathbb{E}\|G\|_{op}/d$ for large d. On the other hand, it is a well-known strengthening of Wigner's semicircle law that $\mathbb{E}\|G\|_{op}/\sqrt{d} \to 2$ as $d \to \infty$. This shows the second inequality in (7) with 2 replaced by $2+o(1)$. We sketch the argument that gives the exact number 2 in Appendix F (it follows from known facts, but appears to have been overlooked in the random matrix theory), yet we will not dwell on it as it intervenes only in the lower estimate in (2) and, in any case, the constants in our final results are not meant to be optimal. Indeed, (7) combined with (4) implies that the volume radius of $\mathcal{D}$ is between $\frac{1}{2}d^{-1/2}$ and $2d^{-1/2}$, while the formulae from [24] allow to deduce that it is equivalent to $\frac{1}{e^{1/4}}d^{-1/2}$ as $d \to \infty$.

We now pass to the analysis of the the volume radius of Σ. We shall show that
$$1/d^{1+\alpha} \le (\operatorname{vol}\Sigma/\operatorname{vol}B_{HS})^{1/d^2} \le C\sqrt{\log d \log\log d}/d^{1+\alpha}, \qquad (9)$$
where α is the same as in (2). Our main result (2) follows then by combining (7), (9) and (6). [To be precise, one obtains *a priori* $1/d^2$ in the exponent, but a more careful analysis of lower order factors such as $2/\sqrt{d}$ and $1/(n+1)$ appearing in (4)-(6) allows to replace d^2 by $\dim\mathcal{S} = d^2 - 1$ without any loss in the constants.]

Before proceeding, let us compare (9) with the results of [3] and [5], which estimate from *below* the in-radius of $\mathcal{S}$ in the Hilbert-Schmidt metric by a quantity that is of order of $d^{-\eta}$, where $\eta = \log 20/\log 4 \approx 2.160964$ and $3/2$ respectively. The trivial upper bound on that radius is the in-radius of $\mathcal{D}$, which equals $1/\sqrt{d(d-1)} = O(d^{-1})$. The second inequality in (9) yields (for large N) a better upper estimate that roughly corresponds to $\eta = 1+\alpha \approx 1.094361$; we elaborate on these and related issues in Appendix H.

The first step towards showing (9) will be to replace the sets Σ_N by their affine images which are more "balanced;" this will also explain the appearance of the mysterious number α in the exponents.

Consider first the sets in question when $N = 1$. As is well known, $\mathcal{S}_1$ and $\mathcal{D}_1$ both coincide with the Bloch "ball," which geometrically is a (solid) Euclidean ball of radius $1/\sqrt{2}$, the boundary of which is the Bloch sphere $\mathcal{T}_1$ consisting of pure states on $\mathcal{B}(\mathbb{C}^2)$ (further identifiable with rank 1 projections on $\mathbb{C}^2$). Accordingly, $\Sigma_1 = \Delta_1$ is a 4-dimensional cylinder whose base

is the Bloch ball and whose axis is the segment $[-I_2/2, I_2/2]$ of Euclidean length $\sqrt{2}$. For definiteness, let us identify $\mathcal{M}_2^{sa}$ with $\mathbb{R}^4$ via the usual basis $\{I_2/\sqrt{2}, \sigma_x/\sqrt{2}, \sigma_y/\sqrt{2}, \sigma_z/\sqrt{2}\}$, where σ_x, σ_y and σ_z are the Pauli matrices (the factors $1/\sqrt{2}$ make this basis orthonormal in the Hilbert-Schmidt sense). Let now A be a linear map on $\mathcal{M}_2^{sa}$ which is diagonal in that basis and whose action is defined by $AI_2 = I_2/\sqrt{2}$, $A_i = \sqrt{3/2}\,\sigma_i$ for $i = x, y, z$. Set $\tilde{\Sigma}_1 := A\Sigma_1$; the important properties of A and $\tilde{\Sigma}_1$ are

(i) the image of the Bloch sphere $A\mathcal{T}_1 =: \tilde{\mathcal{T}}_1$ is geometrically a 2-dimensional sphere of radius $\sqrt{3}/2$ centered at $I_2/\sqrt{8}$ and, as the Bloch sphere itself, it is contained in the *unit* Euclidean sphere S_{HS}; this implies that $\tilde{\Sigma}_1 \subset B_{HS}$

(ii) $\det A = \sqrt{27/16}$ and so $\mathrm{vol}\,\tilde{\Sigma}_1 = \sqrt{27/16}\,\mathrm{vol}\,\Sigma_1$

(iii) vertices of any regular tetrahedron inscribed in $\tilde{\mathcal{T}}_1$ form an orthonormal basis in $\mathcal{M}_2^{sa}$.

The geometric property of the set $\tilde{\Sigma}_1$, which arguably is the reason for its relevance, is that the ellipsoid of smallest volume containing it (the so-called Löwner ellipsoid of $\tilde{\Sigma}_1$) is the Euclidean ball. An equivalent and perhaps more natural point of view would be to compare Σ_1 with its own Löwner ellipsoid. This is in turn equivalent to replacing the Hilbert-Schmidt inner product $\langle u, v \rangle = \mathrm{tr}\,uv$ with $\mathrm{tr}((Au)(Av)) = (3\,\mathrm{tr}\,uv - \mathrm{tr}\,u\,\mathrm{tr}\,v)/2$; it is likely that this non-isotropic inner product and objects associated with it play an important role in the theory. In particular, we obtain this way ellipsoids which – from the volumetric point of view – are nearly non-distinguishable from $\mathcal{S}$ or Σ, and which still enjoy certain permanence relations with respect to the action of the unitary group.

If $N > 1$, we set $\tilde{\Sigma} = \tilde{\Sigma}_N := A^{\otimes N}\Sigma_N$. Since $\det A^{\otimes N} = (\det A)^{N \cdot 4^{N-1}} = ((27/16)^{N/8})^{d^2} = (2^{\alpha N})^{d^2} = (d^\alpha)^{d^2}$, we deduce that (9) is equivalent to

$$1/d \leq (\mathrm{vol}\,\tilde{\Sigma}/\mathrm{vol}\,B_{HS})^{1/d^2} \leq C\sqrt{\log d \log \log d}/d, \tag{10}$$

For the lower estimate in (10) we shall produce a simple (and seemingly not very optimal) geometric argument. Let u_1, u_2, u_3, u_4 be vertices of any regular tetrahedron inscribed in $\tilde{\mathcal{T}}_1$. By the property (iii) above, $(u_j)_{j=1}^4$ is an orthonormal basis of $\mathcal{M}_2^{sa}$. Accordingly, the set $\tilde{U} := \{u_{j_1} \otimes u_{j_2} \otimes \ldots \otimes u_{j_N}\}$, where each j_i ranges over $\{1, 2, 3, 4\}$ is an orthonormal basis of $\mathcal{M}_d^{sa}$. The first inequality in (10) follows now from the inclusions $\tilde{U}, -\tilde{U} \subset \tilde{\Sigma}_N$ and $B_{HS}/d \subset \mathrm{conv}(-\tilde{U} \cup \tilde{U})$ (the latter is a consequence of orthogonality of elements of $\tilde{U}$).

The above argument may appear rather ad hoc, and so it may be instructive to rephrase it in the language of geometry of Banach spaces. Let A_1 be a linear map from $\mathbb{R}^4$ to $\mathcal{M}_2^{sa}$ which sends the standard unit vector basis onto vertices of any regular tetrahedron inscribed in $\mathcal{T}_1$. By construction, A_1 is a contraction from ℓ_1^4 to $\mathcal{C}_1^2$ and so its N-th tensor power $A_1^{\otimes N}$ induces a contraction between the respective projective tensor powers of ℓ_1^4 and $\mathcal{C}_1^2$ (where ℓ_1^k denotes $\mathbb{R}^k$ endowed with the norm $\|(x_j)\| = \sum |x_j|$). As the projective tensor product of ℓ_1-spaces is again an ℓ_1-space, it follows that Σ_N contains the image under $A_1^{\otimes N}$ of the unit ball of $\ell_1^{d^2}$, and hence the image of the Euclidean ball of radius $1/d$. In particular, $\mathrm{vol}\,\Sigma/\mathrm{vol}(A_1^{\otimes N}B_{HS}) \geq (1/d)^{d^2}$. On the other hand, one verifies (directly, or by noticing that $A = |A_1^{-1}| = (A_1^{-1*}A_1^{-1})^{1/2}$) that $\mathrm{vol}(A_1^{\otimes N}B_{HS}) = ((16/27)^{N/8})^{d^2}\,\mathrm{vol}B_{HS}$, and by combining the two estimates we obtain the first inequality in (9).

We note that using for the above calculations the larger volume of the image of the $\ell_1^{d^2}$ ball (resp., $\mathrm{conv}(-\tilde{U} \cup \tilde{U})$) would only result in a slightly better constant c in (2) (the value $c = \sqrt{e/8\pi}$ that was mentioned earlier). This is because the volume radius of the unit ball in ℓ_1^m is roughly the same as that of the inscribed Euclidean ball, the ratio between the two is $\sqrt{2e/\pi}\,(1 - O(1/m))$. This property is behind many striking phenomena discovered in the asymptotic theory of finite dimensional normed spaces, and is closely related to our upper estimates for $\mathrm{vol}\,\Sigma$ and $\mathrm{vol}\,\mathcal{S}$, to which we pass now.

To prove the upper estimate in (10), we shall again use the Urysohn inequality. Analogously to (8) and to the reasoning that followed it, we get

$$\left(\frac{\mathrm{vol}\tilde{\Sigma}}{\mathrm{vol}B_{HS}}\right)^{1/d^2} \leq \int_{S_{HS}} \max_{X \in \tilde{\Sigma}} \mathrm{tr}(XA)\,dA = \gamma_{d^2}^{-1}\mathbb{E}\max_{X \in \tilde{\Sigma}} \mathrm{tr}(XG) \qquad (11)$$

and so it remains to show that the above expectation is $O(\sqrt{\log d \log \log d})$. The expression under the expectation can be thought of as a maximum of a Gaussian process indexed by $\tilde{\Sigma}$ (this just means the family $\mathrm{tr}(XG(\omega))$, $X \in \tilde{\Sigma}$, of jointly Gaussian random variables). There are several methods of differing sophistication which can be used to estimate the expectation of such a maximum. The two leading ones are the Fernique-Talagrand majorizing measure theorem, which gives the correct asymptotic order, but is usually difficult to apply, and the Dudley majoration (by the metric entropy integral), which is almost as precise and usually easier to handle; see [10] for a comprehensive exposition. We shall employ here an even simpler "one-level-discretization" method which, in our case, yields approximately the same

result as the Dudley majoration, and which we now describe in elementary language.

Let μ be the standard Gaussian measure on $\mathbb{R}^m$ (i.e., the one given by the density $m(x) = (2\pi)^{-m/2} \exp(-|x|^2/2)$, where $|\cdot|$ is the corresponding Euclidean norm) and let $F \subset \mathbb{R}^m$ be a finite set contained in a ball of radius R. Then

$$\int_{\mathbb{R}^m} \max_{y \in F} \langle y, x \rangle \, d\mu(x) \leq R\sqrt{2\log(\#F)}, \tag{12}$$

where $\#$ stands for the cardinality of a set. The estimate above is usually quoted with a different numerical constant appearing in place of 2, but it is not difficult – even if somewhat tedious – to verify that it holds in the form stated above. The idea is now to construct a finite set $F \subset \tilde{\Sigma}$ such that $\mathrm{conv} F \supset r\tilde{\Sigma}$ for an appropriate $r \in (0,1)$; it will then follow that

$$\mathbb{E} \max_{X \in \tilde{\Sigma}} \mathrm{tr}(XG) \leq r^{-1}\sqrt{2\log(\#F)} \tag{13}$$

(note that the maxima of the type appearing in (11), (12) or (13) do not change if we replace the underlying set by its convex hull or, conversely, by its extreme points). More specifically, F will be a "sufficiently dense" subset of the set of extreme points of $\tilde{\Sigma}$, which is of the form $\tilde{\mathcal{T}} \cup (-\tilde{\mathcal{T}})$ with

$$\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_N = \{A\rho_1 \otimes A\rho_2 \otimes \ldots \otimes A\rho_N\},$$

where each ρ_j is a pure state on $\mathcal{B}(\mathbb{C}^2)$ (i.e., an element of the Bloch sphere). In other words, $\tilde{\mathcal{T}}$ is a tensor product of N copies of $\tilde{\mathcal{T}}_1 = B\mathcal{T}_1$ which, as we noted earlier, is geometrically a 2-dimensional sphere of radius $\sqrt{3}/2$ contained in the unit sphere of the 4-dimensional Euclidean space.

We start by constructing an appropriate dense subset (usually called *net*) in each copy of $\tilde{\mathcal{T}}_1$ and then consider tensor products of those nets. To facilitate references to existing literature we first look at the *unit* Euclidean ball S^2 rather than $\tilde{\mathcal{T}}_1$. Let $\delta \in (0, \sqrt{2})$ and let $\mathcal{N}$ be a δ-net of S^2, i.e., a subset such that the union of balls of radius δ centered at points of $\mathcal{N}$ covers S^2. An elementary argument shows that $\mathrm{conv}\mathcal{N}$ contains then a ball of radius $(1 - \delta^2/2)$ centered at the origin. If now $F_1 \subset \tilde{\mathcal{T}}_1$ is an appropriate dilation of $\mathcal{N}$ (i.e., with the ratio $\sqrt{3}/2$), then $\mathrm{conv} F_1$ contains a ball of radius $(1 - \delta^2/2) \cdot \sqrt{3}/2$ (in the 3-dimensional affine space containing $\tilde{\mathcal{T}}_1$). with the same center as that of of $\tilde{\mathcal{T}}_1$ It follows that $\mathrm{conv}(-F_1 \cup F_1) \supset (1 - \delta^2/2)\tilde{\Sigma}_1$ and consequently if we set $F := (-F_1 \cup F_1)^{\otimes N} = F_1^{\otimes N} \cup (-F_1^{\otimes N})$, then $\mathrm{conv} F \supset (1 - \delta^2/2)^N \tilde{\Sigma}_N$.

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In remains to find a reasonable bound on $\#F = 2(\#F_1)^N = 2(\#\mathcal{N})^N$. A standard argument comparing areas of caps and that of the entire sphere shows that one may have a δ-net of S^2 of cardinality $< 16/\delta^2$. [This bound is not optimal; the asymptotically – as $\delta \to 0$ – correct order for cardinalities of efficient δ-nets of S^2 is $(2/\sqrt{3})^3\pi/\delta^2$, but coverings of Euclidean spheres, even in dimension 2, do not appear to be completely understood.] This leads to an estimate $\#F < 2(16/\delta^2)^N$, which in combination with (13) gives

$$\mathbb{E}\max_{X\in\tilde{\Sigma}} \operatorname{tr}(XG) \leq (1 - \delta^2/2)^{-N}\sqrt{2\log(2(16/\delta^2)^N)}. \qquad (14)$$

Optimizing the expression on the right hand side over $\delta \in (0, \sqrt{2})$ yields a quantity that is of order $\sqrt{2N\log N}$ for large N (choose, for example, $\delta = (N \log N)^{-1/2}$), as required to complete the proof of (9) (and hence of (2)). Moreover, substituting the obtained bound into (11) and verifying numerically small values of N yields

$$\left(\frac{\operatorname{vol}\tilde{\Sigma}}{\operatorname{vol}B_{HS}}\right)^{1/d^2} \leq \frac{\sqrt{4N\ \log_2(4N)}}{d} = \frac{\sqrt{4\log_2 d\ \log_2(4\log_2 d)}}{d} \qquad (15)$$

(note that the inequality is trivial for $N = 2$), which implies that (2) holds with the third member of the form $\frac{4\sqrt{N\log_2 4N}}{d^{1/2+\alpha}}$. This may be somewhat improved for small to moderate values of N by using estimates on cardinalities of nets of S^2 listed in [7], see Appendix G.

Appendix A *The Urysohn inequality.* If K is a convex body in the m-dimensional Euclidean space which contains 0 in its interior, then

$$\left(\frac{\operatorname{vol}K}{\operatorname{vol}B}\right)^{1/m} = \left(\int_{S^{m-1}} \|x\|_K^{-m}dx\right)^{1/m} \geq \int_{S^{m-1}} \|x\|_K^{-1}dx \geq \left(\int_{S^{m-1}} \|x\|_K dx\right)^{-1},$$

where $\|x\|_K$ is the gauge of K (the norm for which K is the unit ball if K is 0-symmetric – which is the case in the main text) and the integration is performed with respect to the normalized Lebesgue measure on the sphere S^{m-1}. If K is 0-symmetric, this may be combined with the Santaló inequality [16] which asserts that

$$\frac{\text{vol}K}{\text{vol}B} \cdot \frac{\text{vol}K^\circ}{\text{vol}B} \leq 1,$$

where $K^\circ := \{x : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}$ is the polar of K, to obtain $(\text{vol}K^\circ/\text{vol}B)^{1/m} \leq \int_{S^{m-1}} \|x\|_K dx$, and the Urysohn's inequality

$$(\text{vol}K/\text{vol}B)^{1/m} \leq \int_{S^{m-1}} \|x\|_{K^\circ} dx = \int_{S^{m-1}} \max_{y \in K} \langle x, y \rangle dx$$

follows by exchanging the roles of K and K°. Moreover, since the Santaló inequality holds also for not-necessarily-symmetric sets after an appropriate translation, the above inequality holds for any (measurable) bounded set K. [Note that the integral on the right does not change if K is replaced by its translation.]

This is not the most elementary proof of the Urysohn inequality, but one that offers a lot of flexibility. For example, the repeated applications of the Hölder inequality in the first chain of inequalities above can be modified to yield as the last expression $\left(\int_{S^{m-1}} \|x\|_K^p dx\right)^{-1/p}$ for an arbitrary $p > 0$ and, letting $p \to 0$, the geometric mean $\exp\left(-\int_{S^{m-1}} \log \|x\|_K dx\right)$. [Similar inequalities also hold if $p \in [-n, 0)$, and the case $p = -n$ is of course the strongest statement of such nature, the Santaló inequality itself.]

The application of the Urysohn inequality in (8) uses implicitly the elementary fact that $\|\cdot\|_{\Delta^\circ} = \|\cdot\|_{op}$. In other words, the trace class norm $\|\cdot\|_1$ and the operator norm $\|\cdot\|_{op}$ are dual with respect to the trace duality. For the sets Δ_N, the Urysohn inequality gives the correct order of the volume radius, cf. (7). However, this is not always the case, even for rather regular convex bodies. For example, if K is the unit ball of ℓ_1^m, then its volume radius – as we have already mentioned – exceeds the radius of the inscribed Euclidean ball by less than $\sqrt{2e/\pi}$, while the upper bound obtained from the Urysohn inequality contains a parasitic factor which is of order $\sqrt{\log m}$. It is thus conceivable that the logarithmic factors in (2) can be replaced by universal numerical constants. On the other hand, if we do use the Urysohn inequality to establish an upper bound for the volume radius of $\tilde{\Sigma}$ (cf. (10), (11), (15)), then our estimates can not be substantially improved. Indeed,

since our argument showed that $\tilde{\Sigma}$ contained a rotation of the unit ball of $\ell_1^{d^2}$, the previous remark implies that the $\sqrt{N} = \sqrt{\log_2 d}$ factor can not be then avoided, and so it is only the $\sqrt{\log \log d}$ factor that can possibly be eliminated by more careful majorizing of $\mathbb{E}\|G\|_{\tilde{\Sigma}^\circ}$.

Finally, we mention that there exist general volume estimates for convex hulls of finite sets which are asymptotically more precise than the one we derive from the Urysohn inequality (see [1] and its references). However, these estimates are equivalent to the ones presented here in the relevant range of parameters and, moreover, the easily available formulations contain universal but unnamed numerical constants, which would make the corresponding bounds useless for specific values of N.

Appendix B *Projective tensor products of normed spaces.* If X and Y are (say, finite dimensional) normed spaces, their tensor product $X \otimes Y$ may be endowed with the projective tensor product norm $\|\cdot\|_\pi$ defined by
$$\|\tau\|_\pi := \inf \left\{ \textstyle\sum_{j=1}^m \|x_j\| \cdot \|y_j\| \ : \ \sum_{j=1}^m x_j \otimes y_j = \tau \right\}.$$
The resulting normed space is usually denoted $X \hat{\otimes} Y$ or $X \otimes_\pi Y$. If B_X and B_Y are unit balls of X and Y respectively, it follows that the unit ball of $X \hat{\otimes} Y$ coincides with

conv $\{x \otimes y : x \in B_X, y \in B_Y\} =$ conv $\{x \otimes y : x \in \text{ext} B_X, y \in \text{ext} B_Y\}$,

where $\text{ext}(K)$ denotes the set of extreme points of K. If $X = Y = \mathcal{C}_1^2$, the analysis is further simplified by the fact that the set of extreme points of Δ_1, the unit ball of $\mathcal{C}_1^2$, is of the form $-\mathcal{T}_1 \cup \mathcal{T}_1$, where $\mathcal{T}_1$ is the set of pure states. The fact that the set Σ_2 is the unit ball of $\mathcal{C}_1^2 \hat{\otimes} \mathcal{C}_1^2$ follows directly from these identifications. Projective tensor products of more than two spaces are defined analogously (or by induction), and one similarly checks that the unit ball of the Nth projective tensor power of $\mathcal{C}_1^2$ is Σ_N. Tensor products involving spaces $\mathbb{C}^k$ with $k > 2$ may be treated in the same way. For example, the symmetrization of the set of separable states on $\mathcal{B}(\mathbb{C}^{k_1} \otimes \mathbb{C}^{k_2} \otimes \ldots \mathbb{C}^{k_m})$ can be identified with the unit ball in $\mathcal{C}_1^{k_1} \hat{\otimes} \mathcal{C}_1^{k_2} \hat{\otimes} \ldots \hat{\otimes} \mathcal{C}_1^{k_m}$. The problem of the relative size of the set of separable states on $\mathcal{B}((\mathbb{C}^D)^{\otimes N})$, or N qudits, was investigated in [15]. While a more definitive treatment of the higher-dimensional case will be presented elsewhere, we offer some comments on the topic in Appendix I.

On the more elementary level, the projective tensor square of a Euclidean space $\mathbb{C}^k \hat{\otimes} \mathbb{C}^k$ can be identified with $(\mathcal{M}_k, \|\cdot\|_1)$ and its self-adjoint part is, in our cotation, $\mathcal{C}_1^k$. We also mention in passing that the dual space to $X \hat{\otimes} Y$ can be identified with the so-called *injective* tensor product of the duals X^*

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and Y^*, and so Σ° can be thought of as a unit ball in the injective tensor power of the Hermitian part of $\mathcal{B}(\mathbb{C}^2)$, but making this identification explicit doesn't seem to help our analysis at the current level of depth.

Appendix C *The Rogers-Shephard results on symmetrizations of convex sets.* Let $W \subset \mathbb{R}^{n+1}$ be an n-dimensional convex set and denote by h the distance from the affine hyperplane H spanned by W to the origin. Let Ω be the symmetrization of W, i.e., $\Omega := \text{conv}(-W \cup W)$. It was shown in [13] that then

$$2h \text{ vol} W \leq \text{vol} \Omega \leq 2h \, \frac{2^n}{n+1} \text{vol} W, \tag{16}$$

where by $\text{vol} W$ and $\text{vol} \Omega$ we mean the n- and the $n+1$-dimensional volume respectively. To explain the factors appearing in (16) we note that the inequalities becomes equalities if W is centrally symmetric for the first one (this is simple) and if W is a simplex for the second (this is the heart of the Rogers-Shephard result).

To further clarify the first inequality in (16) (used in the upper estimates on the volume of separable states, which is the main point of this note) we point out that it is actually a simple consequence of a much older theorem of Brunn-Minkowski, and more specifically of the following corollary of that theorem.

Let K be an $n+1$-dimensional convex body, u – a vector in the ambient space containing K and H – a hyperplane in that space. Then the function $t \to \text{vol}(K \cap (tu+H))^{1/n}$ (the n-dimensional volume) is concave on its support.

If we apply the above fact with $K = \Omega$ and u - a unit vector perpendicular to H, then the function $\phi(t) := \text{vol}(\Omega \cap (tu+H))$, being even on $[-h, h]$, must attain its maximum at 0 and minimum at h and $-h$. The first inequality in (16) follows then from the Cavalieri principle. A version of the second inequality, which would be sufficient for our purposes, follows similarly from the estimate $\phi(0) \leq 2^{-n}\binom{2n}{n}\text{vol} W$, which is the main result of [12].

Appendix D *Working directly with non-symmetric sets.* Similar but slightly more complicated arguments may be used to obtain *upper* estimates for the volumes of the non-symmetric sets $\mathcal{D}$ and $\mathcal{S}$ by studying directly these sets and not their symmetrizations Σ and Δ. [In principle, this could help to avoid the parasitic factors $2^n/(n+1)$ – where $n = d^2 - 1$ – when passing from Σ, Δ to $\mathcal{S}, \mathcal{D}$.] In both cases it is convenient to pass to a translate of the set in question obtained by subtracting the appropriate multiple of the

maximally mixed state I_d/d, and to consider the translates as subsets of H_0, the $d^2 - 1$-dimensional space of matrices with vanishing trace.

For the set $\mathcal{D}$ (translated by I_d/d), the quantity which replaces $\| \cdot \|_{op}$ in the analogue of (8) is $\lambda_1(\cdot)$, the largest eigenvalue of a matrix. This is of course dominated by the norm, and since the (random Gaussian) trace 0 matrix G_0 can be represented as a conditional expectation of the general Gaussian matrix G, it follows – by the convexity of the norm or of the largest eigenvalue – that $\mathbb{E}\,\lambda_1(G_0) \leq \mathbb{E}\|G\|_{op} \leq 2\sqrt{d}$ which, after some work, leads to an upper estimate for the volume radius of $\mathcal{D}$ identical to that of Δ obtained in (7). [To justify the steps above one needs Appendices A and F.]

For the set $\mathcal{S}$, we pass first to the face $\tilde{\mathcal{S}}$ of the rescaled set $\tilde{\Sigma}$ that corresponds to $\mathcal{S}$, and then subtract $I_d/d^{3/2}$ (the difference with respect to the case of $\mathcal{D}$ is due to the rescaling). Next, we "approximate" the translate by sets built from the points corresponding to elements of F. There are several differences between this setting and that described in the main text, but they can be accounted for fairly easily. The good news is that the new points are not on the sphere since the component in the direction of I_d was subtracted, but this improves our estimate on $\mathrm{vol}\,\mathcal{S}$ only by a factor $1 - O(d^{-1})$. A loss which is even less significant is due to the fact that by reducing the dimension by 1 our formulae will involve the quantity γ_{d^2-1} rather than γ_{d^2}. A somewhat more substantial loss comes from the fact that due to the rescaling the width of $\tilde{\Sigma}$ in the direction of I_d is different from that of Σ by a factor of $2^{N/2} = d^{1/2}$; this affects the relationships between volumes of these bodies and those of $\tilde{\mathcal{S}}$ and $\mathcal{S}$, and consequently our estimates, by the same factor. However, since we are in dimension $d^2 - 1$, the loss in the *volume radius* is a not-so-significant $1 + O(\log d/d)$. The final issue that needs to be analyzed is that while we knew that $\mathrm{conv}F \supset (1 - \delta^2/2)^N\tilde{\Sigma}$, it is not *a priori* clear that a similar inclusion holds for the face $\tilde{\mathcal{S}}$ (or, more precisely, for its translate $\tilde{\mathcal{S}}-I_d/d^{3/2}$). While for a general convex set $K \subset H_1$ the relationship between K and its symmetrization $\mathrm{conv}(-K \cup K)$ may be more involved, using the fact that $\tilde{\mathcal{S}}$ is a convex hull of points contained in a *sphere* centered at $I_d/d^{3/2}$ we can infer that $\mathrm{conv}F \supset (1 - \kappa)\tilde{\Sigma}$ implies $\mathrm{conv}(F \cap \tilde{\mathcal{S}}) \supset I_d/d^{3/2} + (1 - 2\kappa)(\tilde{\mathcal{S}} - I_d/d^{3/2})$, and the difference between the factors $1 - \kappa$ and $1 - 2\kappa$ is asymptotically insignificant.

While the above argument allows to avoid the symmetrizations while estimating the volume radii of $\mathcal{D}$ and $\mathcal{S}$ from above, we get the same majorants as in (7), (9). Moreover, we still need to look at the symmetrizations for the lower bounds which are needed to derive (2). [It is also possible to obtain a

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lower bound on the volume radius of $\mathcal{S}$ directly by noticing that $\mathcal{S}$ contains a simplex spanned by the elements of $(A^{\otimes N})^{-1}\tilde{U}$, but since for the simplex the relevant Rogers-Shephard inequalities become equalities, there is again no significant improvement.]

Appendix E *The exact expressions on the volume of $\mathcal{D}$ and the constants in* (2) *as $N \to \infty$.* After a preliminary version of this note has been posted, the author has learned that a closed formula for the volume of $\mathcal{D}$ was derived in a very recent work [24]. While we were able to calculate the volume radius of $\mathcal{D}$ to within a factor of 2 by the same methods that were employed to analyze $\mathcal{S}$ and with very little extra work, it is instructive to compare the so obtained estimates to those that can be deduced from the exact formula, which in our notation reads

$$\mathrm{vol}\mathcal{D}(\mathbb{C}^d) = \sqrt{d}\,(2\pi)^{d(d-1)/2}\,\frac{\prod_{j=1}^d \Gamma(j)}{\Gamma(d^2)} \tag{17}$$

A tedious but routine calculation based on the Stirling formula shows that the volume radius of $\mathcal{D}(\mathbb{C}^d)$ behaves as $\frac{1}{e^{1/4}}d^{-1/2}(1 + O(d^{-1}))$ as $d \to \infty$

We now recall the refinements related to the volume radii of $\mathcal{S}_N$ and Σ_N suggested in the main text. First, we had the argument that gave $c = \sqrt{e/8\pi}$ in (2), based on using the volume of the unit ball in $\ell_1^{d^2}$, i.e., $2^{d^2}/(d^2)!$, as a lower bound for $\mathrm{vol}\widetilde{\Sigma}$ (see the comments following (10)). Next, we noted that, for large N, the expressions in (14) can be majorized by a quantity that is of order $\sqrt{2N \log N}$. Combining these with the improvement related to $\mathcal{D}_N$ we are led to an asymptotic version of (2) with $c_N \to e^{3/4}/\sqrt{2\pi} \approx 0.844561$ and $C_N \to e^{1/4}\sqrt{2/\log 2} \approx 2.1811$.

Appendix F *Norms of GUE matrices and the constant* 2 *in* (7). It has been known for some time (in fact in a much more general setting) that if $G = G(\omega)$ is the random matrix distributed according to the standard Gaussian measure on $\mathcal{M}_d^{sa}$ (usually called the Gaussian Unitary Ensemble), then, for large m, $\|G\|_{op}$ is, with high probability, close to $2\sqrt{d}$. We sketch here a derivation, from known facts, of the arguably elegant inequality $\mathbb{E}\|G\|_{op} < 2\sqrt{d}$, valid for any d, which appears to have been overlooked in the random matrix theory literature. Similar inequalities are known for Gaussian matrices all whose entries are independent or for real symmetric matrices (also known as the GOE ensemble; however, in the latter case the precise inequality seems to have been established only for the largest eigenvalue, and not for the norm), see [4]. Analogous inequalities with the expected value replaced by

the median can probably be deduced – at least for large d – from [20] and [21].

Our starting point are the recurrence formulae for the moments $a_p = a_p(d) := d^{-1}\mathbb{E}\operatorname{tr}((G/2)^{2p})$, $p \in \mathbb{N}$, derived, e.g., in [6] (see also [9], formulae (6) through (9), for a similar argument and a related estimate)

$$a_p = \frac{2p-1}{2p+2}\left(a_{p-1} + \frac{p(p-1)}{4d^2}\frac{2p-3}{2p}a_{p-2}\right),$$

with $a_0 = 1$ and $a_1 = 1/4$. From these one easily derives by induction

$$a_p \leq \frac{1}{2^{2p}(p+1)}\binom{2p}{p}\prod_{j=1}^{p}\left(1 + \frac{j(j-1)}{4d^2}\right).$$

[This estimate is actually asymptotically precise for $p = o(d)$.] Next, using successively the Stirling formula to majorize the binomial coefficient, the inequalities $1 + x \leq e^x$ and $\sum_{j=1}^{p}j(j-1) \leq p^3/3$ to estimate the product, and denoting $t = pd^{-2/3}$, we arrive at

$$\mathbb{E}\operatorname{tr}\left((G/2)^{2p}\right) = da_p \leq d\,\frac{e^{p^3/12d^2}}{\sqrt{\pi}\,p^{3/2}} = \frac{e^{t^3/12}}{\sqrt{\pi}\,t^{3/2}}.$$

Hence

$$\frac{1}{2}\mathbb{E}\|G\|_{op} < \left(\mathbb{E}\operatorname{tr}\left((G/2)^{2p}\right)\right)^{1/2p} \leq \left[\left(\frac{e^{t^3/6}}{\pi\,t^3}\right)^{1/4t}\right]^{1/d^{2/3}}.$$

This is valid for $t > 0$, at least if the corresponding value of $p = td^{2/3}$ is an integer. The minimal value of the expression in brackets over $t > 0$ is attained at $t \approx 1.38319$ and is approximately $0.738542 \approx \exp(-0.303077) < e^{-0.3}$. Since for sufficiently large d the interval corresponding to values which are $< e^{-0.3}$ contains an element of $d^{-2/3}\mathbb{N}$, we deduce that for such d we have $\mathbb{E}\|G\|_{op} < 2e^{-0.3d^{-2/3}}$. A more careful checking shows that in fact the inequality $\mathbb{E}\|G\|_{op} < 2 - 0.6d^{-2/3}$ holds for *all* values of d (in fact, by the above argument, the same upper estimate is valid for $(\mathbb{E}\|G\|_{op}^r)^{1/r}$ with, say, $r = 2$ or $r = d^{2/3}$).

Going back to the issue of having the precise constant 2 in inequality (7), let us note that the other source of difficulty, namely the fact that the parameter γ_k is only asymptotically of order $\sqrt{k}$ but not equal to $\sqrt{k}$, introduces

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an error that is of smaller order than our "margin of safety." As pointed out earlier, we have $\gamma_k > \sqrt{k-1}$ and so $\gamma_k/\sqrt{k} > \sqrt{1-1/k} \approx 1 - 1/2k$. The relevant value of k is d^2, leading to the relative error of order $d^{-2}/2$, as opposed to the margin of safety of $0.3d^{-2/3}$ (note also that 4 is the smallest value of d that is of interest).

Appendix G *Upper estimates on* $\mathrm{vol}\mathcal{S}_N$ *for small to moderate* N. We now indicate how one may use the explicit efficient nets of the sphere S^2 listed in [7] to majorize the volume radius of $\mathcal{S}_N$ if N is not too large. As a demonstration, we will derive bounds for the volume of the set of separable states on 8 qubits (one may say, a qubyte).

The site [7] lists, for $m \in \{4, \ldots, 130\}$, sets $\mathcal{N}_m$ of m points in S^2 such that every point of S^2 is within $\epsilon = \epsilon_m$ (measured in degrees) from one of the points of $\mathcal{N}_m$, with the dependence $m \to \epsilon_m$ "putatively optimal" (and very likely nearly optimal). Noting that $\delta = 2\sin\epsilon/2$ we verify numerically that in most of the interesting range the putatively optimal value δ_m verifies $m\,\delta_m^2 \approx 5$ (more precisely, $5.1\pm 1\%$, still not far from the asymptotic value $(2/\sqrt{3})^3\pi \approx 4.8368$ that we mentioned earlier). Since in the present context the bound in (14) becomes $(1 - \delta_m^2/2)^{-8}\sqrt{2\log(2m^8)} = (\cos\epsilon_m)^{-8}\sqrt{2\log(2m^8)}$, substituting $m = 5/\delta_m^2$ leads to a function $\phi(\delta) = (1 - \delta^2/2)^{-8}\sqrt{2\log(2(5/\delta^2)^8)}$, which attains its minimum very near $\delta = .15$, which suggests that the optimal value of m should be around 222. This is beyond the range of the tables from [7], but using the largest available $m = 130$ and the corresponding $\epsilon_{130} = 11.3165625°$ yields an upper bound of 10.417406, which is less than 2% larger that the majorant that would presumably be given by $m = 222$. Plugging in the obtained bound into (11) and using the fact that $(\mathrm{vol}\tilde{\Sigma}/\mathrm{vol}\Sigma)^{1/d^2} = d^\alpha = (27/16)^{N/8}$ we are led to

$$v := (\mathrm{vol}\Sigma/\mathrm{vol}B_{HS})^{1/d^2} \leq (16/27)\cdot 10.417406/\gamma_{d^2} < 0.02411446$$

Taking into account (5) and substituting the explicit expression for $\mathrm{vol}B_{HS}$ we obtain

$$\mathrm{vol}\mathcal{S} \leq \frac{\sqrt{d}}{2} \frac{\pi^{d^2/2}\, v^{d^2}}{\Gamma(d^2/2 + 1)}.$$

Finally, using the closed formula for $\mathrm{vol}\mathcal{D}$ from (17) we get

$$\frac{\mathrm{vol}\mathcal{S}}{\mathrm{vol}\mathcal{D}} \leq \frac{(2\pi)^{d/2}\, v^{d^2}\Gamma(d^2)}{2^{d^2/2\,+1}\,\Gamma(d^2/2+1)\prod_{j=1}^d \Gamma(j)} < 8.6\cdot 10^{-19996},$$

which (modulo rounding errors) is equivalent to a much less impressive bound $(\text{vol}\mathcal{S}/\text{vol}\mathcal{D})^{1/\dim\mathcal{D}} \leq 0.49534$.

The smallest N for which an argument such as above gives a nontrivial upper bound appears to be 6; we get $(\text{vol}\mathcal{S}_6/\text{vol}\mathcal{D}_6)^{1/\dim\mathcal{D}_6} < 0.95$. We refer to [17] and its references for extensive (largely numerical) treatment of the case $N = 2$.

Appendix H *The in-radii of $\mathcal{S}_N$ and Σ_N.* The papers [3] and [5] estimate from *below* the in-radius of $\mathcal{S}$ in the Hilbert-Schmidt metric by a quantity that is of order of $d^{-\eta}$, where $\eta = \log 20/\log 4 \approx 2.160964$ and $3/2$ respectively. The "trivial" upper bound on that radius is the in-radius of $\mathcal{D}$, which – by a rather elementary argument – equals $1/\sqrt{d(d-1)} = O(d^{-1})$. By comparing volumes we see that the second inequality in (9) yields an asymptotically better upper estimate that (up to logarithmic factors) corresponds to $\eta = 1 + \alpha \approx 1.094361$. This follows by taking into account (5) or by observing that, by a simple geometric argument, the Euclidean in-radius of Σ is at least as large as that of $\mathcal{S}$ (the latter considered in the hyperplane H_1 of trace one matrices). By tinkering with the argument it is possible to remove the logarithmic factors and, indeed, to obtain an upper bound on the in-radius of Σ (and hence of $\mathcal{S}$) which is $o(d^{-1-\alpha})$, but to improve the exponent new ideas would be necessary – if that is at all possible, that is.

Let us also note that our argument yields as well a lower bound $6^{-N/2}$ on the in-radius of Σ which corresponds to $\eta = \log 6/\log 4 \approx 1.3863$, and so is stronger that those that can be formally derived from [3] and [5]. To see this it is enough to combine the "trivial" lower estimate $d^{-1} = 2^{-N}$ on the in-radius of the set $\tilde{\Sigma}_N$ defined in what follows (cf. the paragraph following (10)) with the known value $(3/2)^{N/2}$ of the norm of the related map $A^{\otimes N}$. However, since the in-radius of $\mathcal{S}$ may *a priori* be (and probably is, except when $N = 1$) smaller than that of Σ, this doesn't improve the lower bounds from [3] and [5]. It is also likely that the methods from those papers may yield a lower bound on the in-radius of Σ that is better than $6^{-N/2}$.

While our calculations narrow down the potential range of the in-radii of $\mathcal{S}$ and Σ, and while further progress along the same lines may be possible, it seems likely that to obtain the exact values of exponents a more careful calculation involving, e.g., spherical harmonics may be necessary. On the other hand, as we have already noted, it may be more natural to consider in this context the inner product norm, different from the Hilbert-Schmidt norm, and induced by the inner product $(u, v) \rightarrow (3\,\text{tr}uv - \text{tr}u\,\text{tr}v)/2$. The

(solid) ellipsoid $\mathcal{E} := \{u : 3\,\mathrm{tr}(u^2) - (\mathrm{tr}u)^2 \leq 2\}$, which is the unit ball with respect to that norm, verifies $\mathcal{E}/d \subset \Sigma \subset \mathcal{E}$ (this is just a restatement of $B_{HS}/d \subset \tilde{\Sigma} \subset B_{HS}$) and these inclusions are essentially optimal. On the one hand, every pure state clearly belongs to the boundary of $\mathcal{E}$. On the other hand, it follows from, say, (10) that the volume radius of Σ is, on the power scale, the same than that of $\mathcal{E}/d$. This means that – from the volumetric point of view – Σ and $\mathcal{E}/d$ are nearly non-distinguishable.

Appendix I *Separable states on N qudits.* (See also [15].) Most of the elements of our analysis can be generalized to tensor products involving spaces $\mathbb{C}^D$ with $D > 2$, leading to non-trivial but not definitive results. As a demonstration, let $D \geq 3$ and consider the family of spaces $\mathcal{H}_N = (\mathbb{C}^D)^{\otimes N}$. We shall employ analogous notation to that of the main text, in particular $d = \dim\mathcal{H}_N = D^N$. The set of pure states on $\mathcal{B}(\mathbb{C}^D)$ coincides with the projective space $\mathbb{C}P^{D-1}$ whose real dimension is $2D - 2$ and which admits, for $\delta > 0$, δ-nets of cardinality not exceeding $(C'/\delta)^{2D-2}$, where C' is a universal constant. This leads to a bound on $(\mathrm{vol}\Sigma_N/\mathrm{vol}B_{HS})^{1/d^2}$ which is of order $(1 - \delta)^{-N}\sqrt{ND\log(C'/\delta)}/d$. Choosing, say, $\delta = 1/N$ and using the same bound on $\mathrm{vol}\mathcal{D}_N$ as earlier combined with the "easy" part of (5) we obtain

$$(\mathrm{vol}\mathcal{S}_N/\mathrm{vol}\mathcal{D}_N)^{1/\dim\mathcal{D}} = O\left(\sqrt{ND\log N}/d^{1/2}\right).$$

Since $d = D^N$, this leads to a nontrivial bound even for $N = 2$ if D is large enough. It is also possible to improve slightly the exponent of d by working (as we did for $D = 2$) with a more balanced affine image of Σ_N. The resulting improvement $\alpha = \alpha_D$ decreases as D increases; for example, $\alpha_3 = \frac{8\log 2}{9\log 3} - \frac{1}{2} \approx 0.0608264$ and, for large D, $\alpha_D \sim (2D\log D)^{-1}$. However, showing optimality of the so obtained exponents requires – for $D > 2$ – new ideas and will be presented elsewhere.

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