A BRIEF INTRODUCTION TO GROUP REPRESENTATIONS AND CHARACTER THEORY

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About these notes

When I teach the abstract algebra sequence for first-year graduate students, I finish with a short unit on group representations and character theory, for two reasons:

- Representation theory brings together many of the topics that appear throughout the algebra course.
- Representation theory is one of the parts of algebra most likely to turn out to be useful for students who don't specialize in algebra.

Unfortunately, many of the standard textbooks for a first graduate course in abstract algebra don't cover representation theory. On the other hand, many of the excellent brief introductions to the subject develop as much representation theory as possible using only elementary linear algebra and the bare rudiments of group theory, which doesn't fit well with my first reason for covering the topic. Hence these notes, which take advantage of the efficiencies offered by the reader's assumed familiarity with groups, rings, fields, modules, and canonical forms — but not any special preparation for representation theory in any of those topics.[1] Reflecting my personal taste, these brief notes emphasize character theory rather more than general representation theory.

1. Definition and examples of group representations

Given a vector space V, we denote by $GL(V)$ the **general linear group** over V, consisting of all invertible linear endomorphisms of V, with the operation of composition.

The following is our basic object of study in these notes.

Definition 1.1. Let G be a group and $\mathbb{F}$ be a field. An **$\mathbb{F}$-representation** of G consists of an $\mathbb{F}$-vector space V and a group homomorphism $\rho : G \to GL(V)$.

The **dimension** of a representation (V, ρ) is the dimension of V.

This definition simply says that a representation of G is a group action of G on some vector space V, in which each element of G acts as a linear map on V. For brevity, a representation (V, ρ) will frequently be identified by either V or ρ alone, with the other left implicit. We will also write $\rho_g \in GL(V)$ for $g \in G$ to avoid a profusion of parentheses. The fact that ρ is a group homomorphism then says that $\rho_g \rho_h = \rho_{gh}$ for each $g, h \in G$.

Group representations are important partly because they simply come up throughout mathematics. Unless you're studying group theory for its own sake, if you're working with a group, it's probably because it's acting on some structure you're more directly interested in. More often than not, that structure is a vector space, or is a subset of a vector space,

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[1]For example, chapters I-V and VII of Hungerford's *Algebra* contain more than enough background.

and the group acts via linear maps. (As some of the examples below demonstrate, even group actions which are defined without mentioning vector spaces are closely associated with representations.) Moreover, as we will see later on, group representations are useful even if you are studying group theory for its own sake.

Example 1.2. Any multiplicative group of $n \times n$ matrices has a natural representation on $\mathbb{F}^n$; ρ_g in this case is just the linear map given by the matrix g. Examples include:

(1) $GL_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid A \text{ is invertible}\}$.
(2) $SL_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid \det A = 1\}$.
(3) The group of upper triangular matrices in $M_n(\mathbb{F})$ with diagonal entries all equal to 1.
(4) The orthogonal group $O(n) = \left\{A \in M_n(\mathbb{R}) \mid A^t A = I_n\right\}$.
(5) The unitary group $U(n) = \{A \in M_n(\mathbb{C}) \mid A^* A = I_n\}$.
(6) The group of $n \times n$ permutation matrices. (Here $\mathbb{F}$ can be any field.)

This last example also gives a representation, called the **natural representation**, of the symmetric group S_n over $\mathbb{F}$: the action of S_n on $\mathbb{F}^n$ is given by $\rho_\sigma(e_i) = e_{\sigma(i)}$.

Example 1.3. Suppose that G acts on a set X. Denote by $\mathbb{F}^X = \{f : X \to \mathbb{F}\}$ the free vector space on X, with basis elements e_x given by $e_x(y) = \delta_{x,y}$. Then the map $\rho : G \to GL(\mathbb{F}^X)$ given by $\rho_g(e_x) = e_{gx}$ is a representation.

The simplest instance of this is again the natural representation of S_n, induced by the standard action of S_n on the set $\{1, \dots, n\}$.

Example 1.4. The symmetric group S_n acts on the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$ in the following way: if $\sigma \in S_n$, then $\sigma(x_{i_1} \cdots x_{i_n}) = x_{\sigma(i_1)} \cdots x_{\sigma(i_k)}$. This action then extends to a representation on $\mathbb{F}[x_1, \dots, x_n]$ (which is, in particular, the free $\mathbb{F}$-vector space on the set of monomials in $x_1, \dots, x_n$).

Example 1.5. Recall that the dihedral group D_{2n} of order $2n$ is the group of symmetries of a regular n-gon. If the n-gon is centered at the origin of $\mathbb{R}^2$, then these symmetries can be extended to be linear endomorphisms of $\mathbb{R}^2$. This defines a representation of D_{2n} on $\mathbb{R}^2$.

Example 1.6. Define $\rho : \mathbb{R} \to GL_2(\mathbb{R})$ by $\rho_t = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$. That is, ρ_t is a counterclockwise rotation of $\mathbb{R}^2$ by t radians (deliberately blurring the distinction between an endomorphism of $\mathbb{R}^2$ and the matrix representing it). Then $\rho_{s+t} = \rho_s \rho_t$. Thus ρ defines a representation of the additive group $\mathbb{R}$ on the vector space $\mathbb{R}^2$.

Definition 1.7. The **trivial** $\mathbb{F}$-representation of G is given by $\tau(g) = \mathrm{Id}_{\mathbb{F}}$, on the 1-dimensional vector space $\mathbb{F}$.

Note that the trivial representation is *not* defined on the trivial vector space $\{0\}$. The unique representation on the zero space is too degenerate to have a standard name.

Example 1.8. The symmetric group S_n also has a nontrivial one-dimensional representation ε (at least when $\mathbb{F}$ doesn't have characteristic 2), give by $\varepsilon(\sigma) = \operatorname{sgn}(\sigma)$, the sign of the permutation σ.

In the rest of these notes we will mostly restrict attention to finite-dimensional representations of finite groups, always on nonzero vector spaces. Eventually we will also restrict attention to $\mathbb{F} = \mathbb{C}$, for reasons that will be explained, but for now we will allow $\mathbb{F}$ to be arbitrary.

Exercises.

1.1. If $f : G_1 \to G_2$ is a group homomorphism and $\rho : G_2 \to GL(V)$ is a representation of G_2, the **pullback** of ρ by f is the representation

$$f^*\rho := \rho \circ f : G_1 \to GL(V)$$

of G_1.

Recall that (the proof of) Cayley's theorem involves the homomorphism $f : G \to S_G$ (the symmetric group on the set G) given by $\big(f(g)\big)(h) = gh$. Describe explicitly the pullback of the natural representation of S_G on $\mathbb{F}^G$ by this homomorphism.

1.2. Let (V, ρ) be an $\mathbb{F}$-representation of G, and let $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ be the dual vector space of V. Define $\rho^* : G \to \mathrm{End}(V^*)$ by

$$\rho^*(g) = (\rho_{g^{-1}})^* \in \mathrm{End}(V^*),$$

where the latter denotes the adjoint of the linear map $\rho_{g^{-1}} \in \mathrm{End}(V)$. Show that ρ^* is an $\mathbb{F}$-representation of G. This representation is called the **dual representation** to (V, ρ).

1.3. Show that if $H < G$ is an abelian subgroup and ρ is a one-dimensional representation of G, then $\rho|_H$ is the trivial representation of H.

2. Representations as modules

Recall that if G is a group and $\mathbb{F}$ is a field, the **group algebra** $\mathbb{F}(G)$ is the free $\mathbb{F}$-vector space $\mathbb{F}^G$ spanned by elements e_g, equipped with a multiplication operation given by $e_g e_h = e_{gh}$. That is, elements of $\mathbb{F}(G)$ are sums of the form $\sum_{g\in G} a_g e_g$, where $a_g \in \mathbb{F}$ is nonzero for only finitely many $g \in G$, and

$$\left(\sum_{g\in G} a_g e_g\right)\left(\sum_{h\in G} b_h e_h\right) = \sum_{g,h\in G} (a_g b_h) e_{gh}.$$

This multiplication operation makes $\mathbb{F}(G)$ into both a ring and a vector space, with compatible structures, hence an algebra.

(Because of the frequent use of e to denote basis vectors in vector spaces, we will use 1 to denote the identity of multiplicative groups.)

Proposition 2.1. *An $\mathbb{F}$-representation of G is "the same as" a module over the ring $\mathbb{F}(G)$. More precisely, there is a one-to-one correspondence between $\mathbb{F}$-representations of G and $\mathbb{F}(G)$-modules, given as follows:*

Let (V, ρ) be an $\mathbb{F}$-representation of G. Then V can be given an $\mathbb{F}(G)$-module structure by:

$$\left(\sum_{g\in G} a_g e_g\right) v := \sum_{g\in G} a_g \rho_g(v).$$

Conversely, if V is an $\mathbb{F}(G)$-module, then V is in particular an $\mathbb{F}$-vector space via $av := (a1_G)v$, and we can define a representation $\rho : G \to GL(V)$ by

$$\rho_g(v) := (1_{\mathbb{F}} g)v.$$

Sketch of proof. What needs to be checked here is:

(1) in the first case, that V is in fact an $\mathbb{F}(G)$-module;
(2) in the second case, that ρ is in fact a group homomorphism;
(3) finally, that these associations are inverse to each other.

All these follow easily from definitions; checking the details yourself is a far more valuable experience than seeing them written down. □

The equivalence in Proposition 2.1 is a great example of the value of introducing multiple viewpoints on a mathematical object. The reasons for thinking about group representations in the first place mostly have to do with the original definition (group action by linear maps); coming from that perspective, considering modules over the somewhat exotic-seeming ring $\mathbb{F}(G)$ may sound like a strange thing to do. But once we realize that group actions by linear maps are really the same thing as modules over group algebras, we can use our knowledge of modules to clarify lots of basic facts and constructions that may seem more mysterious in terms of the group action perspective. A first instance is the following fundamental example of a representation, which is immediately suggested by the module-theoretic perspective.[2]

Example 2.2. The **regular representation** of G over $\mathbb{F}$ is given by the $\mathbb{F}(G)$-module $\mathbb{F}(G)$ itself. Thus $V = \mathbb{F}(G)$ has a basis $\{e_g \mid g \in G\}$ and $\rho : G \to GL(V)$ is defined by $\rho_g(e_h) = e_{gh}$.

As we will see, the regular representation of G turns out to play a central role in the general theory of representations of G.

The module perspective also helps to make the following notions a bit more transparent.

Definition 2.3. A **subrepresentation** of an $\mathbb{F}$-representation of G is an $\mathbb{F}(G)$-submodule.

Equivalently, a subrepresentation of (V, ρ) is an $\mathbb{F}$-subspace of V which is invariant under all the linear maps $\{\rho_g \mid g \in G\}$.

Example 2.4. $V_0 = \{x \in \mathbb{F}^n \mid \sum_{i=1}^n x_i = 0\}$ is a subrepresentation of the natural representation of S_n, referred to as the **standard representation** of S_n.

Definition 2.5. The **direct sum** of two $\mathbb{F}$-representations of G is their direct sum as $\mathbb{F}(G)$-modules.

Equivalently, the (external) direct sum of (V_1, ρ_1) and (V_2, ρ_2) is $(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$, where $\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2)$ is defined by $(\rho_1 \oplus \rho_2)_g = (\rho_1)_g \oplus (\rho_2)_g$; that is,

$$(\rho_1 \oplus \rho_2)_g(v_1, v_2) = (\rho_1(v_1), \rho_2(v_2)).$$

The notation can start to get a bit hairy, but notice that in essence, everything is the simplest thing it possibly could be.

Notice in particular that if $V_1 = \mathbb{F}^n$ and $V_2 = \mathbb{F}^m$, so that $(\rho_1)_g$ is given by an $n \times n$ matrix and $(\rho_2)_g$ is given by an $m \times m$ matrix, then $(\rho_1 \oplus \rho_2)_g$ is given by the $(n+m) \times (n+m)$ block matrix

$$\begin{bmatrix} (\rho_1)_g & 0 \\ 0 & (\rho_2)_g \end{bmatrix}.$$

[2]Exercise **1.1.** suggests a more group-theoretic way to arrive at the regular representation.

Definition 2.6. An **intertwining operator** between $\mathbb{F}$-representations V_1 and V_2 of G is an $\mathbb{F}(G)$-module homomorphism.

Equivalently, if (V_1, ρ_1) and (V_2, ρ_2) are $\mathbb{F}$-representations of G, an **intertwining** operator (or **intertwiner**) is a linear map $\varphi : V_1 \to V_2$ such that

$$(\rho_2)_g \circ \varphi = \varphi \circ (\rho_1)_g \tag{1}$$

for each $g \in G$.

The set of intertwining operators is denoted $\operatorname{Hom}^G(V_1, V_2)$.

The idea of (1) is that φ "commutes with the action of G". The scare quotes are there since there are in general actually two different actions of G on two different spaces; this commutation is in the sense of the commutative diagram

$$\begin{tikzcd} V_1 \arrow[r, "(\rho_1)_g"] \arrow[d, "\varphi"] & V_1 \arrow[d, "\varphi"] \\ V_2 \arrow[r, "(\rho_2)_g"] & V_2 \end{tikzcd}$$

as opposed to the sense of elements of a group or ring commuting with each other. We sometimes abuse notation and write g for the linear map ρ_g when working with a representation ρ; by a further abuse of notation we may then write (1) more simply as

$$g\varphi = \varphi g.$$

The next lemma gives a useful way to produce intertwining operators.

Lemma 2.7. *Let (V_1, ρ_1) and (V_2, ρ_2) be two $\mathbb{F}$-representations of a finite group G and let $\varphi : V_1 \to V_2$ be an $\mathbb{F}$-linear map. Define $\widetilde{\varphi} : V_1 \to V_2$ by*

$$\widetilde{\varphi} = \sum_{g \in G} (\rho_2)_g \circ T \circ (\rho_1)_g^{-1}.$$

Then $\widetilde{\varphi} \in \operatorname{Hom}^G(V_1, V_2)$.

Proof. Let $g \in G$. Then, making the substitution $k = g^{-1}h$,

$$\widetilde{\varphi} \circ (\rho_1)_g = \sum_{h \in G} (\rho_2)_h \circ \varphi \circ (\rho_1)_{h^{-1}g} = \sum_{k \in G} (\rho_2)_{gk} \circ \varphi \circ (\rho_1)_{k^{-1}} = (\rho_2)_g \circ \widetilde{\varphi}$$

since ρ_1 and ρ_2 are group homomorphisms. □

If $\operatorname{char} \mathbb{F} = 0$, or more generally if $\operatorname{char} \mathbb{F} \nmid |G|$, we often work with

$$\varphi_G = |G|^{-1} \sum_{g \in G} (\rho_2)_g \circ T \circ (\rho_1)_g^{-1}, \tag{2}$$

which in the characteristic 0 case is an average over G. This technique of averaging over G is indispensable in this subject.

Definition 2.8. Two $\mathbb{F}$-representations (V_1, ρ_1) and (V_2, ρ_2) of G are **isomorphic** if the corresponding $\mathbb{F}(G)$-modules are isomorphic.

Equivalently, (V_1, ρ_1) and (V_2, ρ_2) are isomorphic if there is an intertwining isomorphism $\varphi : V_1 \to V_2$ of $\mathbb{F}$-vector spaces.

The following proposition gives a practical criterion for isomorphism. The proof is left as an exercise.

Proposition 2.9. *Let (V_1, ρ_1) and (V_2, ρ_2) be two $\mathbb{F}$-representations of G, and let $\mathcal{B}_i$ be a basis of V_i for $i = 1, 2$. Then V_1 and V_2 are isomorphic representations of G if and only if there exists a matrix $S \in GL_n(\mathbb{F})$ such that for every $g \in G$,*

$$[(\rho_1)_g]_{\mathcal{B}_1} = S[(\rho_2)_g]_{\mathcal{B}_2} S^{-1}.$$

(Here $[T]_{\mathcal{B}}$ denotes the matrix of a linear operator $T \in \mathrm{End}(V)$ with respect to the basis $\mathcal{B}$ of V.)

Example 2.10 (Complex representations of finite cyclic groups)**.** Let's classify all finite-dimensional complex representations (V, ρ) of all finite cyclic groups up to isomorphism. It is more convenient to use multiplicative notation for groups here, so let $G = \langle g \rangle$, where g has order n. Up to isomorphism we can assume that $V = \mathbb{F}^m$, so that ρ_g is represented by a matrix $A \in M_m(\mathbb{F})$. Since $\rho_{g^k} = (\rho_g)^k$, it is enough to determine the form of A.

Since $g^n = 1$, we have $\rho_g^n = I$, so $A^n = I_m$. Therefore the minimal polynomial p of A divides

$$x^n - 1 = \prod_{k=0}^{n-1} (x - e^{2\pi ik/n}),$$

and thus p is a product of distinct linear factors. This implies that the Jordan canonical form of A is diagonal, with eigenvalues of the form $e^{2\pi ik/n}$. Therefore ρ is isomorphic to a representation given by

$$\rho_g = \mathrm{diag}(e^{2\pi ik_1/n}, \dots, e^{2\pi ik_m/n})$$

for some $k_1, \dots, k_m \in \{0, \dots, n-1\}$.

That is, any complex representation of a cyclic group $G = \langle g \rangle$ of order n is (isomorphic to) a direct sum of 1-dimensional representations, which are each given by $\rho_g = e^{2\pi ik/n}$ for some $k \in \{0, \dots, n-1\}$.

It's important not to get too carried away by the module perspective, however. The next definition involves introducing an $\mathbb{F}(G)$-module structure after tensoring modules only over $\mathbb{F}$. This is different from what you would get by, say, tensoring $\mathbb{F}(G)$-modules over $\mathbb{F}(G)$. (A similar phenomenon is illustrated by the dual representation V^* introduced in Exercise **1.2.**; the vector space V^* is $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$, not $\mathrm{Hom}_{\mathbb{F}(G)}(V, \mathbb{F}(G))$.)

Lemma 2.11. *Suppose that (V_1, ρ_1) and (V_2, ρ_2) are $\mathbb{F}$-representations of G. Then the map*

$$\rho_1 \otimes \rho_2 : G \to \mathrm{End}(V_1 \otimes_{\mathbb{F}} V_2)$$

given by

$$(\rho_1 \otimes \rho_2)(g) = (\rho_1)_g \otimes (\rho_2)_g$$

defines an $\mathbb{F}$-representation of G on $V_1 \otimes_{\mathbb{F}} V_2$.

Equivalently, if V_1 and V_2 are $\mathbb{F}(G)$-modules, then the $\mathbb{F}$-vector space $V_1 \otimes_{\mathbb{F}} V_2$ has an $\mathbb{F}(G)$-module structure such that

$$\left(\sum_{g \in G} a_g g\right)(v_1 \otimes v_2) = \sum_{g \in G} a_g (gv_1) \otimes (gv_2) \tag{3}$$

for any $v_1 \in V_1$ and $v_2 \in V_2$.

Proof. Here it's easier to prove the first statement and use Proposition 2.1 to conclude the second. (The difficulty with proving the second statement directly is the same one that always arises with tensor products: since not all elements of $V_1 \otimes_{\mathbb{F}} V_2$ are simple tensors $v_1 \otimes v_2$, and representations of arbitrary elements as sums of simple tensors are not unique, the well-definedness of the operation given in (3) is not obvious.)

To verify the first statement, given $g, h \in G$,

$$\begin{aligned}(\rho_1 \otimes \rho_2)(gh) &= (\rho_1)_{gh} \otimes (\rho_2)_{gh} \\ &= \big((\rho_1)_g(\rho_1)_h\big) \otimes \big((\rho_2)_g(\rho_2)_h\big) \\ &= \big((\rho_1)_g \otimes (\rho_2)_g\big)\big((\rho_1)_h \otimes (\rho_2)_h\big) \\ &= \big((\rho_1 \otimes \rho_2)(g)\big)\big((\rho_1 \otimes \rho_2)(h)\big),\end{aligned} \tag{4}$$

by the definition of $\rho_1 \otimes \rho_2$, the fact that ρ_1 and ρ_2 are homomorphisms, and general properties of the tensor product of linear maps. By the last three lines in (4), it also follows that

$$\big((\rho_1 \otimes \rho_2)(g)\big)\big((\rho_1 \otimes \rho_2)(g^{-1})\big) = \big((\rho_1)_g(\rho_1)_{g^{-1}}\big) \otimes \big((\rho_2)_g(\rho_2)_{g^{-1}}\big) = \mathrm{Id}_{V_1} \otimes \mathrm{Id}_{V_2} = \mathrm{Id}_{V_1 \otimes V_2}\,. \tag{5}$$

It follows from (5) that $\rho_1 \otimes \rho_2$ maps into $GL(V_1 \otimes_{\mathbb{F}} V_2)$, and then from (4) that $\rho_1 \otimes \rho_2$ is a group homomorphism. □

Definition 2.12. The **tensor product** of two $\mathbb{F}$-representations (V_1, ρ_1) and (V_2, ρ_2) of G is the representation $(V_1 \otimes_{\mathbb{F}} V_2, \rho_1 \otimes \rho_2)$ defined as in Lemma 2.11.

A different but related kind of tensor product of representations is introduced in Exercise **2.9.**.

Exercises.

2.1. Check the details in the proof of Proposition 2.1.

2.2. Prove in detail the equivalence of the two definitions in Definition 2.6.

2.3. Prove in detail the equivalence of the two definitions in Definition 2.8.

2.4. Prove Proposition 2.9.

2.5. Let $G = \langle g \rangle$ be a cyclic group of prime order p. Show that, up to isomorphism, any finite-dimensional $\mathbb{Q}$-representation of G is a direct sums of trivial representations and the representation on $\mathbb{Q}^{p-1}$ given by

$$\rho_g = \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \in M_{p-1}(\mathbb{Q}).$$

Hint: For an arbitrary representation ρ, consider the rational canonical form of ρ_g, where g, and recall that $x^{p-1} + \cdots + x + 1$ is irreducible over $\mathbb{Q}$.

2.6. An $\mathbb{F}$-representation V of G is called **indecomposable** if V is not (isomorphic to) the direct sum of two nonzero subrepresentations.

Let G be cyclic group of prime order p. Classify up to isomorphism all indecomposable finite-dimensional representations of G over $\mathbb{F}_p$.

Hint: In this case consider the Jordan canonical form of ρ_g.

2.7. Show that the natural representation of S_n is (isomorphic to) the direct sum of the standard representation V_0 defined in Example 2.4 and the trivial representation.

2.8. A real or complex representation (V, ρ) of a group G is called a **unitary representation** if V is an inner product space and for each $g \in G$, the linear map $\rho_g \in GL(V)$ is unitary.

Prove that if G is finite and (V, ρ) is any finite-dimensional real or complex representation of G, then there exists an inner product on V such that V is a unitary representation of G.

Hint: Start by picking *any* inner product on V (you can do this by picking any basis and declaring it to be orthonormal) and then average the inner product over G in a suitable way.

2.9. Let (V_i, ρ_i) be an $\mathbb{F}$-representation of G_i for $i = 1, 2$. Define $\rho_1 \otimes \rho_2 : G_1 \times G_2 \to \operatorname{End}(V_1 \otimes_{\mathbb{F}} V_2)$ by

$$(\rho_1 \otimes \rho_2)(g_1, g_2) = (\rho_1)_{g_1} \otimes (\rho_2)_{g_2} \in \operatorname{End}(V_1 \otimes_{\mathbb{F}} V_2).$$

(a) Show that $\rho_1 \otimes \rho_2$ is a representation of $G_1 \times G_2$.

(b) Explain how this type of tensor product representation relates to the tensor product of two representations of a single group G as in Lemma 2.11 and Definition 2.12.

2.10. Prove that if V is an $\mathbb{F}(G)$-module, then the $\mathbb{F}$-vector space $V^* = \operatorname{Hom}_{\mathbb{F}}(V, \mathbb{F})$ has an $\mathbb{F}(G)$-module structure such that

$$\left[\left(\sum_{g \in G} a_g g\right)(f)\right](v) = f\left(\left(\sum_{g \in G} a_g g^{-1}\right)(v)\right)$$

for any $f \in V^*$ and $v \in V$.

Hint: See Exercise **1.2.**.

3. Irreducibility

From this point on, we will assume that all representations are finite-dimensional.

Definition 3.1. A representation is **irreducible** if it has no proper nonzero subrepresentations. It is **completely reducible** if it is (isomorphic to) a direct sum of irreducible representations.

Notice that in particular, all 1-dimensional representations are irreducible.

Example 3.2. Example 2.10 showed that every complex representation of a cyclic group is, up to isomorphism, a direct sum of 1-dimensional representations. Thus all irreducible complex representations of a cyclic group are 1-dimensional, and all complex representations of a cyclic group are completely reducible.

Theorem 3.3. *Suppose that G is finite and $\operatorname{char} \mathbb{F} \nmid |G|$. If W is a subrepresentation of a representation (V, ρ) of G, then there is a subrepresentation $W' \subseteq V$ such that $V = W \oplus W'$.*

Proof. Let $P : V \to V$ be a projection onto W; that is, an $\mathbb{F}$-linear map such that $P^2 = P$ and $P(V) = W$. Such a map can be constructed using bases: let $(v_1, \dots, v_m)$ be any basis of W, and extend it to a basis $(v_1, \dots, v_n)$ of V. The linear map $P : V \to V$ defined by

$$P(v_i) = \begin{cases} v_i & \text{if } 1 \le i \le m, \\ 0 & \text{if } i > m \end{cases}$$

is a projection onto W.

Now given any projection P onto W, $\ker P$ is a subspace of V, and $\dim W + \dim \ker P = \dim V$ by the Rank–Nullity Theorem. Furthermore, if $w \in W$, then for some $v \in V$, we have $w = P(v) = P^2(v) = P(w)$; thus $W \cap \ker P = \{0\}$. Therefore $V = W \oplus \ker P$.

So we would be done if we could take $W' = \ker P$. The trouble is that $\ker P$ is only an $\mathbb{F}$-subspace of V, and not necessarily an $\mathbb{F}(G)$-submodule. Put another way, P is an $\mathbb{F}$-linear map, but not necessarily an $\mathbb{F}(G)$-module endomorphism of V; that is, an intertwiner. To remedy this problem, we use Lemma 2.7.

Specifically, we define

$$Q := |G|^{-1}\,\widetilde{P} = |G|^{-1} \sum_{g \in G} \rho_g \circ P \circ \rho_g^{-1}.$$

Then $Q \in \operatorname{Hom}^G(V, V)$ by Lemma 2.7.

Since W is a subrepresentation of V, if $w \in W$ then $\rho_g^{-1}(w) \in W$ for every $g \in G$, so $P(\rho_g^{-1}(w)) = \rho_g^{-1}(w)$. It follows that $Q(w) = w$. Moreover, for each $v \in V$ and $g \in G$, $P(\rho_g^{-1}(v)) \in W$, and so $\rho_g \circ P \circ \rho_g^{-1}(v) \in W$. It follows that $Q(v) \in W$; thus $Q^2 = Q$ and $Q(V) = W$.

We now let $W' = \ker Q$. Since Q is an intertwiner, W' is a subrepresentation of V. Moreover, the arguments above show that $V = W \oplus W'$. □

(Did you see where we needed the assumption that $\operatorname{char} \mathbb{F} \nmid |G|$?)

Applying Theorem 3.3 iteratively yields the following result.

Corollary 3.4. *If G is finite and $\operatorname{char} \mathbb{F} \nmid |G|$, then every representation of G is completely reducible.*

In particular, if $\operatorname{char} \mathbb{F} = 0$, then every representation of every finite group is completely reducible.

The next result is almost trivial to prove, but turns out to be immensely important.

Theorem 3.5 (Schur's lemma)**.** *Let $\mathbb{F}$ be algebraically closed. Suppose V is an irreducible $\mathbb{F}$-representation of G, and $\varphi \in \operatorname{Hom}^G(V, V)$. Then $\varphi = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{F}$.*

Proof. Since $\mathbb{F}$ is algebraically closed, φ has an eigenvalue $\lambda \in \mathbb{F}$. Then the eigenspace $E = \ker(\varphi - \lambda \operatorname{Id}_V)$ is a nonzero $\mathbb{F}(G)$-submodule of V. Since V is irreducible, we must have $E = V$, and thus $\varphi = \lambda \operatorname{Id}_V$. □

Corollary 3.4 and Schur's lemma indicate that representation theory works most smoothly over an algebraically closed field of characteristic 0. When restricted to finite groups, we may as well work over the smallest such field, namely the field $\overline{\mathbb{Q}}$ of algebraic numbers (the algebraic closure of $\mathbb{Q}$). But it does no harm to say we're working over the larger but more familiar field $\mathbb{C}$.

For the rest of these notes, we will always assume that $\mathbb{F} = \mathbb{C}$.

Corollary 3.6. *Suppose that V and W are irreducible representations of G. If V and W are isomorphic representations, then $\mathrm{Hom}^G(V,W)$ is 1-dimensional, and every nonzero intertwiner is an isomorphism. Otherwise, $\mathrm{Hom}^G(V,W) = \{0\}$.*

Proof. Let $\varphi \in \mathrm{Hom}^G(V,W)$. Then $\ker\varphi$ and $\varphi(V)$ are subrepresentations of V, so by irreducibility, either $\varphi = 0$ or else $\ker\varphi = \{0\}$ and $\varphi(V) = W$. So if there exists any nonzero intertwiner, then it is an isomorphism.

Now suppose that $0 \neq \varphi \in \mathrm{Hom}^G(V,W)$. For any $\psi \in \mathrm{Hom}^G(V,W)$, Schur's lemma implies that $\varphi^{-1} \circ \psi = \lambda \,\mathrm{Id}_V$ for some $\lambda \in \mathbb{C}$, and so $\psi = \lambda\varphi$. Thus $\mathrm{Hom}^G(V,W)$ is spanned by φ. $\square$

Corollary 3.7. *Let (V,ρ) and (W,σ) be irreducible representations of a finite group G, and let $\varphi : V \to W$ be linear. Define φ_G as in (2). Then:*

(1) If V and W are irreducible and not isomorphic, then $\varphi_G = 0$.
(2) If $(V,\rho) = (W,\sigma)$, then $\varphi_G = \frac{\operatorname{tr}\varphi}{\dim V} I$.

Proof. Recall that by Lemma 2.7, $\varphi_G \in \mathrm{Hom}^G(V,W)$. The first statement follows from Corollary 3.6.

Now if $V = W$, then Schur's lemma implies that $\varphi_G = \lambda I$. Taking traces, $\operatorname{tr}\varphi_G = \lambda \dim_V$. Furthermore,

$$\operatorname{tr}\varphi_G = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}(\rho_g\varphi\rho_g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\operatorname{tr}\varphi = \operatorname{tr}\varphi,$$

which proves the second statement. $\square$

Corollary 3.8. *If G is finite and abelian, then every nonzero irreducible complex representation of G is 1-dimensional.*

Proof. If (V,ρ) is a representation of a finite abelian group G, then $\rho_g \in \mathrm{Hom}^G(V,V)$ for every $g \in G$. If V is irreducible, then by Schur's lemma, each ρ_g is a scalar operator on G, which implies that every subspace of V is invariant under $\{\rho_g \mid g \in G\}$. That is, every subspace of V is a subrepresentation. By irreducibility, this implies that $\dim V \leq 1$. $\square$

Exercises.

3.1. Show that $\rho_n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ defines a representation of $\mathbb{Z}$ on $\mathbb{C}^2$ which is not completely reducible.

3.2. Let $f : G_1 \to G_2$ be a group epimorphism and $\rho : G_2 \to GL(V)$ be a representation of G_2. Prove that the pullback representation $f^*\rho$ (defined in Exercise **1.1.**) is irreducible if and only if ρ is irreducible.

3.3. Let $H < G$ and let $\rho : G \to GL(V)$ be a representation of G.
(a) Show that if $\rho|_H$ is an irreducible representation of H, then ρ is irreducible.
(b) Show that the standard representation of S_3 is irreducible.
(c) Show that the converse of part (a) is false.

3.4. Determine all irreducible finite dimensional representations of the group $\mathbb{Z}_p$ over the field $\mathbb{F}_p$.

3.5. Suppose that G is a finite group and $A < G$ is abelian. Prove that each irreducible representation of G over $\mathbb{C}$ of G has dimension at most $[G : A]$.

Hint: If (V, ρ) is an irreducible representation of G, then $\rho|_A$ defines a representation of A. For an irreducible subrepresentation W of A, consider the subspaces $\rho(g)(W)$.

3.6. Show that if (V, ρ) is an irreducible representation of G and $g \in Z(G)$ (the center of G), then ρ_g is a scalar transformation of V.

4. Characters

Definition 4.1. Let (V, ρ) be a representation of G. The **character** of V is the function $\chi_V = \chi_\rho : G \to \mathbb{C}$ given by $\chi_V(g) = \operatorname{tr} \rho_g$.

If V is an irreducible representation of G, then χ_V is called an **irreducible character**.

Lemma 4.2. *If V_1 and V_2 are isomorphic representations of G, then $\chi_{V_1} = \chi_{V_2}$.*

Proof. By Proposition 2.9, if V_1 and V_2 are isomorphic, then $(\rho_1)_g$ and $(\rho_2)_g$ are represented by similar matrices with respect to any bases of V_1 and V_2. □

Remarkably, the converse of Lemma 4.2 is also true, as we will see in Corollary 5.6 below. Thus, in a sense, the character of a representation, which is a scalar-valued function on G, contains all of the information about the representation.

Note that if ρ is a 1-dimensional representation of G, then $\chi_\rho = \rho$.

Lemma 4.3. *Let V be a representation of G. Then:*

(1) $\chi_V(hgh^{-1}) = \chi_V(g)$ for every $g, h \in G$.
(2) $\chi_V(1_G) = \dim V$.
(3) If G is finite, then $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ for every $g \in G$.

Proof. The first two parts are immediate consequences of basic properties of the trace. For the third, observe that since $g \in G$ is of finite order, $\rho_g^m - I = 0$ for some m, so the eigenvalues $\lambda_1, \dots, \lambda_n$ of ρ_g are all roots of unity. Therefore

$$\chi_V(g^{-1}) = \operatorname{tr} \rho_g^{-1} = \sum_{j=1}^{n} \lambda_j^{-1} = \sum_{j=1}^{n} \overline{\lambda_j} = \overline{\chi_V(g)}.$$ □

Recall that a **class function** on G is a function $\varphi : G \to \mathbb{C}$ such that $\varphi(hgh^{-1}) = \varphi(g)$ for every $g, h \in G$. So Lemma 4.3(1) says that characters of G are class functions.

The next two results show that characters algebraic operations on the level of representations in a particularly simple way. The proof of Lemma 4.4 is left as an exercise.

Lemma 4.4. *Let V and W be representations of G. Then $\chi_{V\oplus W} = \chi_V + \chi_W$ and $\chi_{V\otimes W} = \chi_V \chi_W$.*

In Lemma 4.5, V^* denotes the dual representation defined in Exercise **1.2.**.

Lemma 4.5. *If (V, ρ) is a representation of a finite group G, then $\chi_{V^*} = \overline{\chi_V}$.*

Proof. Let $g \in G$. Since g is of finite order, say m, $\rho_g^m = \mathrm{Id}_V$, which implies that each eigenvalue λ of ρ_g is a root of unity. Therefore $\lambda^{-1} = \overline{\lambda}$. The claim follows since $\chi_V(g)$ is the sum (with multiplicities) of the eigenvalues of ρ_g. □

Exercises.

4.1. Prove Lemma 4.4.

4.2. Prove that if χ is any character of a finite group G, then χ is a linear combination of irreducible characters with nonnegative integer coefficients.

4.3. Let V and W be representations of groups G and H respectively, and let $V \otimes W$ be the representation of $G \times H$ defined in Exercise **2.9.**. Show that $\chi_{V\otimes W}(g,h) = \chi_V(g)\chi_W(h)$ for every $g \in G$, $h \in H$.

4.4. Let G be a finite group. Show that the following two statements are equivalent.
(a) Every (complex) character of G is real-valued.
(b) For every $g \in G$, g is conjugate to g^{-1}.

4.5. Let G be abelian, and define $\widehat{G}$ to be the set of all irreducible characters of G. Show that $\widehat{G}$ is an abelian group, with the binary operation given by pointwise multiplication.

5. Orthogonality of characters

From this point on we will always assume that G is finite.

Definition 5.1. For $\varphi, \psi : G \to \mathbb{C}$, define

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g\in G} \varphi(g)\overline{\psi(g)}.$$

This is an inner product on the space of functions $G \to \mathbb{C}$.

Observe that if $\varphi, \psi : G \to \mathbb{C}$ are class functions, then

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{C} |C|\, \varphi(C)\overline{\psi(C)},$$

where the sum is over the conjugacy classes C of G, and that if V and W are representations of G, then

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g\in G} \chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|} \sum_{g\in G} \chi_V(g)\chi_W(g^{-1}).$$

Theorem 5.2 (First orthogonality relation for characters)**.** *If (V,ρ) and (W,σ) are irreducible representations of G, then*

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \textit{if } V \textit{ and } W \textit{ are isomorphic,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since isomorphic representations have equal characters, we may assume that (V,ρ) and (W,σ) are either equal, or else nonisomorphic.

Pick bases $(v_1,\dots,v_n)$ of V and $(w_1,\dots,w_m)$ of W, and let $[\rho_g]$ and $[\sigma_g^{-1}]$ denote the matrices representing ρ_g and σ_g^{-1} with respect to the appropriate bases. Denote by E_{ij} the

$m \times n$ matrix whose i, j entry is 1 and whose other entries are all 0. Then

$$\begin{aligned}\langle \chi_V, \chi_W \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_W(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \big(\operatorname{tr} \rho_g\big) \big(\operatorname{tr} \sigma_g^{-1}\big) \\ &= \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{m} \sum_{j=1}^{n} [\rho_g]_{ii} [\sigma_g^{-1}]_{jj} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{|G|} \sum_{g \in G} \big([\rho_g] E_{ij} [\sigma_g^{-1}]\big)_{ij}.\end{aligned}$$

Now E_{ij} is the matrix, with respect to our bases, of the linear map $\varphi : V \to W$ defined by

$$\varphi(v_k) = \delta_{ik} w_j,$$

and then

$$\frac{1}{|G|} \sum_{g \in G} [\rho_g] E_{ij} [\sigma_g^{-1}]$$

is the matrix of φ_G, as defined in (2). If V and W are not isomorphic, then this matrix is 0 by Lemma 3.7. On the other hand, if $(V, \rho) = (W, \sigma)$, then Lemma 3.7 implies that

$$\frac{1}{|G|} \sum_{g \in G} [\rho_g] E_{ij} [\sigma_g^{-1}] = \frac{\operatorname{tr} E_{ij}}{m} I_m = \frac{\delta_{ij}}{m} I_m,$$

and so $\langle \chi_V, \chi_W \rangle = 1$. □

Theorem 5.2 has a multitude of important applications.

Corollary 5.3. *The number of distinct (up to isomorphism) irreducible representations of G is less than or equal to the number of conjugacy classes in G.*

In particular, there exist only finitely many distinct irreducible representations of G.

Proof. Theorem 5.2 implies that distinct irreducible characters are linearly independent elements of the vector space of class functions on G. The dimension of that vector space is clearly equal to the number of conjugacy classes in G. □

We will see in Corollary 6.6 that the number of irreducible representations of G is actually *equal* to the number of conjugacy classes in G.

Fix a complete list $V_1, \ldots, V_m$ of distinct (up to isomorphism) irreducible representations of G. If V is any representation, then since V is completely reducible,

$$V \cong \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{n_k} V_k \tag{6}$$

for some $n_1, \ldots, n_k \geq 0$. (A term in which $n_k = 0$ is understood to be omitted.) We abbreviate this as $V \cong \bigoplus_{k=1}^{m} n_k V_k$. This is called a **canonical decomposition** of V, and the summands $n_k V_k = \bigoplus_{j=1}^{n_k} V_k$ are the **isotypical components** of V. Note that by Lemma 4.4,

$$\chi_V = \sum_{k=1}^{m} n_k \chi_{V_k}. \tag{7}$$

Corollary 5.4. *Let W be an irreducible representation of G, and let V be any representation of V. Let*

$$V \cong \bigoplus_{k=1}^{m} W_k$$

be any decomposition of V as a direct sum of irreducible representations. Then $\langle \chi_V, \chi_W \rangle$ is equal to the number of k such that $W_k \cong W$.

In particular, if $V = \bigoplus_{k=1}^m n_k V_k$ is a canonical decomposition of V, then $n_k = \langle \chi_V, \chi_{V_k} \rangle$.

Proof. This follows directly from Lemma 4.4 and Theorem 5.2. □

Corollary 5.4 shows that the numbers n_k in a canonical decomposition of V depend only on V. The canonical decomposition is unique in this sense.

We call the quantity $\langle \chi_V, \chi_W \rangle$ in Corollary 5.4 the the **multiplicity** of V in W. The following result is immediate.

Corollary 5.5. *The multiplicity in V of the trivial representation of G is $\frac{1}{|G|} \sum_{g \in G} \chi_V(g)$.*

Corollary 5.4 allows us to prove the converse of Lemma 4.2.

Corollary 5.6. *Let V and W be representations of G. Then V and W are isomorphic if and only if $\chi_V = \chi_W$.*

Proof. Let $V \cong \sum_{k=1}^m n_k V_k$ and $W \cong \sum_{k=1}^m n_k' V_k$ be canonical decompositions. If $\chi_V = \chi_W$, then Corollary 5.4 implies that $n_k = n_k'$ for every k, and hence $V \cong W$. □

Corollary 5.7. *Let $V \cong \bigoplus_{k=1}^m n_k V_k$ be a canonical decomposition of V. Then $\langle \chi_V, \chi_V \rangle = \sum_{k=1}^m n_k^2$.*

Proof. This follows immediately from from Theorem 5.2 and (7). □

Corollary 5.8. *A representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.*

Proof. If V is irreducible, the $\langle \chi_V, \chi_V \rangle = 1$.

Now suppose $\langle \chi_V, \chi_V \rangle = 1$. Let $V \cong \bigoplus_{k=1}^m n_k V_k$ be a canonical decomposition of V. Corollary 5.7 implies that $\sum_{k=1}^m n_k^2 = 1$, which implies that and so $V \cong V_k$ for some k. Hence V is irreducible. □

Example 5.9. We will use Corollary 5.8 to show that the standard representation V_0 of S_n (defined in Example 2.4) is irreducible.

The case $n = 3$ of this, which is easy to do without characters, is Exercise **3.3.**. The general case can also be done without characters, but Corollary 5.8 reduces it to an easy computation.

First, by Exercise **2.7.**, the natural representation of S_n is the direct sum of V_0 and the trivial representation τ. Therefore the character χ of the natural representation satisfies

$\chi = \chi_\tau + \chi_{V_0}$. The trivial character χ_τ is always equal to 1, and $\chi(\pi)$ is equal to the number $f(\pi)$ of fixed points of π. Hence

$$\chi_{V_0}(\pi) = f(\pi) - 1.$$

Now

$$\langle \chi_{V_0}, \chi_{V_0} \rangle = \langle f-1, f-1 \rangle = \langle f, f \rangle - 2\langle f, 1 \rangle + 1.$$

Now

$$\begin{aligned}\langle f, 1 \rangle &= \frac{1}{n!}\sum_{\pi \in S_n} f(\pi) = \frac{1}{n!}\sum_{\pi \in S_n}\sum_{k=1}^{n} \delta_{k,\pi(k)} = \frac{1}{n!}\sum_{k=1}^{n}\sum_{\pi \in S_n} \delta_{k,\pi(k)} \\ &= \frac{1}{n!}\sum_{k=1}^{n} |\{\pi \in S_n \mid \pi(k) = k\}| = \frac{1}{n!}\sum_{k=1}^{n}(n-1)! = 1,\end{aligned}$$

and

$$\begin{aligned}\langle f, f \rangle &= \frac{1}{n!}\sum_{\pi \in S_n} f(\pi)^2 = \frac{1}{n!}\sum_{\pi \in S_n}\left(\sum_{k=1}^{n}\delta_{k,\pi(k)}\right)^2 = \frac{1}{n!}\sum_{\pi \in S_n}\sum_{j=1}^{n}\sum_{k=1}^{n}\delta_{j,\pi(j)}\delta_{k,\pi(k)} \\ &= \frac{1}{n!}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{\pi \in S_n}\delta_{j,\pi(j)}\delta_{k,\pi(k)} = \frac{1}{n!}\sum_{j=1}^{n}\sum_{k=1}^{n}|\{\pi \in S_n \mid \pi(j) = j \text{ and } \pi(k) = k\}| \\ &= \frac{1}{n!}\sum_{j=1}^{n}(n-1)! + \frac{1}{n!}\sum_{j=1}^{n}\sum_{k \neq j}(n-2)! = \frac{1}{n!}n(n-1)! + \frac{1}{n!}n(n-1)(n-2)! = 2.\end{aligned}$$

It follows that $\langle \chi_{V_0}, \chi_{V_0} \rangle = 1$, and therefore V_0 is an irreducible representation.

Corollary 5.10. *A representation V is irreducible if and only if the dual representation V^* is irreducible.*

Proof. This follows from Corollary 5.8 and Lemma 4.5. □

Exercises.

5.1. Use Schur's lemma to give another proof of the uniqueness of the canonical decomposition without using characters.

5.2. Show that if (V, ρ) is an irreducible representation of G and (W, σ) is a 1-dimensional representation of G, then the representation $V \otimes W$ of G is irreducible.

5.3. Show that if V and W are irreducible representations of G and H, respectively, then the representation $V \otimes W$ of $G \times H$ (defined in Exercise **2.9.**) is also irreducible.

5.4. Use character theory to prove both of the following statements.

(a) For a function $f : \{0, 1, \ldots, n-1\} \to \mathbb{C}$, the **discrete Fourier transform** is the function $\widehat{f} : \{0, 1, \ldots, n-1\} \to \mathbb{C}$ defined by

$$\widehat{f}(k) = \frac{1}{n}\sum_{j=0}^{n-1} f(j)e^{2\pi ijk/n}.$$

Then for any such function f,

$$f(j) = \sum_{k=0}^{n-1} \widehat{f}(k) e^{-2\pi i jk/n}.$$

(b) For a function $f : \{0,1\}^n \to \mathbb{C}$, the **Walsh–Hadamard transform** is the function $\widehat{f}$ defined on subsets $J \subseteq \{1, \dots, n\}$ by

$$\widehat{f}(J) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} f(x)(-1)^{\sum_{j\in J} x_j}.$$

Then for any such function f,

$$f(x) = \sum_{J \subseteq \{1,\dots,n\}} \widehat{f}(J)(-1)^{\sum_{j\in J} x_j}.$$

6. The number and dimensions of irreducible representations

We next consider the decomposition of the regular representation into irreducible representations.

Lemma 6.1. *The character of the regular representation of G is given by*

$$\chi_R(g) = \begin{cases} |G| & \text{if } g = 1_G, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Letting $(\mathbb{F}^G, \rho)$ denote the regular representation, with respect to the basis $\{e_g\}$ of $\mathbb{F}^G$, ρ_g is represented by the matrix $[\rho_g]_{h,h'} = \delta_{h', g^{-1}h}$. The $|G|$ diagonal entries $[\rho_g]_{h,h} = \delta_{h,g^{-1}h}$ are all 1 if $g = 1_G$ and all 0 otherwise. □

Theorem 6.2. *Let V be any irreducible representation of G. Then V is isomorphic to a subrepresentation of the regular representation R of G, with multiplicity equal to* $\dim V$*.*

Proof. By Lemma 6.1,

$$\langle \chi_V, \chi_R \rangle = \frac{1}{|G|} \chi_V(1_G)\chi_R(1_G) = \chi_V(1_G) = \dim V,$$

so the theorem follows from Corollary 5.4. □

Corollary 6.3 (Frobenius's theorem)**.** *Let $V_1, \dots, V_m$ be a complete set of representatives of isomorphism classes of irreducible representations of G. Then*

$$\sum_{k=1}^{m} (\dim V_k)^2 = |G|\,.$$

Proof. By Theorem 6.2,

$$\chi_R = \sum_{k=1}^{m} (\dim V_i) \chi_{V_i},$$

and so

$$\sum_{k=1}^{m} (\dim V_k)^2 = \langle \chi_R, \chi_R \rangle = |G|$$

by Theorem 5.2 and Lemma 6.1. □

Although it's tempting on first exposure to module theory to think that it's just like linear algebra, only with more general sorts of scalars, Frobenius's theorem shows how radically differently module theory can be depending on the scalar ring. In ordinary linear algebra over a field $\mathbb{F}$, every module (vector space) is a direct sum of copies of $\mathbb{F}$. On the other hand, Frobenius's theorem shows that each irreducible module over the group ring $\mathbb{C}(G)$ *is contained in* the scalar ring $\mathbb{C}(G)$!

Example 6.4. Let $d_1, \ldots, d_m$ be the dimensions of the distinct irreducible representations of S_3. These satisfy $d_1^2 + \cdots + d_m^2 = 6$ by Frobenius's theorem. We already know the trivial representation with $d_1 = 1$, the sign representation with $d_2 = 1$, and the standard representation with $d_3 = 2$. Since $1^2 + 1^2 + 2^2 = 6$, these are all of the distinct irreducible representations of S_3.

Theorem 6.5. *Let $\chi_1, \ldots, \chi_m$ be the distinct irreducible characters of G. If $\varphi : G \to \mathbb{C}$ is a class function, then*

$$\varphi = \sum_{k=1}^{m} \langle \varphi, \chi_k \rangle \, \chi_k.$$

Proof. Let

$$\psi = \varphi - \sum_{k=1}^{m} \langle \varphi, \chi_k \rangle \, \chi_k.$$

Then ψ is a class function and $\langle \chi_k, \psi \rangle = 0$ for every k. Let (V, ρ) be any representation of G, and define $\xi_V : V \to V$ by

$$\xi_V = \frac{1}{|G|} \sum_{g \in G} \psi(g) \rho_g. \tag{8}$$

Then $\xi_V \in \operatorname{Hom}^G(V, V)$ (the proof is left as an exercise — notice that we're averaging again!). If V is irreducible, then by Schur's lemma, $\xi_V = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{C}$. But

$$\operatorname{tr} \xi_V = \frac{1}{|G|} \sum_{g \in G} \psi(g) \chi_V(g) = \langle \psi, \chi_{V^*} \rangle = 0,$$

so $\lambda = 0$ and hence $\xi_V = 0$. By complete reducibility, it follows that $\xi_V = 0$ also when V is not irreducible. In particular, when V is the regular representation of G,

$$0 = \xi_V e_1 = \frac{1}{|G|} \sum_{g \in G} \psi(g) e_g,$$

which implies that $\psi = 0$. $\square$

Corollary 6.6. *The number of distinct irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. Clearly the dimension of the space of class functions of G is equal to the number of conjugacy classes. Theorem 5.2 implies that the irreducible characters of G are linearly independent, and Theorem 6.5 implies that they span the space of class functions, so that they form a basis of that space. $\square$

Example 6.7. Since S_3 has three conjugacy classes (given by the distinct cycle types, which are themselves given by distinct partitions of 3: $3 = 2 + 1 = 1 + 1 + 1$), this gives another way to see that the three irreducible representations of S_3 that we've already seen are in fact all of them.

Example 6.8. S_4 has five conjugacy classes ($4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1+1$), and therefore five distinct irreducible representations. Their dimensions $d_1, \dots, d_5$ satisfy $d_1^2 + \cdots + d_5^2 = 24$. We already know the one-dimensional trivial and sign representations, and the three-dimensional standard representation. That means there are two additional representations whose dimensions satisfy $d_4^2 + d_5^2 = 13$. The only way to write 13 as a sum of two squares is as $2^2 + 3^2$, so the other two irreducible representations are two- and three-dimensional.

We can actually identify the other three-dimensional representation. If ε denotes the sign representation and ψ denotes the standard representation, then by Exercise **5.2.**, $\psi \otimes \varepsilon$ is a three-dimensional irreducible representation. We can verify that $\psi \otimes \varepsilon$ is not isomorphic to ψ by considering the character. As observed in Example 5.9, $\chi_\psi(\pi) = f(\pi) - 1$ for a permutation π. Therefore $\chi_\psi((12)) = 1$, but $\chi_\varepsilon((12)) = \operatorname{sgn}(12) = -1$, so $\chi_{\psi\otimes\varepsilon}((12)) = -1 \neq \chi_\psi((12))$.

We also have enough information to describe the two-dimensional representation σ. Recall that $V = \{\iota, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4. Then S_4/V is a group of order 6, which must be noncyclic because every element of S_4 has order at most 4. Thus $S_4/V \cong S_3$, so there exists a group epimorphism $\varphi : S_4 \to S_3$. By Exercise **3.2.**, the pullback by φ of the standard representation of S_3 is an irreducible two-dimensional representation of S_4.

Recall that the **commutator subgroup** of G is the subgroup G' generated by all elements of the form $ghg^{-1}h^{-1}$. The commutator subgroup is normal and G/G' is abelian.

Theorem 6.9. *The number of one-dimensional representations of G is $[G : G']$.*

Proof. Write $m = [G : G']$. Since G/G' is abelian, it has m distinct one-dimensional irreducible representations, and their pullbacks (via the canonical projection $G \to G/G'$) give m distinct one-dimensional representations of G.

On the other hand, if $\rho : G \to GL(\mathbb{C}) = \mathbb{C}^*$ is a one-dimensional representation of G, then $\rho = 1$ on G', and so ρ induces a representation $\widetilde{\rho} : G/G' \to \mathbb{C}^*$ whose pullback to G is ρ. Thus the m representations given above are all the distinct one-dimensional representations of G. $\square$

Notice that the proof of Theorem 6.9 implicitly describes all the one-dimensional representations of G: they are all pullbacks via the canonical projection of representations of the abelian group G/G', which are easy to describe (Exercise **6.3.**).

Exercises.

6.1. Prove that the linear map $\xi_V \in \operatorname{End}(V)$ defined in (8) is an intertwiner.

6.2. Show that every irreducible representation of $G \times H$ is isomorphic to one of the form $V \otimes W$ (as defined in Exercise **2.9.**), where V and W are irreducible representations of G and H, respectively.

6.3. Let G be a finite abelian group. Describe all irreducible representations of G.
Hint: Decompose G in terms of invariant factors or elementary divisors, and use Exercise **6.2.**.

6.4. Prove that if every irreducible representation of G is one-dimensional, then G is abelian.

6.5. The **quaternion group** Q_8 is the set $\{1, -1, i, -i, j, -j, k, -k\}$ with multiplication defined by

$$\begin{aligned} i^2 = j^2 = k^2 &= -1, \\ ij = k &= (-1)ji, \\ jk = i &= (-1)kj, \\ ki = j &= (-1)ik, \end{aligned}$$

with 1 as the identity and -1 behaving as you'd expect ($(-1)^2 = 1$, $(-1)i = -i = i(-1)$, etc.).
Determine the dimensions of all the irreducible representations of Q_8.

6.6. Find an explicit group epimorphism $\varphi : S_4 \to S_3$, and use it to give an explicit description of an irreducible representation of S_4 on the two-dimensional space $\{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}$.

6.7. Show that for any $n \geq 2$, the trivial representation τ and the sign representation ε are the only one-dimensional representations of S_n.

6.8. Determine all the one-dimensional representations of A_4.

6.9. A conjugacy class C in G is called **self-inverse** if, for every $g \in C$, $g^{-1} \in C$ as well. Show that the number of real-valued irreducible characters of G is equal to the number of self-inverse conjugacy classes C.

7. Character tables

Definition 7.1. The **character table** of a group G is a two-dimensional array with rows corresponding to the irreducible characters χ of G and columns corresponding to the conjugacy classes C of G, whose entry in row χ and column C is given by $\chi(C)$.

Note that by Corollary 6.6, the character table has equal numbers of rows and columns.
Since every character of G is a linear combination with nonnegative integer coefficients of the irreducible characters of G (Exercise **4.2.**), the character table completely describes all the characters of G.

Example 7.2. If $G = \langle g \rangle$ is cyclic of order n, then each conjugacy class contains a single element, and we've seen that there is a one-dimensional (hence irreducible) representation given by $\rho_j(g^k) = \rho_j(g)^k = e^{2\pi ijk/n}$ for each $j = 0, \ldots, n-1$. Thus the character table for G is the $n \times n$ matrix whose (j,k) entry is $e^{2\pi ijk/n}$.

Example 7.3. We saw in the last section that the irreducible representations of S_3 are the trivial representation τ, the sign representation ε, and the standard representation ψ. Recalling that $\chi_\psi(\pi) = f(\pi) - 1$, where $f(\pi)$ is the number of fixed points of ψ, we get the following character table for S_3 (we designate each conjugacy class by a single representative):

	ι	(12)	(123)
τ	1	1	1
ε	1	-1	1
ψ	2	0	-1

Theorem 7.4 (Second orthogonality relation for characters)**.** *Let $\chi_1, \dots, \chi_m$ be the distinct irreducible characters of G. Then*

$$\sum_{j=1}^m \chi_j(g)\overline{\chi_j(h)} = \begin{cases} \frac{|G|}{|C(g)|} & \text{if } g \sim h, \\ 0 & \text{otherwise,} \end{cases}$$

where $g \sim h$ means that g is conjugate to h, and $C(g)$ denotes the conjugacy class of g.

Proof. By Corollary 6.6, G has m conjugacy classes $C_1, \dots, C_m$. Since characters are class functions, Theorem 5.2 states that

$$\delta_{j,k} = \frac{1}{|G|}\sum_{g\in G} \chi_j(g)\overline{\chi_k(g)} = \sum_{\ell=1}^m \frac{|C_\ell|}{|G|}\chi_j(C_\ell)\overline{\chi_k(C_\ell)} = \sum_{\ell=1}^m \sqrt{\frac{|C_\ell|}{|G|}}\chi_j(C_\ell)\overline{\sqrt{\frac{|C_\ell|}{|G|}}\chi_k(C_\ell)}.$$

That is, the $m \times m$ matrix $\left[\sqrt{\frac{|C_k|}{|G|}}\chi_j(C_k)\right]_{j,k=1}^m$ has orthonormal rows, and is therefore a unitary matrix and also has orthonormal columns. That is,

$$\sum_{j=1}^m \sqrt{\frac{|C_k|}{|G|}}\chi_j(C_k)\overline{\sqrt{\frac{|C_\ell|}{|G|}}\chi_j(C_\ell)} = \delta_{k,\ell},$$

which is equivalent to the claim. □

The two orthogonality relations for characters (Theorems 5.2 and 7.4) and the numerical information in Frobenius's theorem (Theorem 6.3) and Corollary 6.6 together are enough to determine a great deal of information about the characters of a finite group. For small, groups, they often give enough information to determine the character table completely.

Example 7.5. We already saw the character table for S_3, constructed using previous knowledge of what the representations are. Let's see how we could come up with it without any knowledge of the representations themselves.

First, S_3 has three conjugacy classes, represented by ι, (12), and (123), of sizes 1, 3, and 2, respectively. Therefore S_3 has three irreducible characters whose dimensions satisfy $d_1^2 + d_2^2 + d_3^2 = 6$. The only possibility, up to reordering, is $d_1 = d_2 = 1$ and $d_2 = 2$. One of the one-dimensional representations must be the trivial representation τ (which is defined for every group), for which $\chi_\tau = 1$. Since also $\chi_V(\iota) = \dim V$ for any representation, this gives us the partial character table:

	ι	(12)	(123)
τ	1	1	1
ρ_1	1		
ρ_2	2		

If we now write x and y for the second and third entries in the ρ_1 row, applying the first orthogonality relation to the first two rows implies that $1 + 3x + 2y = 0$. Moreover, since ρ_1 is one-dimensional, $\rho_1 = \chi_{\rho_1}$, and so $\chi_{\rho_1}(\pi^k) = (\chi_{\rho_1}(\pi))^k$. Thus $x^2 = 1$ (since $(12)^2 = \iota$) and $y^2 = y$ (since $(123)^2 = (132) \sim (123)$). So $x = \pm 1$ and $y \in \{0, 1\}$. The only such values satisfying $1 + 3x + 2y = 0$ are $x = -1$ and $y = 1$.

We now have:

	ι	(12)	(123)
τ	1	1	1
ρ_1	1	-1	1
ρ_2	2		

We could fill in the last row using the first orthogonality relation to determine two linear equations satisfied by its two missing entries, but it's simpler to use the second orthogonality relation to determine them one at a time. First,

$$1 - 1 + 2\chi_{\rho_2}((12)) = 0,$$

so $\chi_{\rho_2}((12)) = 0$, and then

$$1 + 1 + 2\chi_{\rho_2}((123)) = 0,$$

so $\chi_{\rho_2}((123)) = -1$. This fully recovers the character table we found above.

Example 7.6. In the last section we gave an almost complete description of the irreducible representations of S_4. Exercise **6.6.** asked for an explicit description of the two-dimensional irreducible representation σ, but using the second orthogonality relation and the information we already have:

	1	(12)	(123)	(1234)	$(12)(34)$
τ	1	1	1	1	1
ε	1	-1	1	-1	1
σ	2				
ψ	3	1	0	-1	-1
$\varepsilon \otimes \psi$	3	-1	0	1	-1

we can trivially determine at least the character χ_σ:

$$\begin{gathered}
1 - 1 + 2\chi_\sigma((12)) + 3 - 3 = 0, \\
1 + 1 + 2\chi_\sigma((123)) + 0 + 0 = 0, \\
1 - 1 + 2\chi_\sigma((1234)) - 3 + 3 = 0, \\
1 + 1 + 2\chi_\sigma((12)(34)) - 3 - 3 = 0,
\end{gathered}$$

giving us the complete character table:

	1	(12)	(123)	(1234)	$(12)(34)$
τ	1	1	1	1	1
ε	1	-1	1	-1	1
σ	2	0	-1	0	2
ψ	3	1	0	-1	-1
$\varepsilon \otimes \psi$	3	-1	0	1	-1

Example 7.7. A_4 has conjugacy classes represented by ι, (12)(34), (123), and (132). There are three one-dimensional representations (Exercise **6.8.**) which are all pullbacks of representations of the abelianization A_4/V. The remaining irreducible representation must have dimension $\sqrt{12-3}=3$. We therefore have the partial character table

	1	(12)(34)	(123)	(132)
ψ_1	1	1	$e^{2\pi i/3}$	$e^{4\pi i/3}$
ψ_2	1	1	$e^{4\pi i/3}$	$e^{2\pi/3}$
φ	3			

The second orthogonality relation allows us to trivially complete the last row:

	1	(12)(34)	(123)	(132)
φ	3	-1	0	0

Exercises.

7.1. Let G be a finite group. Show that

$$\sum_V (\dim V)\chi_V(g) = \begin{cases} |G| & \text{if } g=1, \\ 0 & \text{otherwise,} \end{cases}$$

where the sum runs over all distinct irreducible complex representations of G.

7.2. Determine the character table of the quaternion group Q_8.

7.3. Determine the character table of the dihedral group of order 8.

7.4. Show that two groups with the same character table need not be isomorphic.
Hint: See the last two exercises.

7.5. (a) Determine the "multiplication table" for irreducible representations of S_4. That is, for each pair ρ_1, ρ_2 of irreducible representations of S_4, find the canonical decomposition of $\rho_1 \otimes \rho_2$.
(b) Do the same for S_3, A_4, Q_8, and D_8.

7.6. Prove that the three-dimensional irreducible representation φ of A_4 is the restriction to A_4 of the standard representation of S_4.

8. A sample application: the 5/8 theorem

In addition to its importance for applications to other fields (only vaguely hinted at in these notes), representation and character theory is an important tool for understanding the structure of groups. The results of section 6 already give a hint of how this can be, since they relate basic properties of G to basic properties of the family of irreducible characters of G. Here is one simple consequence, whose statement makes no reference to characters or representations.

Lemma 8.1. *Let G be a group with n elements and m conjugacy classes, and let $k=[G:G']$ be the index of the commutator subgroup. Then*

$$n+3k \geq 4m.$$

Proof. By Corollary 6.6, G has m irreducible representations. By Theorem 6.9, exactly k of them are one-dimensional, and so the others have dimensions $d_1,\dots,d_{m-k}\geq 2$. Therefore by Frobenius's theorem (Theorem 6.3),

$$n = k + \sum_{j=1}^{m-k} d_j^2 \geq k + 4(m-k),$$

which is equivalent to the claim. □

The following further consequence has a more appealing, less technical statement. (Although this result is also not hard to prove without character theory, there are other theorems in group theory whose only known proofs use character theory.[3])

Theorem 8.2 (The 5/8 theorem)**.** *Let G be a finite group. If*

$$\frac{\left|\left\{(g,h) \in G^2 \;\middle|\; gh = hg\right\}\right|}{|G|^2} > \frac{5}{8}, \tag{9}$$

then G is abelian.

This theorem has a simple probabilistic interpretation: If the probability that two randomly selected elements of G commute is greater than 5/8, then the group must be abelian.

Proof. Recall that the subgroup $C_G(g) = \left\{h \in G \;\middle|\; g = hgh^{-1}\right\}$ is the **centralizer** of g in G, and that $[G : C_G(g)] = |C(g)|$ (where $C(g)$ is the conjugacy class of G). Therefore

$$\begin{aligned}\left|\left\{(g,h) \in G^2 \;\middle|\; gh = hg\right\}\right| &= \sum_{g \in G} \left|\{h \in G \mid gh = hg\}\right| = \sum_{g \in G} |C_G(g)| \\ &= \sum_{g \in G} \frac{|G|}{|C(g)|} = |G|\, m,\end{aligned}$$

where m is the number of conjugacy classes in G. In the notation of Lemma 8.1, the hypothesis therefore says that $\frac{m}{n} > \frac{5}{8}$. By Lemma 8.1, this implies that $n + 3k > \frac{5}{2}n$, so $k > n/2$, and therefore

$$\left|G'\right| = \frac{n}{k} < 2.$$

Hence G' is trivial, so G is abelian. □

Exercises.

8.1. Give an example of a nonabelian group for which the left hand side of (9) is equal to 5/8.

8.2. Let p be the probability on the left hand side of (9). Show that $1/p$ is the average value of the dimension squared of an irreducible representation of G.

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[3] I learned this proof from `https://mathoverflow.net/a/91686/`.