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On meromorphic parameterizations of real algebraic curves

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Abstract. A singular flat geometry may be canonically assigned to a real algebraic curve Γ; namely, via analytic continuation of unit speed parameterization of the real locus $\Gamma_{\mathbb{R}}$. Globally, the metric $\rho = |Q| = |q(z)|dzd\bar{z}$ is given by the meromorphic quadratic differential Q on Γ induced by the standard complex form $dx^2 + dy^2$ on $\mathbb{C}^2 = \{(x, y)\}$. By considering basic properties of Q, we show that the condition for local arc length parameterization along $\Gamma_{\mathbb{R}}$ to extend meromorphically to the complex plane is quite restrictive: For curves of degree at most four, only lines, circles and Bernoulli lemniscates have such meromorphic parameterizations.

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1. Introduction

Historical explanations of the term *elliptic integral* often begin with the computation of arc length along an ellipse and *elliptic functions* are said, casually, to be obtained by 'inversion of elliptic integrals'.

Actually, such a narrative applies better to the quartic curve known as the lemniscate of Bernoulli. As is well known, discoveries of Count Fagnano and Euler on the integral for arc length along the lemniscate led eventually to the addition theorem for elliptic integrals (see [13,17]). Inversion of the lemniscatic integral, an elliptic integral of the first kind, yields a unit speed parameterization of the lemniscate by certain Jacobi elliptic functions—the so-called lemniscatic functions.

But the elegant lemniscate of Bernoulli is a rare exception in this respect (and in other ways described in [9,10]). The first paragraph notwithstanding, the elliptic integral of the second kind for arclength along an ellipse is not likewise globally invertible. In fact, the main point of the present paper is to explain in geometric terms why inversion of arc length along a real algebraic curve hardly ever leads to meromorphic coordinate functions defined on the complex plane. To state a precise theorem to this effect, we first need to recall some terminology from the classical theory of curves. In the meantime, a version of the result for curves of low degree may be stated concisely:

Theorem 1.1. *Let $f(x, y) = 0$ define an irreducible, real algebraic curve of degree $n \leq 4$ with non-trivial real locus $\Gamma_{\mathbb{R}}$, and let $x(s), y(s)$, $a < s < b$ parameterize an arc of $\Gamma_{\mathbb{R}}$ by unit speed: $f(x(s), y(s)) = 0$ and $x'(s)^2+y'(s)^2=1$. If $x(s), y(s)$ extend meromorphically to all complex parameter values, $\zeta = s + it \in \mathbb{C}$, then $\Gamma_{\mathbb{R}}$ is a line, a circle, or a Bernoulli lemniscate.*

The similar Theorem 5.7 applies to curves of arbitrary degree, but with an additional assumption. The theorem concludes an investigation which requires us to consider curves from a number of points of view and to visit special topics which might nowadays be described as 'quaint'—at least not entirely standard. With this in mind, the present exposition is intended to provide a relatively self-contained introduction to the relevant topics, developing heuristics through key examples, explicit computations and graphics.

Before describing the organization of the paper, we list here the various notations for a curve which are meant to signal the appropriate context: $\Gamma \subset \mathbb{C}^2$ is a real, affine algebraic curve (with real locus $\Gamma_{\mathbb{R}} \subset \mathbb{R}^2 \simeq \mathbb{C}$); $\Gamma^* \subset \mathbb{C}P^2$ is the corresponding projective curve; $\tilde{\Gamma}^*$ is the underlying compact Riemann surface; $(\tilde{\Gamma}^*, Q)$ and $(\tilde{\Gamma}^*, \rho)$ add geometric structure to the latter via $Q = dzdw$, a (canonically defined) meromorphic quadratic differential. (Likewise, $\tilde{Q} = \tilde{\omega}^2$ is an orientable quadratic differential defined by lifting Q to a certain branched double cover of $\tilde{\Gamma}^*$.) Since all objects are uniquely determined by $\Gamma_{\mathbb{R}}$, however, we may occasionally find it expedient to deviate from such explicit notation.

In Sect. 2 we consider planar visualizations of real algebraic curves $\Gamma : f(x, y) = 0$ via arc length parameterizations $x(s) + iy(s)$ of $\Gamma_{\mathbb{R}}$; here we interpret analytic continuation to complex values of the arc length parameter $\zeta = s + it$ in terms of planar curve dynamics. For key examples Γ, we give explicit, global arc length parameterizations—or detect the failure of such a parameterization in terms of the evolving curve developing unacceptable singularities in finite time. The examples demonstrate that isotropic coordinates, $z = x + iy$, $w = x - iy$, are especially well suited for viewing relevant features of Γ.

Real, projective algebraic curves $\Gamma^* : F[X, Y, Z] = 0$ are the context for Sect. 3, where we review properties of isotropic projection—that is, projection from a circular point $c_{\pm} = [1, \pm i, 0]$ onto the real plane. This discussion assigns geometric meanings to: isotropic points, foci, defoci, circular foci, etc. It becomes apparent that curves Γ^* containing $c_{\pm}$ play a special role; in fact, singularities of Γ^* at $c_{\pm}$ are the main concern, eventually, in the proof of Theorem 5.7.

Beginning in Sect. 4, the underlying structure of Γ^* as a compact Riemann surface $\tilde{\Gamma}^*$ is required. In this context, the quadratic differential $Q = dzdw$

on $\tilde{\Gamma}^*$ is defined using (essentially) notation of Sect. 2 and meromorphic extension to isolated singularities (at ideal points). Then we have access to elements of the theory of quadratic differentials on Riemann surfaces: the formula $4g - 4 = Z(Q) - P(Q)$, relating the numbers of zeros and poles of Q to the genus $g = g(\tilde{\Gamma}^*)$; the geometric interpretation of Q, theory of its singularities, trajectories, etc.

In Sect. 5, the assumption on arc length parameterization $x(\zeta), y(\zeta)$ is first shown to imply $Z(Q) = 0$, hence, $g = 0$ or $g = 1$. An assumption limiting 'high order inflections' of Γ^* at circular points then rules out $g = 1$ and, with $g = 0$, the three possible pole patterns—$P = 4,\ P = 2 + 2$ and $P = 1 + 1 + 1 + 1$—lead to the three possible curves in the conclusion of Theorem 5.7: line, circle, and lemniscate. Curves of degree $n \leq 4$ already constitute a diverse family of curves; but the assumption holds for all such curves, so Theorem 1.1 follows as a corollary.

2. Basic examples and heuristics

A real, affine algebraic curve $\Gamma \subset \mathbb{C}^2 = \{(x, y)\}$ is defined by a polynomial equation $0 = f(x, y) = \sum_{i=0}^{k} \sum_{j=0}^{l} c_{ij} x^i y^j$, with real coefficients c_{ij}; with $c_{kl} \neq 0, \Gamma$ has degree $n = k + l$. Reality of Γ may be expressed $\bar{f} = f$, where the bar operation is defined by complex conjugation of coefficients of f.

We will also work with isotropic (conjugate) coordinates $z = x + iy, w = x - iy$ and the corresponding polynomial $g(z, w) = f(\frac{z+w}{2}, \frac{z-w}{2i})$. Note $(x, y) \in \mathbb{R}^2 \Leftrightarrow w = \bar{z}$, and the reality condition for the curve is now $g(z, w) = \bar{g}(w, z)$. Casually applying the identification $\mathbb{R}^2 \simeq \mathbb{C}, (x, y) \leftrightarrow z = x + iy$, we may regard $\Gamma_{\mathbb{R}}$ as a subset of $\mathbb{C}$; namely, the z-locus of the equation $g(z, \bar{z}) = 0$. We may then view non-real points of Γ via *isotropic projection* onto the real plane, $(z, w) \mapsto z \in \mathbb{C} \simeq \mathbb{R}^2$ (to be described more geometrically in Sect. 3). This is how the examples of the present section may be graphically (if incompletely) understood.

We will assume Γ has non-trivial real locus $\Gamma_{\mathbb{R}} = \Gamma \cap \mathbb{R}^2$; that is, $\Gamma_{\mathbb{R}}$ contains an arc. Such an analytic arc may be assigned an arclength parameter $s \in (a, b)$ and corresponding coordinate functions $x(s), y(s)$, which are necessarily real analytic. Replacing s by $\zeta = s + it$, analytic continuation gives $x(\zeta), y(\zeta)$ defined on some connected domain $U \subset \mathbb{C}$ containing the interval (a, b). Such a 'complex arc length parameterization' $x(\zeta), y(\zeta)$—alternatively, $z(\zeta), w(\zeta)$—automatically satisfies:

$$f(x(\zeta), y(\zeta)) = 0, \quad x'(\zeta)^2 + y'(\zeta)^2 = 1, \quad \zeta \in U \tag{2.1}$$

$$g(z(\zeta), w(\zeta)) = 0, \quad z'(\zeta)w'(\zeta) = 1, \quad \zeta \in U \tag{2.2}$$

A basic question: How large can U be chosen?

For heuristics, it is useful to narrow the question by considering a planar, dynamical interpretation of the above continuation problem (effectively limiting the allowable domains U). We start with $z(s) = x(s) + iy(s),\ s \in \mathbb{R}$, a unit speed parameterization of the 'initial curve' $\Gamma_0 = \Gamma_{\mathbb{R}}$; for simplicity, let $\Gamma_{\mathbb{R}}$ consist of a single, immersed curve. Analytic continuation to $\zeta = s + it \mapsto z(\zeta) = x(\zeta) + iy(\zeta),\ s \in \mathbb{R},\ -\epsilon < t < \epsilon$ then describes an evolution of analytic curves Γ_t in the complex plane with initial curve Γ_0 and 'initial velocity field' given by the unit normal along Γ_0. We remark that Γ_t is a *geodesic* in the sense of the infinite dimensional symmetric space geometry developed in [2,3]. (The initial velocity field for such a geodesic is allowed to vary in length and to vanish as it switches between 'inward' and 'outward'—so the present case is very special.)

There is a maximal time domain $-\tau < t < \tau$, where τ is positive or infinite; symmetry of the time domain is an automatic consequence of the underlying symmetry of the real algebraic curve Γ. Further, there are limiting curves $\Gamma_{\pm\tau}$ which are necessarily singular. In fact, these singularities occur at the (real) *foci* and *defoci* of the initial curve Γ_0, to be described in the context of projective geometry, below. For the moment, we note that such singularities come in various flavors; in particular, the foci of a quadric are the usual ones, while certain, more exotic, foci (e.g., the lemniscate's foci) do not interrupt the continuation of $z(\zeta)$.

Example. **Line** Let $\mathcal{L}$ be the complex line with equation $y = 0$. In conjugate coordinates, $\mathcal{L}$ has equation $z = w$ and unit speed parameterization $z(\zeta) = w(\zeta) = \zeta,\ \zeta \in \mathbb{C}$. The horizontal lines $\Gamma_t : s \mapsto z(s + it) = s + it$ describe the curve evolution for $-\infty < t < \infty$. Anticipating later developments, it is useful to view this on the Riemann sphere $S^2 \subset \mathbb{R}^3$, where a 'dipole' singularity appears at the north pole; namely, Γ_t pulls back via stereographic projection $\pi : S^2 \to \mathbb{C} \cup \{\infty\}$ to a family of circles mutually tangent at the north pole.

Example. **Circle** Let $\mathcal{C}$ be the unit circle $x^2 + y^2 = 1$. With $x = \cos\zeta,\ y = \sin\zeta$, the circle equation and unit speed condition $x'^2 + y'^2 = 1$ hold for all $\zeta \in \mathbb{C}$. In conjugate coordinates, $z = e^{i\zeta},\ w = e^{-i\zeta}$ satisfy $zw = 1, z'w' = 1$. The punctured plane $\mathbb{C} \setminus \{0\}$ is foliated by concentric circles $\Gamma_t : s \mapsto z_t(s) = e^{-t}e^{is},\ -\infty < t < \infty$. The "missing" origin is in fact the circle's focus. On the Riemann sphere, $\pi^{-1}(\Gamma_t)$ has a pair of 'simple pole' singularities at north and south poles.

Example. **Ellipse** Let $\mathcal{E}$ be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ 0 < b < a$; in isotropic coordinates, $0 = g(z, w) = \frac{(z+w)^2}{4a^2} - \frac{(z-w)^2}{4b^2} - 1$. As explained in Sect. 3, the foci $\pm c = \pm\sqrt{a^2 - b^2}$ are the z-values determined by the system $0 = g(z, w) = g_w(z, w)$; likewise, the defoci $\pm\frac{a^2+b^2}{c}$ of $\mathcal{E}$ may be obtained by solving for z in the system $0 = g(z, w) = g_z(z, w)$.

Using preliminary parameterization $z(\phi) = x(\phi) + iy(\phi) = a\sin\phi + ib\cos\phi, w(\phi) = x(\phi) - iy(\phi) = a\sin\phi - ib\cos\phi$, one may compute arclength along $\mathcal{E}$ as an elliptic integral of the second kind with modulus $m = k^2 = c^2/a^2$:

$$s(\phi) = \int \sqrt{z'w'}d\phi = \int \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}\ d\phi = aE(\phi, \frac{c^2}{a^2}) \qquad (2.3)$$

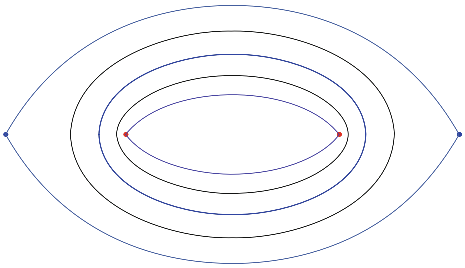

FIGURE 1 Ring domain of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ with boundary singularities at foci and defoci

The length of $\mathcal{E}$ is the complete elliptic integral $L = 4s(\pi/2) = 4aE(c^2/a^2)$. Since $\frac{ds}{d\phi} > 0$, $\phi \in \mathbb{R}$, it follows that $s(\phi)$ is invertible and $z(\phi(s))$ maps $[0, L)$ onto $\mathcal{E}_{\mathbb{R}}$ with unit speed.

Now let us consider analytic continuations $\zeta(\phi)$, $z(\phi(\zeta))$, where complex values of $\zeta = s + it$, ϕ are allowed. One finds four points at which $\frac{d\zeta}{d\phi} = 0$. Namely, $z' = 0$ when $\tan\phi = -ia/b$, i.e., $z(\phi) = \pm c$; likewise, $w' = 0$ when $\tan\phi = ia/b$, i.e., $z(\phi) = \pm\frac{a^2+b^2}{c}$. Thus, $\zeta(\phi)$ is not locally invertible near such values of ϕ, and $z(\phi(\zeta))$ is not defined. In other words, foci and defoci mark the limits of the curve evolution $t \mapsto z(\phi(s + it))$, $-\tau < t < \tau$; that is, the maximal time

$$\tau = |\zeta(\arctan(\pm ia/b)) - \zeta(\pi/2)| = \Im[aE(\arctan(ia/b), \frac{c^2}{a^2})] \tag{2.4}$$

may be interpreted as the time it takes to get from vertex $z = a$ to defocus $z = \frac{a^2+b^2}{c}$, or from focus $z = c$ to $z = a$.

The evolving curve foliates a topological annulus $\mathcal{A}$ with singular boundary. Figure 1 shows the case $a = 5, b = 3, c = 4$, for which the relevant dimensions are: $L = 20E(16/25) \approx 25.5$, $\tau = \Im[5E(\arctan(i5/3), 16/25)] \approx 1.66$. Here, $\mathcal{A}$ is in fact conformally equivalent to a round annulus $1 = r < |z| < R$, where $R = e^{2\pi M}$ is determined by the annular modulus $M = \frac{1}{2\pi}\ln\frac{R}{r} = \frac{L}{2\tau} \approx 7.67$.

As explained below, $\mathcal{A}$ corresponds, via isotropic projection, to a *singular ring domain* in the Riemann surface $\mathcal{E}$ in the sense of [18]. Given that $z(\phi(\zeta))$ is not globally defined, it is not surprising that the real parametrization $z(\phi(s))$ is already awkward to represent analytically; techniques are developed in [7] for the purpose of numerically computing global versions of Fig. 1 for real algebraic curves.

Example. **Lemniscate** The equation $(x^2 + y^2)^2 = x^2 - y^2$ defines a Bernoulli lemniscate $\mathcal{B}$. As $\mathcal{B}$ may be obtained by *inversion* $(x, y) \mapsto (x, -y)/(x^2 + y^2)$ of the hyperbola $x^2 - y^2 = 1$, a rational parameterization of the genus zero curve $\mathcal{B}$ is easily found: $x = \frac{u+u^3}{1+u^4}, y = \frac{u-u^3}{1+u^4}$.

Applying the polar arclength element $ds^2 = dr^2 + r^2 d\theta^2$ to the equation $r^2 = \cos 2\theta$ of $\mathcal{B}$ gives the famous lemniscatic integral for arclength along $\mathcal{B}$:

$$s(r) = \int \frac{dr}{\sqrt{1-r^4}} = F(\arcsin(r), -1) \tag{2.5}$$

Historical note: In [13] one may find an account of the remarkable and little known study by Serret of curves whose arc lengths may be similarly represented by elliptic integrals of the first kind.

Inversion of the elliptic integral of the first kind $s(r)$ yields a Jacobi elliptic sine function, $r(s) = \mathrm{sn}(s, -1)$. The present case—with parameter $m = k^2 = -1$— is known also as the lemniscatic sine $\mathrm{sl}s = \mathrm{sn}(s, -1)$. Using $r = \mathrm{sl}s$ in $x = r\cos\theta, y = r\sin\theta, r^2 = \cos 2\theta$ leads to the following arclength parameterization of $\mathcal{B}$:

$$x(s) = \frac{1}{\sqrt{2}}\,\mathrm{sl}s\,\mathrm{dl}s, \quad y(s) = \frac{1}{\sqrt{2}}\,\mathrm{sl}s\,\mathrm{cl}s, \quad 0 \le s \le 4K \tag{2.6}$$

Here, $\mathrm{cl}s = \sqrt{1-\mathrm{sl}^2 s}, \mathrm{dl}s = \sqrt{1+\mathrm{sl}^2 s}$ are the lemniscatic cosine and delta functions, and $K = K(-1)$ is the complete elliptic integral of the first kind. Using the formulas for $\mathrm{cl}s$, $\mathrm{dl}s$ and the derivatives, $\frac{d}{ds}\mathrm{sl}s = \mathrm{cl}s\mathrm{dl}s$, $\frac{d}{ds}\mathrm{cl}s = -\mathrm{sl}s\mathrm{dl}s$, $\frac{d}{ds}\mathrm{dl}s = \mathrm{sl}s\mathrm{cl}s$, one may verify that $x(s), y(s)$ satisfy $(x^2+y^2)^2 = x^2 - y^2$ and $x'(s)^2 + y'(s)^2 = 1$, as required.

In contrast to the ellipse example, the arc length parameterization 2.6 extends meromorphically to all $\zeta \in \mathbb{C}$. For the sake of brevity, let it suffice to recall: $\mathrm{sl}\zeta$ maps the interior of the square with vertices $i^j K, j = 0, 1, 2, 3$, conformally onto the open unit disk; $\mathrm{sl}\zeta$ extends, by the Schwarz reflection principle, to an adjacent square, which is mapped onto the exterior of the disk; further use of the reflection principle gives $\mathrm{sl}\zeta$ as a doubly periodic function on $\mathbb{C}$. Picturing a checkerboard tiling on the ζ-plane, $\mathrm{sl}\zeta$ has zeros at 'white square centers' $(2n + 2mi)K,\ n, m \in \mathbb{Z}$, and poles at 'black square centers' $((2n+1)+(2m+1)i)K$. The coordinate functions $x(\zeta), y(\zeta)$ are likewise doubly periodic, meromorphic functions, whose mutual poles at black square centers will be seen to correspond to ideal points of $\mathcal{B}$.

For visualization, we consider again the first isotropic coordinate:

$$z(\zeta) = x(\zeta) + iy(\zeta) = \frac{1}{\sqrt{2}}\,\mathrm{sl}\zeta(\mathrm{dl}\zeta\ + i\mathrm{cl}\zeta) = \frac{\sqrt{2}\,\mathrm{sl}(\frac{1+i}{2}\zeta)}{1+\mathrm{sl}^2(\frac{1+i}{2}\zeta)}, \quad \zeta = s + it \in \mathbb{C} \tag{2.7}$$

(The alternative expression for $z(\zeta)$ on the right is tricky to verify directly by elliptic function identities, but follows from the above rational parameterization of $\mathcal{B}$, as in [11].) The resulting planar curve evolution $z_t(s) = z(s+it), -\infty < t < \infty$, is depicted in Fig. 2, which shows the lemniscate (initial curve) $z_0(s)$ and "evolved" curves $z_{t_j}(s),\ t_j = jK/4,\ -4 \le j < 4$.

The interpretation of Fig. 2 may seem less transparent than Fig. 1, due to the many intersections; but these will be resolved by a more intrinsic view

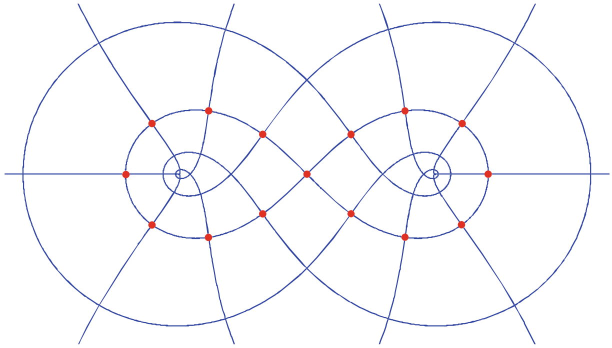

FIGURE 2 Lemniscate curve family $z_t(s) = z(s + it)$

of $\mathcal{B}$. One may also be concerned about the visible singularities at the lemniscate's foci $z = \pm\frac{\sqrt{2}}{2}$; but such foci represent a different type of singularity from the ellipse foci (as explained below), and do not disrupt the curve evolution. While $z(\zeta)$ has poles at $\pm(1+i)K$, unexpected removable singularities $z(\pm(1-i)K) = \pm\frac{\sqrt{2}}{2}$ account for the lemniscate's foci. (Using the first formula for $z(\zeta)$, *Mathematica* returns the incorrect value $z(\pm(1-i)K) = 0$, and behaves erratically at nearby points; this may be disconcerting, but is not a serious issue for plotting Fig. 2, since *Mathematica* handles the alternate expression without difficulty.)

Once again, isotropic coordinates quickly turn up clues to the continuation problem; but the subtleties of the present example may be taken as initial motivation for Sects. 3, 4 and 5, where algebraic curves are considered with the benefit of projective geometry and Riemann surfaces.

Remark. We mention some interesting facts related to the above example which are not strictly relevant: The family $z_t(s)$ is self-orthogonal; for uniform time increments $t_j = jK/n$, the curves $z_{t_j}(s)$ subdivide the lemniscate $z_0(s)$ into arcs of equal length; the subdivision points (shown in Fig. 2 for $n = 4$) are constructible by ruler and compass for precisely the same integers $n = 2^j p_1 p_2 \dots p_k$ for which the regular n-gon is constructible (here, $p_i = 2^{2^{m_i}} + 1$ are distinct Fermat primes). We refer the interested reader to [9–11,13,14].

3. Projective curves: circular and isotropic points, foci, defoci

As just illustrated, some relevant features of a curve may not be visible within the affine plane $\mathbb{C}^2$; that is, one needs to consider also the curve's 'points at infinity'. For this purpose, one introduces the complex projective plane $\mathbb{C}P^2 = \{[X, Y, Z]\}$; here, $X, Y, Z \in \mathbb{C}$ are not all zero, and $[X, Y, Z] \sim [\lambda X, \lambda Y, \lambda Z]$ for any $\lambda \neq 0$. Thus, a point in $\mathbb{C}P^2$ is represented by a (complex) line through the origin in $\mathbb{C}^3$. The 'finite points' of $\mathbb{C}P^2$ are identified with the affine plane

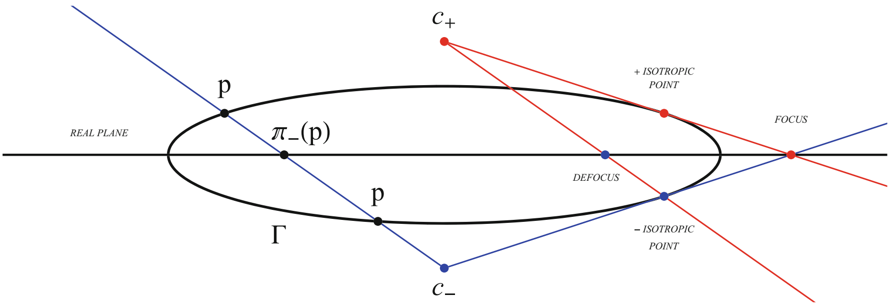

Figure 3 Isotropic projection, foci and defoci

via $(x, y) \mapsto [x, y, 1]$ and $[X, Y, Z] \mapsto (X/Z, Y/Z), Z \neq 0$; the ideal points are those with $Z = 0$.

A line in $\mathbb{C}P^2$ is defined by an equation of the form $AX + BY + CZ = 0$, where $A, B, C \in \mathbb{C}$ are not all zero. Except for the ideal line $Z = 0$, every line is the projective completion of an affine line $Ax + By + C = 0$, whose ideal point is defined by $AX + BY = 0$. Likewise, the homogeneous equation for an n^{th} degree curve $\Gamma^* \subset \mathbb{C}P^2$ is written $0 = F[X, Y, Z] = Z^n f(\frac{X}{Z}, \frac{Y}{Z})$; Γ^* is the projective completion of the affine curve $f(x, y) = 0$—this curve Γ is the 'affine part' of Γ^*. A line and an n^{th} degree curve Γ^* intersect in n points, counting multiplicities; in particular, Γ^* meets the ideal line n *times*.

Setting $Z = 0$ in the equation of a circle $(X - aZ)^2 + (Y - bZ)^2 - R^2Z^2 = 0$ gives $X^2 + Y^2 = 0$; thus, the two ideal points of a circle are always $c_\pm = [1, \pm i, 0]$. Accordingly known as the *circular points*, $c_\pm \in \mathbb{C}P^2$ play a distinguished role in the 'metrical theory of curves' [15]. Likewise, the features of algebraic curves of interest here are those preserved by the group of real similarities; this is the subgroup of the projective group $PGL(3, \mathbb{C})$ preserving the real plane and fixing the circular points. In particular, such transformations preserve the families of *isotropic lines*—lines passing through c_+ or c_-. Such lines have equations of the form $X \pm IY - z_0 Z = 0$. This equation describes the unique line $\overline{c_\pm \mathbf{p}}$ passing through $c_\pm$ and the point in the real plane $\mathbf{p} = [x_0, y_0, 1] \in \mathbb{R}^2$, where $z_0 = x_0 + iy_0$.

Now consider Γ^*, a real algebraic curve of degree n. To each finite point $p \in \Gamma$, let $\overline{c_\pm p}$ denote the isotropic line through $c_\pm$ and p. Define isotropic projections:

$$\pi_\pm : \Gamma \to \mathbb{R}^2, \quad \pi_\pm(p) = \mathbf{p} = \overline{c_\pm p} \cap \mathbb{R}^2. \tag{3.1}$$

It will often suffice to use just $\pi = \pi_+$; the involution $[X, Y, Z] \overset{\sigma}{\leftrightarrow} [\bar{X}, \bar{Y}, \bar{Z}]$ on Γ^* relates the two projections by $\pi_-(p) = \pi_+(\sigma(p))$.

Recall that a real curve is said to be *circular* if it contains the circular points (by real symmetry, it contains both $c_\pm$ if it contains either). Assume (for the next several paragraphs) that the degree n curve Γ^* is non-singular and non-circular. Then the situation, represented schematically in Fig. 3, may be described as follows. All but finitely many $\mathbf{p} \in \mathbb{R}^2$ have preimage set $\pi^{-1}(\mathbf{p}) \subset$

Γ consisting of n distinct points, at each of which $\overline{c_+\mathbf{p}}$ meets Γ transversely and nearby which π is locally one-to-one. Each of the $m < \infty$ exceptions $\mathbf{f} \in \mathbb{R}^2$ is a *focus* of Γ^* $(m \leq n(n-1)$, as will be seen shortly).

Such a focus is associated with a (*positive*) *isotropic point* $f_+ \in \pi^{-1}(\mathbf{f})$ ('point of isotropic tangency'), where the line $\overline{c_+\mathbf{f}} = \overline{c_+f_+}$ meets Γ with higher multiplicity, and the number of distinct preimages $|\pi^{-1}(\mathbf{f})| < n$ is thereby reduced. Similar comments apply to π_- and corresponding (negative) isotropic points f_-, yielding the same foci $\mathbf{f} = \pi_-(f_-)$. Isotropic points come in pairs, $f_+ \overset{\sigma}{\longleftrightarrow} f_-$, and foci $\mathbf{f} = \pi_+(f_+)$ are paired with *defoci* $\mathbf{d} = \pi_+(f_-) = \pi_-(f_+)$.

To relate present definitions to the notation of the previous section, note that any point $p \in \mathbb{C}^2$ is the intersection of a pair of isotropic lines: $p = \overline{c_+p} \cap \overline{c_-p}$. If the equations of the two lines are $X+IY-z_0Z = 0$ and $X-IY-w_0Z = 0$, then the corresponding affine equations for $x = X/Z, y = Y/Z$ are $x + iy = z_0$ and $x-iy = w_0$; in other words, (affine) isotropic lines are defined simply by equating one of the isotropic coordinates to a constant: $z = z_0$ or $w = w_0$. In view of the identification $[x, y, 1] \leftrightarrow z = x + iy$, we may write $\mathbf{z} = \pi_+(p)$, $\mathbf{w} = \pi_-(p)$; here we have used boldface to signify that $\mathbf{z}$, $\mathbf{w}$ are to be regarded as points in the real plane.

The homogeneous equation for the tangent line to Γ^* at a nonsingular point p is given by: $F_X(p)X + F_Y(p)Y + F_Z(p)Z = 0$. At an isotropic point, $(F_X, F_Y, F_Z) = \lambda(1, \pm i, -Z_0)$ for some $Z_0, \lambda \neq 0$. Using $0 = F[X, Y, Z] = Z^n f(\frac{X}{Z}, \frac{Y}{Z})$, a finite isotropic point is found to satisfy $f_x \pm i f_y = 0$. Alternatively, application of the chain rule to $g(z, w) = f(\frac{z+w}{2}, \frac{z-w}{2i})$ gives $g_w = 0$ or $g_z = 0$, respectively, for isotropic points f_+ or f_-. Thus, the method used in the ellipse example to locate foci $\mathbf{z} = \pi(f_+)$ and defoci $\mathbf{w} = \pi(f_-)$ has been explained. Not only that, we have the bound $m \leq n(n-1)$ on the number of such foci (or defoci), since Bézout's Theorem says that the pair of curves $g = 0$ and $g_w = 0$ intersect in $n(n-1)$ points (counting multiplicities and ideal points). We remark that the curve $g_w = 0$ is the *first polar curve* of the point c_+ with respect to the original curve $g = 0$.

We turn now to the circular curves, where strictly fewer than $n(n-1)$ foci are to be expected. To see why, we first consider the circle:

Example. **Circle** An irreducible, circular quadric $(n = 2)$ is necessarily a circle $F = (X - aZ)^2 + (Y - bZ)^2 - R^2Z^2 = 0$. The two isotropic tangents $0 = F_X(c_\pm)X + F_Y(c_\pm)Y + F_Z(c_\pm)Z = 2(X \pm iY - (a \pm ib)Z)$ intersect the real plane at the circle's center. All other isotropic lines meet the circle at exactly one finite point. One may compare π_+ with stereographic projection π from north pole on the sphere; the two extend, via $\pi_+(c_+) = \mathbf{0}$ and $\pi(n.p.) = \infty$, to $1-1$, nonsingular maps. Partly for this reason, we will not use traditional modifiers *singular* or *double* for a focus like the circle's (which may be said to result from the collision of a pair of foci of an increasingly round ellipse).

Now consider a real, degree n curve Γ^*, and let $1 \leq k \leq n/2$. Γ^* is *k-circular* if it contains $c_\pm$ with multiplicity k. To revisit the picture of isotropic projection, all but finitely many isotropic lines $X \pm IY - z_0 Z = 0$ intersect Γ^* transversely in $n-k$ finite points. Foci are again associated with isotropic tangent lines; but as the above example illustrates, tangency may occur at $c_\pm$ itself, which may or may not be a point of ramification of $\pi_\pm$. Either way, we will refer to the tangent line's intersection with the real plane as a *circular focus*—not a standard term, but suiting our purposes. In case $n = 2k$, the even degree curve Γ^* is *totally circular*, i.e., it contains no ideal points other than $c_\pm$. Totally circular curves may appear to be rather special; but understanding these curves and their singularities at $c_\pm$ is of primary concern below (and singularities at other points will be seen not to require much attention).

Totally circular quartics ($n = 4$) include many well-known curves such as limaçons, cardioids, Cassinians, lemniscates, Neumann's curve, hyperbolic ellipses. The diversity of such quartics is suggested already by the possible numbers of *nodes* δ and *cusps* κ consistent with the Clebsch genus formula $g = \frac{(n-1)(n-2)}{2} - \delta - \kappa$. The Bernoulli lemniscate illustrates a node—a point of transverse self-intersection—while the origin on the cardioid $(x^2 + y^2 - x)^2 = x^2 + y^2$ is a cusp.

More generally, a totally circular quartic is either: (a) a *bicircular quartic*, which has nodes at $c_\pm$; (b) a *cartesian*, which has cusps at $c_\pm$; or (c) tangent to the ideal line at (nonsingular) circular points. By real symmetry, singularities at $c_\pm$ must match, and a possible third singularity must be real. With $n = 4$, the genus formula $g = 3 - \delta - \kappa$ shows that a bicircular quartic or cartesian has genus *one* if it has only the two singularities $c_\pm$ ($\delta = 2, \kappa = 0$ or $\delta = 0, \kappa = 2$), whereas Γ^* is rational ($g = 0$) if it has a third singularity. The properties of such families of quartics are studied extensively in classical treatises (see [1, Ch. IX], [15, pp. 240–263]).

Remark. Rational quartics of types (a), (b) may in fact be obtained from quadrics via inversion (*quadratic transformation*), as in the lemniscate example. Specifically, a cartesian or bicircular quartic results, depending on whether the point of inversion (not on the curve) is a focus of the quadric or not; in either case, the third singularity appears at the center of inversion. The same construction applies to higher degree curves: A real, non-circular, degree k curve is transformed into a totally circular curve of degree $n = 2k$ via inversion about a point not on the curve. This fact will not be used directly, but may help to put the special role of totally circular curves in perspective.

Example. **Lemniscate** Here we discuss foci and isotropic projection for the lemniscate $\mathcal{B} : f(x, y) = (x^2 + y^2)^2 - x^2 + y^2 = 0$. First note that the origin is a node with pair of tangents represented by the lowest order term, $(y - x)(y + x) = 0$. The origin is in fact a *biflecnode*—where each branch of the curve has an inflection point. In fact, the line $x(t) = t, y(t) = \pm t$ meets the lemniscate *four times* at the origin; three zeros of $f(t, \pm t) = 4t^4$ are accounted for by the tangent branch, the fourth by the transverse branch. One easily verifies that there are no other finite critical points and there are no finite isotropic points.

The homogeneous equation $F(X, Y, Z) = (X^2+Y^2)^2-X^2Z^2+Y^2Z^2 = 0$ shows the lemniscate is totally circular, the circular points being the remaining two singularities of the rational quartic $\mathcal{B}$. In fact, $c_\pm$ are also biflecnodes—being images of the origin under projective symmetries of the lemniscate. To see this, introduce (homogenous) isotropic coordinates

$$\alpha = X + iY,\ \beta = X - iY, \eta = \frac{Z}{\lambda};\ X = \frac{\alpha+\beta}{2},\ Y = \frac{\alpha-\beta}{2i},\ Z = \lambda\eta \qquad (3.2)$$

with $\lambda = i\sqrt{2}$ and note that $0 = F(X, Y, Z) = G(\alpha, \beta, \eta) = \alpha^2\beta^2+\eta^2\alpha^2+\beta^2\eta^2$ is an invariant equation under permutations of α, β, η (also under sign changes, so an *octahedral* subgroup of $PGL(3, \mathbb{C})$ acts on $\mathcal{B}$, as discussed at length in [9,10]).

In particular, the tangent line pair $\alpha^2 + \beta^2 = 0$ at the origin may be sent to the pair of tangents $\alpha^2 + \eta^2 = 0$ at $c_+(\alpha = \eta = 0)$; converting back to the original coordinates, the two isotropic lines are found to intersect the real plane at the lemniscate's foci $x = \pm 1/\sqrt{2}, y = 0$. These are 'circular foci' in the above sense, but qualitatively different from a circle's foci: Because a tangent line at c_+ makes *three point contact* with its branch, a nearby isotropic line meets the branch in two nearby points (and nowhere else, besides the double point c_+ itself). In other words, c_+ accounts for the two points of ramification for the double covering π_+. The two foci $\pi_+(c_+)$ are readily identified in Fig. 2, the two sheets of π_+ at every other point being associated with a transverse curve-pair. (The full interpretation of the figure will be given in the next section.)

As a further illustration of the isotropic coordinates α, β, η, we now establish a characterization of $\mathcal{B}$ which we will need in Sect. 5.

Proposition 3.1. *A real, trinodal quartic with biflecnodes at the circular points $c_\pm$ is a Bernoulli lemniscate.*

Proof. Let us use curly brackets to denote points in the above isotropic coordinates. The vertices of the *triangle of reference* $\alpha\beta\eta = 0$ are the circular points $c_+ = \{0, 1, 0\}$, $c_- = \{1, 0, 0\}$, and the origin $O = \{0, 0, 1\}$. By assumption, Γ^* has nodes $c_\pm$ and, by real translation, we may take O to be the real node. Then the equation of Γ^* has the form:

$$0 = G(\alpha, \beta, \eta) = A\beta^2\eta^2 + B\eta^2\alpha^2 + C\alpha^2\beta^2 + \alpha\beta\eta(a\alpha + b\beta + c\eta) \qquad (3.3)$$

The vanishing of $\alpha^4, \beta^4, \eta^4$ terms is required for Γ^* to 'circumscribe' the triangle of reference, while the vanishing of $\alpha^3, \beta^3, \eta^3$ terms is needed for Γ^* to be singular at triangular vertices. The further condition $ABC \neq 0$ must hold in order for Γ^* not to be decomposable.

Now consider the equation of the pair of tangents at the node c_+, which is obtained by equating the coefficient of β^2 to zero: $H_+(\alpha, \eta) = A\eta^2 + b\alpha\eta + C\alpha^2 = 0$. There will be a pair of independent solutions $(\alpha_i, \eta_i),\ i = 1, 2$ precisely when $b^2 - 4AC \neq 0$ (otherwise, c_+ would be a cusp). Further, in this case, $\alpha_i \neq 0$ and $\eta_i \neq 0$—otherwise A or C vanishes.

For c_+ to be a biflecnode, both of the tangent lines must meet Γ^* four times at $c_+ = L_i(0)$. Substitution of the parameterized tangent lines $L_i(t) = \{\alpha_i t, 1, \eta_i t\},\ i = 1, 2$ into G gives $G(L_i(t)) = \beta^2 H_+(\alpha_i, \eta_i)t^2 + \alpha_i\eta_i(a\alpha_i + c\eta_i)t^3 + B\alpha_i^2\eta_i^2 t^4$. Here, the first term vanishes, and so must the second, in order for $G(L_i(t))$ to vanish four times at $t = 0$. Further, the first factor in the second term $\alpha_i\eta_i$ is nonzero. But the only solution to the nonsingular system $a\alpha_i + c\eta_i = 0,\ i = 1, 2$ is $a = c = 0$. Since the same argument applies to the biflecnode c_-, it follows likewise that $b = 0$, so Γ^* has equation: $A\beta^2\eta^2 + B\eta^2\alpha^2 + C\alpha^2\beta^2 = 0$. Up to scaling of variables, this is the above equation for $\mathcal{B}$; in other words, the real curve Γ^* is projectively equivalent to $\mathcal{B}$. But a real projective transformation which preserves the circular points is a real similarity, so Γ^* is in fact a Bernoulli lemniscate. $\square$

Remark. The proposition gives a classical result as stated in [1,6]. Note, however, that our proof did not require the third singularity to be a node. Though there are indeed bicircular quartics with real cusps, we have actually shown: *A real, rational quartic with biflecnodes at* $c_\pm$ *is a Bernoulli lemniscate.*

4. The clinant quadratic differential $Q = dzdw$ on $\tilde{\Gamma}^*$

The *Schwarz function* of a regular analytic curve $\Gamma \subset \mathbb{C}$ is the analytic function $S(z)$ uniquely defined near Γ by the requirement that $\bar{z} = S(z)$ holds for $z \in \Gamma$. The derivative of the Schwarz function is unimodular along Γ; that is, for $z \in \Gamma, S'(z) = \frac{d\bar{z}}{dz} = e^{-2i\theta}$. Given an orientation along Γ, θ may be defined as the angle between the real axis and the unit tangent vector $\frac{dz}{ds} = \mathbf{T} = e^{i\theta}$. The complex unit $e^{-2i\theta}$ along Γ is called the *clinant*. From $\mathbf{T} = 1/\sqrt{S'}$ follows the formula for signed curvature of Γ: $\kappa = \frac{i}{2}S''/S'^{3/2}$. The theory and numerous interesting applications of the Schwarz function are extensively developed in [4] (see also [16]).

In its original context, the Schwarz function tends to be regarded locally (near Γ), and multivalued functions are generally avoided. But in the case of an affine algebraic curve $\Gamma \subset \mathbb{C}^2$ given in isotropic coordinates by $g(z, w) = 0$, it is reasonable for one to define the Schwarz function as the algebraic function $w = S(z)$ satisfying $g(z, S(z)) = 0$.

Actually, we will use the fact that an algebraic curve $\Gamma^* \subset \mathbb{C}P^2$ has an induced structure as a compact Riemann surface $\tilde{\Gamma}^*$, and presently adopt the point of view that the isotropic coordinates z, w define meromorphic functions on $\tilde{\Gamma}^*$. Subsequently, taking quotient and product of the corresponding meromorphic differentials dz, dw yields, respectively, a meromorphic 'clinant' function q and quadratic differential Q on $\tilde{\Gamma}^*$. (See [5,12,18] for background on Riemann surfaces and quadratic differentials.)

To be explicit, we use isotropic coordinates to express the *meromorphic clinant*

$$q = \frac{dw}{dz} = -\frac{g_1}{g_2} \tag{4.1}$$

and the *clinant quadratic differential* ([8])

$$Q = qdz^2 = dzdw = S'(z)dz^2 = -\frac{g_1}{g_2}dz^2 = -\frac{g_2}{g_1}dw^2. \tag{4.2}$$

By these formulas we are introducing useful notation, the precise meaning of which will be addressed in the statement and proof of Proposition 4.1.

We are especially interested in the zeros and poles of Q. We denote the singular set of Q by $Q_{\text{sing}} = \text{zeros(Q)} \cup \text{poles(Q)} \subset \tilde{\Gamma}^*$ and its complement, the regular set, by $Q_{\text{reg}} = \tilde{\Gamma}^* \setminus Q_{\text{sing}}$ (notation chosen to avoid confusion with the singular and regular sets Γ^*_{sing}, Γ^*_{reg} of Γ^* itself.) Also, we adopt the following notational shorthand: Given $p \in \Gamma^*$, let $\tilde{p} \in \tilde{\Gamma}^*$ denote *any one* of the corresponding points; on $\Gamma^*_{\text{reg}}, p \leftrightarrow \tilde{p}$ is a one-to-one correspondence and the *tilde* may be omitted.

Proposition 4.1. *Consider $\Gamma^* \subset \mathbb{C}P^2$, a real algebraic curve with the structure of a compact Riemann surface $\tilde{\Gamma}^*$. There is a meromorphic quadratic differential Q on $\tilde{\Gamma}^*$, represented at all but finitely many points by formula 4.2. Assuming $p \in \Gamma^*$ is not a circular point:*

(i) *p is an ideal point if and only if $\tilde{p}$ is a pole of Q.*
(ii) *If p is nonsingular, it is isotropic if and only if $\tilde{p}$ is a zero of Q.*

Proof. Consider first a finite regular point $p = (z, w) \in \Gamma$; the partial derivatives $g_1(p) = \frac{\partial g}{\partial z}(z, w)$ and $g_2(p) = \frac{\partial g}{\partial w}(z, w)$ do not both vanish. If, say, $g_2(p) \neq 0$, then z restricts to a valid analytic coordinate near p and the equation $g(z, w) = 0$ implicitly defines an analytic function $w = S(z)$ with differential $dw = S'(z)dz = -\frac{g_1}{g_2}dz$ near p. Likewise, if $g_1(p) \neq 0$, then w, z and $dw, dz = -\frac{g_2}{g_1}dw$ define local analytic functions/differentials on Γ. It is only a matter of re-interpretation to insert *tildes*—$\tilde{p} \in \tilde{\Gamma}^*$, etc.—though we omit *tildes* on variables, as in Formulas 4.1, 4.2.

If p is one of the finitely many singular or ideal points of Γ^*, then $\tilde{p}$ may be treated as an isolated singularity of z and w. Near a finite point $p \in \Gamma^*, z$ and w are locally bounded and $\tilde{p}$ is a removable singularity; in the vicinity of a non-circular ideal point p, z and w approach infinity and $\tilde{p}$ must be a pole of both functions. Here we use the complex structure on $\tilde{\Gamma}^*$; that is, we refer z or w to an analytic chart about $\tilde{p} \in \tilde{\Gamma}^*$ in order to obtain analytic functions on a punctured disk $D^\circ \subset \mathbb{C}$ and invoke the Riemann extension theorem. (The chart amounts to local resolution of the singularity, but all we require is that it exists, a simpler result than the Puiseux expansion itself.)

Thus we have meromorphic functions z, w on $\tilde{\Gamma}^*$ and, in turn dz, dw, $q = \frac{dw}{dz}, Q = dzdw$ are automatically well-defined and meromorphic on $\tilde{\Gamma}^*$. Further, Q is analytic at finite points, and has a pole of order at least *four* at $\tilde{p}$ when p is a non-circular ideal point.

To prove (ii), consider a finite, non-singular point $p \in \Gamma$. Then at least one of the last two expressions in 4.2 gives a valid local coordinate representation.

Further, p is a positive isotropic point if and only if $g_1(p) \neq 0, g_2(p) = 0$, which is precisely the condition for the last representation to be valid with $Q(p) = 0$. (We note that $\pi_+(p)$ is a focus, and a branch point of $S(z)$.) On the other hand, p is a negative isotropic point if and only if $g_1(p) = 0, g_2(p) \neq 0$, which is precisely the condition for the earlier representations to be valid with $Q(p) = 0$. (In this case, $\pi_+(p)$ is a defocus, and $S'(z) = 0$.) □

Remark 4.2. The proposition does not describe singularities of Q at $\tilde{p}$ when p is a finite singularity or a circular point of Γ^*. Finite singularities will be easily ruled out for the curves of interest to us. On the other hand, singularities of Q resulting from circular points are subtle and will require our attention in the next section.

Next, we review some relevant definitions and results from the theory of quadratic differentials on Riemann surfaces. In terms of a local coordinate z on a Riemann surface Σ, a meromorphic quadratic differential has expression $Q = q(z)dz^2$ with $q(z)$ a local meromorphic function. Under change of variable, $z = f(\tilde{z}), Q$ transforms like the square of an ordinary differential: $q(z)dz^2 = \tilde{q}(\tilde{z})d\tilde{z}^2$, where $\tilde{q}(\tilde{z}) = q(f(\tilde{z}))(f'(\tilde{z}))^2$. Q determines a flat geometry on Q_{reg} with Riemannian metric $\rho = |Q| = |q(z)|dzd\bar{z}$, and an orthogonal pair of geodesic foliations on Q_{reg} by *horizontal* and *vertical trajectories* of Q. Horizontal arcs may be described by parameterized curves $z(u)$ satisfying $Q > 0$, i.e., $q(z(u))\frac{dz}{du}^2 > 0$. Similarly, vertical arcs are defined by $Q < 0$. Trajectories are maximal arcs.

The flatness of ρ may be seen by taking a branch of the square root of Q at a regular point to get a linear differential ω and its dual vectorfield W:

$$\omega = \sqrt{q}dz \quad \longleftrightarrow \quad W = \frac{1}{\sqrt{q}}\frac{\partial}{\partial z} = (u + iv)\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

The real vectorfields $\Re[W] = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ and $\Im[W] = -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}$ (whose integral curves are horizontal/vertical arcs) are orthonormal with respect to ρ, and commute as a consequence of the Cauchy–Riemann equations (equivalently, ω is closed). Actually, a meromorphic vectorfield interpretation applies globally in the special case of an *orientable* quadratic differential $Q = \omega^2 = h(z)^2dz^2$, whose behavior is therefore simpler.

Locally, the geometry (Q_{reg}, ρ) is best described using a *natural parameter* ζ, given by a branch of the following integral near a regular point $p_0 \in Q_{\mathrm{reg}}$:

$$\zeta = \Phi(p) = \int_{p_0}^{p} \sqrt{Q} = \int_{p_0}^{p} \sqrt{q}dz \tag{4.3}$$

($\tilde{\zeta} = \pm\zeta$+const would also be a natural parameter near p_0.) By the inverse function theorem, Φ maps a small neighborhood $V \subset Q_{\mathrm{reg}}$ of p_0 conformally onto a domain in $U \subset \mathbb{C}$. In fact, it follows from $d\zeta^2 = Q$ that Φ defines an isometry between (V, ρ) and $(U, d\zeta d\bar{\zeta})$; in particular, Φ takes geodesics in Σ to straight lines in $\mathbb{C}$.

We will also want to consider the local parameterization $\zeta \mapsto \gamma(\zeta)$ obtained by inversion of Φ. Note that restriction of γ to horizontal (vertical) lines $s + it_0$ ($s_0 + it$) in $\mathbb{C}$ gives unit speed parameterizations of horizontal (vertical) arcs of Q. In case p_0 lies on a closed horizontal trajectory, $\gamma(\zeta)$ extends analytically to a maximal horizontal strip $S : -\infty \leq a < \Im[\zeta] < b \leq \infty$, and $\gamma : S \to Q_{\mathrm{reg}}$ defines a regular infinite covering (with multivalued inverse Φ). The image $R = \gamma(S)$ is a *Euclidean cylinder* in (Q_{reg}, ρ) each of whose boundary components contains a critical point of Q. It is also possible for p_0 to lie on a horizontal trajectory which extends infinitely far in both directions, and then $\Phi : R \to S$ may define an isometry of Euclidean half-planes, or horizontal strips. Such *Euclidean domains* are 'building blocks' which may be glued together along critical trajectories to form the flat geometry (Q_{reg}, ρ).

But in fact the latter geometry may reasonably be extended to include zeros and first order poles in a *singular flat geometry* on the set $Q_{\mathrm{fin}} = \Sigma \setminus Q_\infty$ obtained by deleting only the second and higher order poles. As developed comprehensively in [18], the theory preserves many familiar local, as well as global, results about geodesics in Riemannian geometry (with minor exceptions related to first order poles). The local theory builds on normal forms for Q, which we now recall.

The local normal form at a point $p_0 \in \Sigma$ provides a natural parameter ζ in terms of which Q and the isometry Φ have simple representations. Excluding even-order poles, for the moment, the following expressions for Q and Φ hold, with $\zeta = 0$ at p_0 and $0 \leq |\zeta| < \epsilon$ sufficiently small:

$$Q = \left(\frac{n}{2} + 1\right)^2 \zeta^n d\zeta^2, \quad \tilde{\zeta} = \Phi(\zeta) = \zeta^{\frac{n}{2}+1}, \quad n \neq -2, -4, \ldots, \tag{4.4}$$

(Following [18], the scaling is chosen to make Φ simple as possible.) We note that the two expressions are related by $d\tilde{\zeta}^2 = Q$—which should not be regarded as an even simpler form for Q; unlike the natural parameter ζ itself, $\tilde{\zeta}$ is not a valid local coordinate on Σ (except in the earlier case $n = 0$, where $\tilde{\zeta} = \zeta$). The normal form for even order poles is similar, except that $\Phi(\zeta)$ may include a logarithmic term.

Now we see that Q_∞ is indeed the set of "infinitely distant" points of (Σ, ρ), which need to be deleted to define the singular geometry on "finitely distant" points (Q_{fin}, ρ). That is, $\lim_{\zeta \to 0} |\Phi(\zeta)| = \infty$ for $n \leq -2$, while $\lim_{\zeta \to 0} \Phi(\zeta) = 0$ for $n \geq -1$. In the same vein, the lengths of certain geodesics attached to p_0 are easily computed (assuming no logarithmic term in Φ), giving $L = \infty$ and $L < \infty$ in the two cases. Namely, using the normal form 4.4, the ray $\zeta(r) = re^{i\theta}$ is seen to be horizontal when $\zeta^n \zeta'^2 = (\frac{n}{2}+1)^2 r^n e^{i(n+2)\theta} > 0$, i.e., $\theta = \theta_k = \frac{2\pi k}{n+2}$ for integers k.

The geodesic rays $\theta = \theta_k$ are also key to the 'local portraits' for horizontal arcs near critical points of Q. In the case $n \geq -1$ these rays divide the neighborhood $|\zeta| < \epsilon$ into $n + 2$ sectors, each of which is isomorphic to the upper half of a Euclidean disc, foliated by horizontal lines. A simple pole, for instance, can be pictured as the vertex created by bending an index card and taping one half of the lower edge to the other. Likewise, a first order zero resembles the vertex formed by three index cards whose lower edges are creased at right angles so that each half-edge can be taped to the half-edge of another card. In both examples, the lines on the index cards are horizontal arcs.

On the other hand, a higher order pole p_0 serves as ∞, for $|n|-2$ half-planes separated by the rays $\theta=\theta_k$. The most familiar such portrait is that of $Q=\frac{d\zeta}{\zeta^4}$; the dipole at $\zeta=0$ is interpreted as two half planes separated by the rays $\theta=0$ and $\theta=\pi$. In contrast to the previous case, note that nearby horizontal trajectories tend to p_0 like the rays $\theta=\theta_k$. For the most part we have ignored cases with logarithmic terms, but the important example $Q=-\frac{d\zeta}{\zeta^2}, \Phi=i\ln\zeta$ will be discussed below, and will be seen to represent a cylindrical end of $(Q_{\rm fin},\rho)$.

Finally, we mention some basic facts about geodesics in the singular geometry $(Q_{\rm fin},\rho)$. As in Riemannian geometry, a locally rectifiable curve $c:(a,b)\to Q_{\rm fin}$ is a *geodesic* if it is locally length-minimizing: Any $t\in(a,b)$ lies in a subinterval $[t_1,t_2]\subset(a,b)$ such that $L(c([t_1,t_2]))=\int_c|Q|=d(p_1,p_2)=\inf_{\tilde c}\int_{\tilde c}|Q|$, the competing curves $\tilde c\subset Q_{\rm fin}$ having the same endpoints $p_j=c(t_j)$. We note that the metric $d(p_1,p_2)$ so defined agrees with the existing topology on $Q_{\rm fin}\subset\Sigma$.

Now, a geodesic through a regular point p_0 is mapped locally by Φ to a straight line in the ζ-plane, i.e., it makes a fixed (counterclockwise) angle $0\le\vartheta<\pi$ with the horizontal foliation: $\arg d\zeta^2=\arg Q=2\vartheta=\theta$. More globally, the latter equation defines a *θ-trajectory* (which is horizontal in case $\theta=0$ and vertical in case $\theta=\pi$). If c passes through the regular point p_0, it follows that c coincides with a θ-trajectory as long as both are defined.

Unlike a θ-trajectory, however, a geodesic $c(t)$ may pass through a zero or first order pole, at which point, θ may jump as $c(t)$ joins up with a new θ-trajectory. A bit more problematically, certain pairs of points arbitrarily close to a simple pole may be joined by two distinct shortest paths; for this reason, many results in [18] are formulated either for holomorphic quadratic differentials or by puncturing at such poles. However, we will require only the following simple consequence of the local theory: Any point $p_0\in Q_{\rm fin}$ has a normal form neighborhood U such that p_0 is joined to any other point $p\in U$ by a unique shortest geodesic—namely, a finite θ-arc (not including p_0 itself, if p_0 is critical).

Applying the above constructions to $\Sigma=\tilde\Gamma^*, Q=dzdw$ results in a globalization and geometric interpretation of the foliations Γ_t considered in Sect. 2. Namely, the horizontal trajectories project isotropically to the latter foliation, and $s=\Re[\zeta], t=\Im[\zeta]$ measure, respectively, ρ-arclength along curves Γ_t, and ρ-distance between these *parallel geodesics*. Along $\Gamma_{\mathbb R}$, ρ-arclength is the usual planar arclength.

Concrete examples may be computed by solving an ODE system derived from Eq. 2.2. By careful consideration of singularities and metric data associated with Q, one can determine the full trajectory structure of Q and singular flat

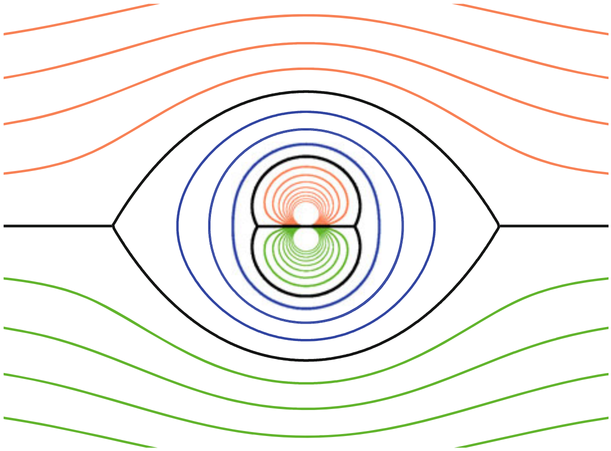

FIGURE 4 Ring domain and four half-planes of the ellipse

geometry of $\tilde{\Gamma}^*$. The relevant computational techniques are developed in [7], where the various graphical examples also require judicious choice of branch cuts defining the sheets of isotropic projection. Full plots—extending those shown in Sect. 2—can be somewhat complicated, but are not necessary for us to consider in detail here. Instead, we briefly describe the singular flat geometry of $\tilde{\Gamma}^*$ for the curves considered in the previous sections, and use these examples to illustrate the local portraits for horizontal arcs near the types of Q-singularities of greatest relevance to us.

Example. **Line** $\mathcal{L} : g = z - w = 0$ has quadratic differential $Q = dz^2$, which has no finite singularity. $(\mathcal{L} \setminus p, \rho)$ is the Euclidean plane, foliated by horizontal and vertical lines. Here, the ideal point $p = [X, Y, Z] = [1, 0, 0]$ corresponds to $\tilde{z} = 1/z = 0$, where $Q = -d\tilde{z}^2/\tilde{z}^4$ has a fourth order pole; near p, the horizontal trajectories exhibit the dipole local portrait anticipated above.

Example. **Circle** $\mathcal{C} : g = zw - 1 = 0$ has $Q = -\frac{dz^2}{z^2} = -\frac{dw^2}{w^2}$, which has second order poles at the circular points corresponding to $z = 0$, and to $w = 0$. $(\mathcal{C} \setminus \{c_+, c_-\}, \rho)$ is a cylinder foliated by parallels of length $\int_{\mathcal{C}} |\sqrt{Q}| = \int_{\mathcal{C}} |dz/z| = 2\pi$ and by verticals which extend infinitely up and down. The local portrait for a second order pole (with negative coefficient) is exhibited by the circles $|z| = e^t$, obtained by isotropic projection of the horizontal trajectories.

Example. **Ellipse** $\mathcal{E} : g = c^2(z^2 + w^2) - 2d^2zw + 4a^2b^2 = 0,\ 0 < b < a,\ c = \sqrt{a^2 - b^2},\ d = \sqrt{a^2 + b^2}$ has $Q = \frac{c^2z - d^2w}{d^2z - c^2w}dz^2 = \frac{c^2w - d^2z}{d^2w - c^2z}dw^2$ with two fourth order poles, at the $[X, Y, Z] = [a, \pm ib, 0]$, and four first order zeros, at $(z, w) = \pm(c, d^2/c)$ and $(z, w) = \pm(d^2/c, c)$. The poles/zeros illustrate (i) and (ii) of Proposition 4.1. In Fig. 4 ($a = 5, b = 3$), one recognizes the local portraits at the four simple zeros, where three horizontal trajectories meet at 120° angles; one also sees two fourth order poles (the other two being at infinity).

In Fig. 4 we have deviated from our usual graphical method. Instead of using isotropic projection, we have pulled back Q conformally to $\mathbb{C}$, so that everything may be viewed on one sheet: One ring domain (compare Fig. 1)—now interpreted as a cylinder of circumference $L = 20E(16/25) \approx 25.5$ and height $h = 2\tau \approx 3.3$—and four half-planes. Topologically, one may picture $\mathcal{E}_{\mathbb{R}}$ as the equator on S^2 and the cylinder as a tropical zone with 'peaks' at the four zeros of Q; the four critical trajectories not part of the cylinder boundary meet in pairs at north and south dipoles and partition the complement of the cylinder into the four half-planes.

Example. **Lemniscate** $\mathcal{B} : g = 2z^2w^2 - z^2 - w^2 = 0$ has exceptionally simple Schwarz function $w = S(z) = iz/\sqrt{1-2z^2}$ and $Q = S'(z)dz^2 = i(1-2z^2)^{-3/2}dz^2$. Note Q has no zeros and it has four first order poles at the double circular points (where $\tilde{z} = 1/z \to 0, w \to \pm 1/\sqrt{2}$ or $\tilde{w} = 1/w \to 0, z \to \pm 1/\sqrt{2}$). First order poles look like endpoints of arcs surrounded by 'hairpin turn' trajectories. The four poles of $\mathcal{B}$ are joined in pairs by two critical trajectories. $\mathcal{B} \simeq S^2$ may be pictured as a striped pillow made from a cylinder of circumference $L(\mathcal{B}) = 4K(-1)$ and height $h = 2K(-1)$ by collapsing each bounding circle to a critical arc.

It is natural also to consider Eq. 2.7 on the square $0 \le s, t < 4K$, i.e., on the underlying elliptic curve Σ, so that $\gamma(\zeta) = (z(\zeta), w(\zeta))$ describes a double covering of the sphere ($\mathcal{B}$) by the torus (Σ), branched over the four pillow corners. (Then $z(\zeta) = \pi_+(\gamma(\zeta))$ is a fourfold covering $z : \Sigma \to \mathbb{C}P^1$, where the branching of π_+ occurs over the two foci.)

Example. **Limaçon** The *limaçon of Pascal* is the bicuspidal quartic with equation $f(x, y) = (x^2 + y^2 - ax)^2 - b^2(x^2 + y^2)$. Inverting a quadric about a focus produces such a rational quartic with cusps at $c_\pm$. The third singularity of the resulting curve is an arcnode (complex node) for the elliptic limaçon ($0 < a < b$), a (real) node for the hyperbolic limaçon ($0 < b < a$), and a cusp for the cardioid ($0 < a = b$)—in case the quadric is a parabola.

The most importance feature of this example is that it illustrates the third order pole of Q resulting from the cusp at c_+. Q has another third order pole at c_- and two isotropic points, corresponding to the visible focus/defocus pair—see Fig. 5—so altogether we have $0 = Z(Q) - P(Q) = 1+1-3-3 = -4$, as expected. (This example also illustrates Plücker's formula for the *class* $m = n(n-1) - 2\delta - 3\kappa = 4$—the degree of the dual curve. A cusp at c_+ reduces the number $m = 4$ of tangents from c_+ by three, and the remaining tangency occurs at the finite +-isotropic point, which projects to the focus.) In the figure, the neighborhood of the circular focus is only partially filled in because it takes infinitely many equally spaced parallel geodesics to fill up the half-plane.

For a heuristic explanation of the behavior of $Q = dzdw$ at a cusp $c_\pm$, it may suffice to consider the origin on the familiar cuspidal cubic, $y^2 = x^3$. On the one hand, projection $z = \pi_p$ from $p = (0, 0)$ onto the line $x = 1$ 'resolves the singularity'. Namely, the line $y = tx$ through a point (x, y) on the curve intersects $x = 1$ at height $t = \pi_p(x, y) = y/x = \sqrt{x}$. Upon inversion, one arrives at the well-known rational parameterization for the cubic, $(x(t), y(t)) = \pi_p^{-1}(t) = (t^2, t^3)$. In other words, p is a regular point for π_p ($dz \neq 0$). On the other hand,

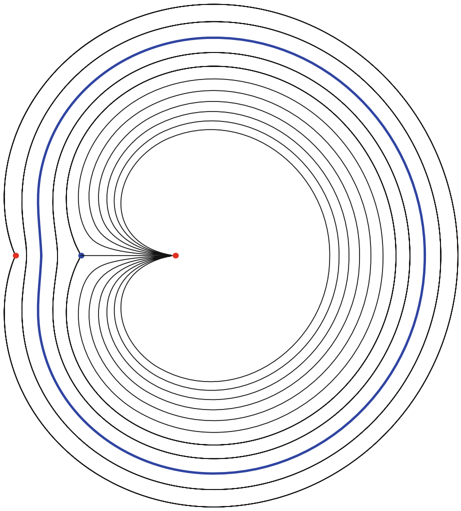

FIGURE 5 Limaçon $(x^2+y^2-3x)^2 = 25(x^2+y^2)$: Ring domain and one half-plane (of two); focus, defocus and circular focus

projection from a point $q \neq p$ is singular at $t = 0$. Specifically, let $q = (0, 1)$ (a convenient point not on the tangent line at p), and let $1/w = \pi_q$ denote projection from q onto the x-axis. Then $\pi_q(x, y) = \frac{x}{1-y} = \frac{t^2}{1-t^3}$ is locally two-to-one near $t = 0$ (dw has a third order pole). Cusps will be treated more formally below.

5. Euclidean maps

As above, let $\Gamma^* \subset \mathbb{C}P^2$ denote an irreducible, real algebraic curve and $(\tilde{\Gamma}^*, Q)$ its underlying Riemann surface, together with the meromorphic quadratic differential $Q = dzdw = dx^2 + dy^2$. Let $U \subset \mathbb{C}$ be a domain in the ζ-plane; points in U are denoted $\zeta = s + it$, and $d\zeta^2$ is the standard 'Euclidean quadratic differential' on U. The following definition formalizes the notion of 'complex arclength parameterization'.

Definition 5.1. A holomorphic map $\gamma = (z, w) : U \to \tilde{\Gamma}^*$ will be called a *Euclidean map* if Q pulls back to the Euclidean quadratic differential: $d\zeta^2 = \gamma^* Q = z'(\zeta)w'(\zeta)d\zeta^2$. In case $U = \mathbb{C}, \gamma$ will be called a *global Euclidean map*.

Remarks:

1. Our notation for Q, γ, etc., uses isotropic coordinates as in Proposition 4.1; that is, z, w are initially defined as analytic functions on the affine curve Γ, but induce meromorphic functions on $\tilde{\Gamma}^*$. Likewise, $\gamma = (z, w)$ is represented by functions $z(\zeta), w(\zeta)$ which are analytic except at poles, and uniquely determine $\gamma = (z, w) : U \to \tilde{\Gamma}^*$, a *holomorphic map*: for any local chart ϕ on $\tilde{\Gamma}^*, \phi(\gamma(\zeta))$ is holomorphic on its domain.

2. A Euclidean map $\gamma(\zeta)$ satisfies Eq. 2.2 on $V = \gamma^{-1}(\Gamma)$. Therefore: (a) $\gamma : V \to \Gamma \subset \mathbb{C}^2$ is an immersion; (b) A singularity of Γ in $\gamma(V)$ is an ordinary multiple point; (c) $\gamma(V) \subset Q_{\text{reg}}$, that is, the image of a Euclidean map cannot contain a finite zero of $Q, p \in \Gamma$ (likewise for poles).
3. The immersion $\gamma : V \to \Gamma$ is a local isometry, that is, $d\zeta d\bar{\zeta} = |\gamma^* Q| = \gamma^* \rho$; in fact, $\zeta = \gamma^{-1}(p)$ defines a natural parameter near any $p_0 \in \gamma(V)$.
4. If $\gamma(s)$ is any real parameterization of a regular algebraic curve $\Gamma_{\mathbb{R}}$ by arc length, then γ extends to a Euclidean map defined on a maximal horizontal strip $S : |\Im[\zeta]| < \tau \leq \infty$; the closure of $\gamma(S)$ contains a critical point of Q. For the curves $\mathcal{L}, \mathcal{C}, \mathcal{B}$ (line, circle, and lemniscate), $\tau = \infty$, i.e., $\gamma(\zeta)$ is a global Euclidean map.

Proposition 5.2. *The image of a global Euclidean map $\gamma : \mathbb{C} \to \tilde{\Gamma}^*$ is precisely the set of finitely distant points: $\gamma(\mathbb{C}) = Q_{\text{fin}}$.*

Proof. Obviously $\gamma(\mathbb{C}) \subset Q_{\text{fin}}$. Since Q_{fin} is connected and $\gamma(\mathbb{C})$ is open, it suffices to show that $\gamma(\mathbb{C})$ is also closed in Q_{fin}. If $p_0 \in Q_{\text{fin}}$ is the limit of points $p_j = \gamma(\zeta_j)$, some p_J belongs to a normal form neighborhood U of p_0, and p_J, p_0 lie at the ends of a half-open θ-arc of finite length, $\gamma(\zeta_j + te^{i\vartheta}),\ 0 \leq t < L$. By continuity, $\gamma(\zeta_j + Le^{i\vartheta}) = p_0$. $\square$

Combining Propositions 4.1, 5.2, we see that the image of a global Euclidean map $\gamma : \mathbb{C} \to \tilde{\Gamma}^*$ contains all finite points $p \in \Gamma$, but no ideal points—except possibly circular points $c_\pm \in \Gamma^*$. Also, by (2) (c), existence of γ now rules out zeros for Q, except possibly at circular points $c_\pm$.

As the lemniscate example shows, a simple pole of Q may indeed occur at a circular point—though it necessarily belongs to $\gamma(\mathbb{C})$. The fact that such a critical point of Q can be "masked" by γ^* reflects the greater variety of behavior of quadratic differentials as compared with linear meromorphic differentials. However, the point of the following lemma is that simple poles are the only critical points of a meromorphic quadratic differential which can be hidden by pull-back.

Lemma 5.3. *Let $Q' = h^* Q$ be the pull-back of a meromorphic quadratic differential Q by a holomorphic map h, p' a regular point of Q' and $p = h(p')$. Then either:*

(a) *p is a regular point of Q and p' is a regular point of h, or*
(b) *p is a simple pole of Q and p' is a simple zero of dh.*

Proof. We use the local normal form: $Q = (\frac{n+2}{2})^2 \xi^n d\xi^2$, where $\xi = 0$ at p. (We neglect cases with logarithmic term in Φ, since $n \geq -1$ in our application). Also, we use the local normal form for h: $\nu \xrightarrow{h} \xi = \nu^m$, where $\nu = 0$ at p', and dh has a zero at p' of order $m - 1 \geq 0$. Then

$$Q' = h^* Q = (\frac{n+2}{2})^2 \nu^{mn} (m\nu^{m-1} d\nu)^2 = m^2 (\frac{n+2}{2})^2 \nu^{mn+2m-2} d\nu^2$$

For p' to be a regular point of Q', integers $m \geq 1, n$ must satisfy $mn + 2m - 2 = 0$, i.e., $m(n+2) = 2$, so (a) $m = 1, n = 0$, or (b) $m = 2, n = -1$. □

A related, useful idea is to lift a given (Σ, Q) to an orientable quadratic differential $\tilde{Q} = \tilde{\omega}^2$ defined on a double cover $\tilde{\Sigma}$ branched over odd-order singularities of Q. Orders of singularities of $\tilde{Q}$ above even order singularities of Q will be the same, but orders above branch points are related by $\mathrm{ord}_{\tilde{p}}\tilde{Q} = 2\mathrm{ord}_p(Q) + 2$. So $\tilde{Q}$ has only even order singularities; in particular, if $\mathrm{ord}_p(Q) = -1$, the singularity at $\tilde{p}$ 'disappears'.

To apply the lemma to our present situation, we assume a circular point $p = c_\pm$ is not a higher order pole of $Q = dzdw$. If $h = \gamma$ is a global Euclidean map, $p = h(p')$ for some p'. It follows that p can only be a regular point or simple pole of Q. Since we have just ruled out the remaining possibility for a zero of Q, the first sentence of the following proposition has been established:

Proposition 5.4. *If $\gamma : \mathbb{C} \to \tilde{\Gamma}^*$ is a global Euclidean map, $Q = dzdw$ has no zeros. Therefore, $\tilde{\Gamma}^*$ has genus zero or one and fits one of the following cases:*

1. *$g(\tilde{\Gamma}^*) = 0$ and Q has one fourth order pole $p \in \mathbb{R}P^2$.*
2. *$g(\tilde{\Gamma}^*) = 0$ and Q has a pair of double poles at $\tilde{c}_\pm$.*
3. *$g(\tilde{\Gamma}^*) = 0$ and Q has four simple poles at $\tilde{c}_\pm$.*
4. *$g(\tilde{\Gamma}^*) = 1$ and Q has no poles.*

Proof. We invoke the degree formula $\deg(Q) = Z(Q) - P(Q) = 4g - 4$. (For an orientable quadratic differential, this follows from the formula for ordinary differentials, which is dual to the Poincaré–Hopf index formula for vector-fields: $\deg(Q) = \deg(\omega^2) = 2\deg(\omega) = 2(2g-2)$. Otherwise, one may lift Q to $\tilde{Q}$, as above, and take account of the Riemann–Hurwitz relation $4\tilde{g} - 4 = 4m(g-1) + 2B$; here, $m = 2$ is the degree of the covering and B is the total branching number.)

Reality of the fourth order pole in case (1) follows by symmetry, and pairing of poles of order less than four, necessarily at $c_\pm$, rules out $P(Q) = 1 + 3$. □

Remark 5.5. Except in case (1), $\tilde{\Gamma}^*$ is totally circular. In cases (1), (2), (4), γ is an unbranched covering onto, respectively, $\tilde{\Gamma}^* \setminus \{p\} \simeq \mathbb{C}$, a cylinder $\tilde{\Gamma}^* \setminus \{c_+, c_-\} \simeq \mathbb{C}/\mathbb{Z}$, or a torus $\tilde{\Gamma}^* \simeq T^2$. In case (3), γ factors as $\gamma = \pi_2 \circ \pi_1$, where $\pi_1 : \mathbb{C} \to T^2$ is a regular covering (parameterization by elliptic functions) and $\pi_2 : T^2 \to \tilde{\Gamma}^*$ is a double covering branched at four points $c^j_\pm$; this corresponds to lifting Q to $\tilde{Q}$, as above.

In general, the limitations on Q still allow for a wide range of behavior for dz, dw at a circular point. Namely, dw might have a pole of order $k \geq 2$ and dz a zero of order $j + k - 2$; then Q will have a double pole, simple pole or regular point, respectively, for $j = 0, 1, 2$—consistent with the proposition. Here, the higher order zeros of dz are made possible by a high order of contact of a tangent line with a branch of Γ^*. It will be convenient to adopt a classical term for such 'higher order inflection points' (though traditional usage is sometimes restricted to regular points, such as the origin on the curve $y = x^4$):

Definition 5.6. A point $p \in \Gamma^*$ will be called an *undulation point* when at least one tangent line at p makes four point contact with its branch.

Theorem 5.7. *Let $\Gamma^* \subset \mathbb{C}P^2$ be an irreducible, real algebraic curve. In case the circular points $c_\pm$ belong to Γ^*, assume $c_\pm$ are not undulation points. Suppose a local arclength parameterization $\gamma : (a, b) \to \Gamma_{\mathbb{R}}$ extends meromorphically to the complex plane. Then $\Gamma_{\mathbb{R}}$ is a line, a circle, or a Bernoulli Lemniscate.*

Proof. The hypothesis implies there exists a global Euclidean map $\gamma : \mathbb{C} \to \tilde{\Gamma}^*$, so $\tilde{\Gamma}^*, Q$ must fit one of the cases of Proposition 5.4. If Γ^* is not circular, it has a unique ideal point $p \in \tilde{\Gamma}^*$. This is because Q has at least fourth order poles at non-circular ideal points, and can only belong to case (1): dz and dw are regular except for simple poles at the one ideal point $p \in \mathbb{R}P^2$, isotropic projection $z : \tilde{\Gamma}^* \to \mathbb{C} \cup \{\infty\}$ is unbranched, and $n = deg(\Gamma^*) = 1$. In other words, Γ^* is a line.

Thus, from now on, we assume Γ^* is circular. In order to discuss the behavior of dz and dw at $\tilde{c}_\pm \in \tilde{\Gamma}^*$, it suffices to consider a local parameterization at $c_+ = \{0, 1, 0\}$:

$$\alpha(t) = \sum_{j=k}^{\infty} \alpha_j t^j, \quad \beta(t) = 1 + \sum_{j=1}^{\infty} \beta_j t^j, \quad \eta(t) = \sum_{j=k}^{\infty} \eta_j t^j, \qquad k \geq 1, \quad \eta_k \neq 0$$

Here we use the isotropic coordinates 3.2 (with $\lambda = 1$), and $k \geq 1$ denotes the smallest integer for which $\eta_k \neq 0$. Note also that we have written $\alpha_k t^k$ as the first term of $\alpha(t)$ (though it may vanish); if a lower order term were non-zero, $z(t) = \frac{\alpha(t)}{\eta(t)}$ would have a pole, and then Q would have fourth order poles at both circular points, contradicting the proposition.

The tangent line is the isotropic line $z = \lim_{t\to 0} \frac{\alpha(t)}{\eta(t)} = \frac{\alpha_k}{\eta_k}$, i.e., $\eta_k \alpha - \alpha_k \eta = 0$. Substitution of $\alpha(t), \eta(t)$ into the latter equation gives: $0 = \eta_k \alpha(t) - \alpha_k \eta(t) = \sum_{j=k}^{\infty} (\eta_k \alpha_j - \alpha_k \eta_j) t^j$. Here, the leading term drops out and, after re-indexing, the equation may be expressed: $0 = t^{k+1} \sum_{i=0}^{\infty} (\eta_k \alpha_{i+k+1} - \alpha_k \eta_{i+k+1}) t^j$.

For c_+ to be non-undulational, $k \leq 2$. In case $k = 2, dz|_{c_+} = \frac{\eta_2\alpha_3 - \alpha_2\eta_3}{\eta_2^2} dt \neq 0$, for the vanishing of the latter quantity would imply c_+ is an undulation point. Since $w(t)$ has a second order pole at $t = 0, Q = dzdw$ has a third order pole, again contradicting the proposition. (This is the case of a cusp at c_+.)

In case $k = 1, \tilde{\Gamma}^*$ is locally transverse to the ideal line; since this holds for each branch, c_+ is either a regular point of Γ^* or an ordinary multiple point. Then we may use $t = \eta$ to parameterize $\tilde{\Gamma}^*$ locally, and write $z(t) = t^{-1}\alpha(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \cdots, \quad w(t) = t^{-1}\beta(t) = t^{-1} + \beta_1 + \beta_2 t + \cdots$. The intersection of $\tilde{\Gamma}^*$ with the tangent line $\alpha - \alpha_1 \eta = 0$ is computed by solving $0 = \alpha(t) - \alpha_1 t = \alpha_2 t^2 + \alpha_3 t^3 + \cdots$ Here, α_2 and α_3 cannot both vanish, since c_+

is non-undulational. It follows that Q has a pole at each branch of each circular point; further, to be consistent with the proposition, Γ^* must be totally circular.

Now the tangent line makes just two point contact with its branch (simple tangency) when $dz|_{c_+} = \alpha_2 dt \neq 0$ and Q has a second order pole (case (2)). This is precisely the case where c_+ is a regular point, the ideal line meets Γ^* once at each circular point, and Γ^* is a circular quadric, i.e., a circle.

It remains to consider the possibility of an inflection point at c_+, where the tangent line makes three point contact with its branch. This case, and only this case, gives $Q = d(\alpha_3 t^2 + \alpha_4 t^3 \cdots)d(t^{-1} + \beta_1 + \cdots)$ a simple pole at c_+. Because a simple pole occurs only in case (3), it is evident that $\tilde{\Gamma}^*$ has exactly two branches at each circular point (thus accounting for the four simple poles of Q), and that Γ^* is a rational quartic with biflecnodes at $c_\pm$. In fact, the real singularity must also be a node (finite cusps being inconsistent with γ), so Proposition 3.1 implies $\tilde{\Gamma}^*$ is a Bernoulli lemniscate. $\square$

Theorem 1.1 now follows as a corollary. For an undulation point in a quartic (no such point exists for $n \leq 3$) must be a regular point $p \in \Gamma^*_{\text{reg}}$; otherwise the tangent line at p would meet Γ^* five times. Similarly, such an undulation point cannot be a point of tangency to the ideal line. Nor can $c_\pm$ be undulation points of Γ^*, because then $c_\pm$ would only account for two of four required meetings with the ideal line and the other two meetings (distinct or not) would result in $P(Q) > 4$. Thus, having ruled out circular undulation points, the theorem applies.

References

[1] Basset, A.: *An Elementary Treatise on Cubic and Quartic Curves*. Merchant Books, New York (2007)

[2] Calini, A., Langer, J.: *Schwarz reflection geometry I: continuous iteration of reflection*. Math. Z. **244**, 775–804 (2003)

[3] Calini, A., Langer, J.: *Schwarz reflection geometry II: local and global behavior of the exponential map*. Exp. Math. **16**(3), 321–346 (2007)

[4] Davis, P.: *The Schwarz function and its applications*. In: The Carus Mathematical Monographs, vol. 17. The Mathematical Association of America (1974)

[5] Farkas, H., Kra, I.: *Riemann Surfaces*. GTM 71. Springer, Berlin (1980)

[6] Hilton, H.: *Plane Algebraic Curves*, 2nd edn. Oxford University Press, Oxford (1932)

[7] Langer, J.: *Plotting Algebraic Curves*. Preprint

[8] Langer, J., Singer, D.: *Foci and foliations of real algebraic curves*. Milan J. Math. **75**, 225–271 (2007)

[9] Langer, J., Singer, D.: *When is a curve an octahedron*? Am. Math. Mon. **117**(10), 889–902 (2010)

[10] Langer, J., Singer, D.: *Reflections on the lemniscate of Bernoulli: The forty eight faces of a mathematical gem*. Milan J. Math. **78**, 643–682 (2010)

[11] Langer, J., Singer, D.: *The lemniscatic chessboard*. Forum Geometricorum (to appear)

[12] Muciño-Raymundo, J.: *Complex structures adapted to smooth vector fields*. Math. Ann. **322**, 229–265 (2002)

[13] Prasolov, V., Solovyev, Y.: *Elliptic functions and elliptic integrals*. In: Translations of Mathematical Monographs. American Mathematical Society (1997)

[14] Rosen, M.: *Abel's theorem on the lemniscate*. Am. Math. Mon. **88**, 387–395 (1981)

[15] Salmon, G.: A *Treatise on the Higher Plane Curves*, 3rd edn. G. E. Stechert & Co. (1934)

[16] Shapiro, H.: *The Schwarz function and its generalization to higher dimensions*. In: University of Arkansas Lecture Notes in the Mathematical Sciences, vol. 9. Wiley, New York (1992)

[17] Stillwell, J.: *Mathematics and its History*, 2nd edn. Springer, Berlin (2002)

[18] Strebel, K.: *Quadratic Differentials*. Springer, Berlin (1984)

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