

Introduction

The present volume analyzes mathematical models of time-dependent physical phenomena on three levels: *microscopic*, *mesoscopic*, and *macroscopic*. We provide a rigorous derivation of each level from the preceding level and the resulting mesoscopic equations are analyzed in detail. Following Haken (1983, Sect. 1.11.6) we deal, “*at the microscopic level, with individual atoms or molecules, described by their positions, velocities, and mutual interactions. At the mesoscopic level, we describe the liquid by means of ensembles of many atoms or molecules. The extension of such an ensemble is assumed large compared to interatomic distances but small compared to the evolving macroscopic pattern. . . . At the macroscopic level we wish to study the corresponding spatial patterns.*” Typically, at the macroscopic level, the systems under consideration are treated as spatially continuous systems such as fluids or a continuous distribution of some chemical reactants, etc. In contrast, on the microscopic level, Newtonian mechanics governs the equations of motion of the individual atoms or molecules.¹ These equations are cast in the form of systems of deterministic coupled nonlinear oscillators. The mesoscopic level² is probabilistic in nature and many models may be faithfully described by stochastic ordinary and stochastic partial differential equations (SODEs and SPDEs),³ where the latter are defined on a continuum. The macroscopic level is described by time-dependent partial differential equations (PDE’s) and its generalization and simplifications.

In our mathematical framework we talk of particles instead of atoms and molecules. The *transition from the microscopic description to a mesoscopic (i.e., stochastic) description* requires the following:

- Replacement of spatially extended particles by point particles
- Formation of small clusters (ensembles) of particles (if their initial positions and velocities are similar)

¹ We restrict ourselves in this volume to “classical physics” (cf., e.g., Heisenberg (1958)).

² For the relation between nanotechnology and mesoscales, we refer to Roukes (2001).

³ In this volume, mesoscopic equations will be identified with SODEs and SPDEs.

- Randomization of the initial distribution of clusters where the probability distribution is determined by the relative sizes of the clusters
- “*Coarse graining*,” i.e., representation of clusters as cells or boxes in a grid for the positions and velocities

Having performed all four simplifications, the resulting description is still governed by many deterministic coupled nonlinear oscillators and, therefore, a simplified *microscopic* model.

Given a probability distribution for the initial data, it is possible, through scaling and similar devices, to proceed to the mesoscopic level, governed by SODEs and SPDEs, as follows:

- Following Einstein (1905), we consider the substance under investigation a “solute,” which is immersed in a medium (usually a liquid) called the “solvents.” Accordingly, the particles are divided into two groups: (1) *Large particles*, i.e., the solute particles; (2) *small particles*, the solvent particles.
- Neglect the interaction between small particles.

- Consider first the interaction between large and small particles. To obtain the Brownian motion effect, increase the initial velocities of the small particles (to infinity). Allow the small particles to escape to infinity after having interacted with the large particles for a macroscopically small time. This small time induces a partition of the time axis into small time intervals. In each of the small time intervals the large particles are being displaced by the interaction with clusters of small particles. Note that the vast majority of small particles have previously not interacted with the large particles and they disappear to infinity after that time step. (Cf. Figs. 1 and 2.) This implies almost independence of the displacements of the large particles in different time intervals and,

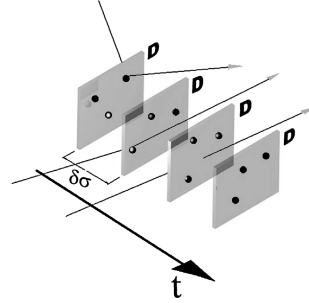


Fig. 1

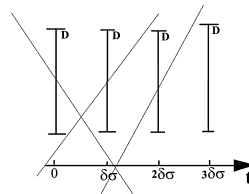


Fig. 2

in the scaling limit, independent increments of the motion of the large particles. To make this rigorous, an infinite system of small particles is needed if the interval size tends to 0 in the scaling limit. Therefore, depending on whether or not friction is included in the equations for the large particles, we obtain that, in the scaling limit, the positions or the velocities of the large particles perform Brownian motions in time.⁴ If the positions are Brownian motions, this model is called

⁴ The escape to infinity after a short period of interaction with the large particles is necessary to generate independent increments in the limit. This hypothesis seems to be acceptable if for

the Einstein-Smoluchowski model, and if the velocities are Brownian motions, then it is called an Ornstein-Uhlenbeck model (cf. Nelson, 1972).

- The interaction between large particles occurs on a much slower time scale than the interaction between large and small particles and can be included after the scaling limit employing fractional steps.⁵ Hence, the positions of the large particles become solutions of a system of SODEs in the Einstein-Smoluchowski model.
- The step from (Einstein-Smoluchowski) SODEs to SPDEs, which is a more simplified mesoscopic level, is relatively easy, if the individual Brownian motions from the previous step are obtained through a Gaussian space–time field, which is uncorrelated in time but spatially correlated. In this case the empirical distribution of the solutions of the SODEs is the solution of an SPDE, independent of the number of particles involved, and the SPDE can be solved in a space of densities, if the number of particles tends to infinity and if the initial particle distribution has a density. The resulting SPDE describes the distribution of matter in a continuum.

The *transition from the mesoscopic SPDEs to macroscopic (i.e., deterministic) PDE's* occurs as follows:

- As the correlation length⁶ in the spatially correlated Gaussian field tends to 0 the solutions of the SPDEs tend to solutions of the macroscopic PDEs (as a weak limit).

The mesoscopic SPDE is formally a PDE perturbed by state-dependent Brownian noise. This perturbation is small if the aforementioned correlation length is small. Roughly speaking, the spatial correlations occur in the transition from the microscopic level to the mesoscopic SODEs because the small particles are assumed to move with different velocities (e.g., subject to a Maxwellian velocity distribution). As a result, small particles coming from “far away” can interact with a given large particle “at the same time” as small particles were close to the large particles. This generates a long-range mean-field character of the interaction between small and large particles and leads in the scaling limit to the Gaussian space–time field, which is spatially correlated. Note that the perturbation of the PDE by state-dependent Brownian noise is derived from the microscopic level. We conclude that the correlation length is a result of the discrete spatially extended structures of the microscopic level. Further, on the mesoscopic level, the correlation length is a measure of the strength of the fluctuations around the solutions of the macroscopic equations.

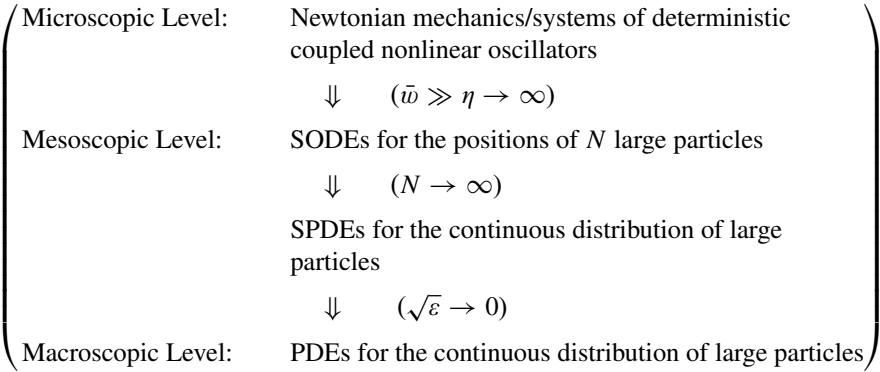
Let \bar{w} denote the average speed of the small particles, $\eta > 0$ the friction coefficient for the large particles. The typical mass of a large particle is $\approx \frac{1}{N}$, $N \in \mathbf{N}$, and

spatially extended particles the interparticle distance is considerably greater than the diameter of a typical particle. (Cf. Fig. 1.) This holds for a gas (cf. Lifshits and Pitayevskii (1979), Ch.1, p. 3), but not for a liquid, like water. Nevertheless, we show in Chap. 5 that the qualitative behavior of correlated Brownian motions is in good agreement with the depletion phenomenon of colloids in suspension.

⁵ Cf. Goncharuk and Kotelenz (1998) and also our Sect. 15.3 for a description of this method.

⁶ Cf. the following Chap. 1 for more details on the correlation length.

$\sqrt{\varepsilon} > 0$ is the correlation length in the spatial correlations of the limiting Gaussian space–time field. Assuming that the initial data of the small particles are coarse-grained into independent clusters, the following scheme summarizes the main steps in the transition from microscopic to macroscopic, as derived in this book:



Next we review the general content of the book. Formally, the book is divided into five parts and each part is divided into chapters. The chapter at the end of each part contain lengthy and technical proofs of some of the theorems that are formulated within the first chapter. Shorter proofs are given directly after the theorems. Examples are provided at the end of the chapters. The chapters are numbered consecutively, independent of the parts.

In Part I (Chaps. 1–3), we describe the transition from the microscopic equations to the mesoscopic equations for correlated Brownian motions. We simplify this procedure by working with a space–time discretized version of the infinite system of coupled oscillators. The proof of the scaling limit theorem from Chap. 2 in Part I is provided in Chap. 3. In Part II (Chaps. 4–7) we consider a general system of Itô SODEs⁷ for the positions of the large particles. This is called “mesoscopic level A.” The driving noise fields are both correlated and independent, identically distributed (i.i.d) Brownian motions.⁸ The coefficients depend on the empirical distribution of the particles as well as on space and time. In Chap. 4 we derive existence and uniqueness as well as equivalence in distribution. Chapter 5 describes the qualitative behavior of correlated Brownian motions. We prove that correlated Brownian motions are weakly attracted to each other, if the distance between them is short (which itself can be expressed as a function of the correlation length). We remark that experiments on colloids in suspension imply that Brownian particles at close distance must have a tendency to attract each other since the fluid between them gets *depleted* (cf. Tulpar et al. (2006) as well as Kotelenetz et al. (2007)) (Cf. Fig. 4

⁷ We will drop the term “Itô” in what follows, as we will always use Itô differentials, unless explicitly stated otherwise. In the alternative case we will consider Stratonovich differentials and talk about Stratonovich SODEs or Stratonovich SPDEs (cf. Chaps. 5, 8, 14, Sects. 15.2.5 and 15.2.6).

⁸ We included i.i.d. Brownian motions as additional driving noise to provide a more complete description of the particle methods in SPDEs.

in Chap. 1). Therefore, our result confirms that correlated Brownian motions more correctly describe the behavior of a solute in a liquid of solvents than independent Brownian motions. Further, we show that the long-time behavior of two correlated Brownian motions is the same as for two uncorrelated Brownian motions if the space dimension is $d \geq 2$. For $d = 1$ two correlated Brownian motions eventually clump. Chapter 6 contains a proof of the flow property (which was claimed in Chap. 4). In Chap. 7 we compare a special case of our SODEs with the formalism introduced by Kunita (1990). We prove that the driving Gaussian fields in Kunita's SODEs are a special case of our correlated Brownian motions. In Part III (*mesoscopic level B*, Chaps. 8–13) we analyze the SPDEs⁹ for the distribution of large particles. In Chap. 8, we derive existence and strong uniqueness for SPDEs with finite initial mass. We also derive a representation of semilinear (Itô) SPDEs by Stratonovich SPDEs, i.e., by SPDEs, driven by Stratonovich differentials. In the special case of noncoercive semilinear SPDEs, the Stratonovich representation is a first order transport SPDE, driven by Stratonovich differentials. Chapter 9 contains the corresponding results for infinite initial mass, and in Chap. 10, we show that certain SPDEs with infinite mass can have homogeneous and isotropic random fields as their solutions. Chapters 11 and 12 contain proofs of smoothness, an Itô formula and uniqueness, respectively. In Chap. 13 we review some other approaches to SPDEs. This section is by no means a complete literature review. It is rather a random sample that may help the reader, who is not familiar with the subject, to get a first rough overview about various directions and models. Part IV (Chap. 14) contains the macroscopic limit theorem and its complete proof. For semi-linear non-coercive SPDEs, using their Stratonovich representations, the macroscopic limit implies the convergence of a first order transport SPDE to the solution of a deterministic parabolic PDE. Part V (Chap. 15) is a general appendix, which is subdivided into four sections on analysis, stochastics, the fractional step method, and frame-indifference. Some of the statements in Chap. 15 are given without proof but with detailed references where the proofs are found. For other statements the proofs are sketched or given in detail.

Acknowledgement

The transition from SODEs to SPDEs is in spirit closely related to D. Dawson's derivation of the measure diffusion for branching Brownian motions and the resulting field of superprocesses (cf. Dawson (1975)). The author is indebted to Don Dawson for many interesting and inspiring discussions during his visits at Carleton University in Ottawa, which motivated him to develop the particle approach to SPDEs. Therefore, the present volume is dedicated to Donald A. Dawson on the occasion of his 65th birthday.

A first draft of Chaps. 4, 8, and 10 was written during the author's visit of the Sonderforschungsbereich "Diskrete Strukturen in der Mathematik" of the University of

⁹ Cf. our previous footnote regarding our nomenclature for SODEs and SPDEs.

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