Integrable and near integrable °uid oscillators

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Abstract

We study 2D ideal Euler °uid truncated to ¯nitely many Fourier modes, and show that some low mode systems are integrable, or near integrable. Those include the classical triad-oscillator, exactly solvable in Jacobi elliptic functions, and some larger models made of coupled triad-oscillators. Our method exploits conserved integrals, Hamiltonian structure, and elliptic (Jacobi-Weiserstrass) parameterization. The motion is shown to be periodic or quasi-period with periods given by the complete elliptic integrals. We also use triad invariants to construct asymptotic solutions of non-integrable forced-dissipative systems.

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1 Introduction

Fluid dynamics is typically associated with complex patterns and phenomena that involve multiple scales, transient structures, chaos and turbulence so one does not expect to see much of regular patterns, or integrability.

Indeed, ideal fluids have but a few conserved integrals, including energy, "functionals of vorticity" (enstrophy) in 2D, and possible conservation laws based on symmetries, e.g. angular momentum. The infinite set of "vorticity invariants" (called Casimirs) does not make fluid an integrable system. Indeed, their joint level sets in the "phase-space" of velocity configurations are still infinitely large. The reduced dynamics could be cast in the Hamiltonian form (Arnold 1978, Kuznetsov-Mikhailov 1980, Marsden-Weinstein 1983), but is not likely to possess any additional (non-trivial) integrals, besides energy, enstrophy, helicity, and the symmetry integrals. This conclusion was partly corroborated in Gurarie 1995, that showed energy and enstrophy be the only quadratic invariants of 2D ideal Euler fluid.

Occasionally one could see integrable motion to creep in the realm of fluids under exceptional conditions, like KdV soliton, Hasimoto vortex filament 1972, or some its recent generalizations (Kuznetsov -, 1999).

Interestingly, integrability reemerges in some small (Galerkin) truncated fluid systems, made of the few lowest interacting modes - the focus of our paper. Truncated fluid systems appear naturally in the study of fluids, theoretical or computational. For instance, forced Kolmogorov flow, made of a single mode with velocity profile $U = (\cos y, 0)$, becomes unstable past critical force value, and evolves (in the first approximation) into a triad of Fourier modes (Dolzhanski, et al 1981).

Truncated Euler systems are derived from the 2D vorticity equation:

$$\zeta_t + J(\psi, \zeta) = F + D \text{ - forcing-dissipation} \tag{1}$$

$\psi(x, y, t)$ - stream-field, $\zeta = \Delta\psi$ - vorticity, $J(f, g) = f_x g_y - f_y g_x$ - Jacobian, expanded in Fourier modes: $\zeta = \sum_k z_k(t) e^{ik\cdot r}$. The resulting coupled nonlinear system

$$\dot{z}_k = \sum_{p+q=k} A_{pq} z_p z_q \tag{2}$$

has structure coefficients

$$A_{p,q} = p \wedge q (1/p^2 - 1/q^2) \tag{3}$$

Confining $\{p, q, k\}$ to a finite lattice set, e.g. $\{p, q, k \leq N\}$, we get a truncated (low mode) system.

In general, truncation leads to the loss of conservation properties, particularly, higher order Casimirs[1]. But two important quadratic invariants, energy and enstrophy remain

$$E = \tfrac{1}{2} \sum |z_k|^2 / k^2; \; Q = \tfrac{1}{2} \sum |z_k|^2$$

[1] Several truncation schemes were proposed, based on modified structure coefficients (??), that retain some vorticity invariants (e.g. Zeitlin 1991).

- a consequence of "triad resummation", and special form of coefficients (3)

$$\frac{d}{dt}E = \sum_{p+q+k=0} \underbrace{\left(A_{p,q}+A_{q,k}+A_{k,p}\right)}_{0} \mathrm{Re}\left(z_p^* z_q^* z_k^*\right)$$

Indeed, one could show using Gurarie 1995, that truncated E and Q are the only quadratic invariants for sufficiently "large" lattices $\{k : |k| \le N\}$.

In stark contrast, "small" fluid oscillators (2), composed of the lowest Fourier modes retain, or gain, sufficient number of conserved invariants, to become completely integrable (or near integrable). We shall list them in the increasing order. A single **real triad** $\{x_p; x_k; x_q\}$,

$$\psi = x_p \cos p\cdot r + x_k \cos k\cdot r + x_q \cos q\cdot r;\ p+k+q=0$$

has two quadratic invariants (energy/ enstrophy), and is solved exactly in Jacobi elliptic functions (reviewed in section 2.1). Next comes **complex triad** $\{z_p; z_k; z_q\}$ (6D real system)

$$\psi = z_p e^{ip\cdot r} + z_k e^{ik\cdot r} + z_q e^{iq\cdot r}$$

identified with four coupled real triads. It possesses natural Hamiltonian structure, has 3 conserved integrals: energy/ enstrophy (quadratic), and cubic Hamiltonian. It also exhibits strictly periodic motion, and a closed form solution in terms of the Weierstrass and Jacobi functions. Finally, we study the octaplet of Fourier modes $\{p; q; \pm p \pm q\}$ - a 4D complex (8D-real) system of coupled triad oscillators, and show it to be near-integrable, with 6 quadratic conserved integrals[2]. It exhibits more complicated quasi-periodic patterns with two basic periods. We conjecture, that this quasi-periodic motion could be parametrized via a suitable generalization of the Weierstrass/Jacobi elliptic theory. As Jacobi functions cn, sn, dn, parametrize joint level curves of two elliptic cylinders in 3-space,

$$\xi = \frac{x_p^2}{a} + \frac{x_k^2}{b};\quad \eta = \frac{x_p^2}{a} + \frac{x_q^2}{c} \qquad (4)$$

the extended theory should do the same for 2D algebraic surfaces in 8-space defined by 6 quadratic invariants of the octaplet.

While the general problem remains open we gain some insight on the octaplet dynamics by reducing it to a single pair of modes (real + imaginary) for either p or q: $\{x_p; y_p\}$, or $\{x_q; y_q\}$. Each pair is shown to describe a 2D particle-motion in a quartic potential well, whose coefficients depend on the quadratic invariants of the octaplet system.

We conclude the paper with the study of forced-dissipative triads, that appear in the perturbed zonal flows. Such system are non-integrable, yet conserved integrals of section 2.1, could be used to construct its asymptotic solutions in certain stable regimes.

Our interest in fluid oscillators is partly motivated by their role in larger "primitive equations", like shallow water. Shallow water could be viewed as coupling between 2D quasigeostrophy (a modified Euler fluid) and the gravity waves modes. Under certain conditions (small Froude-Rossby numbers) vorticity and wave modes evolve on distinct time

[2] In earlier work, Zeitlin 1991, has found three quadratic invariants for a modified octaplet (modified structure coefficients A), based of his Lie symmetry approach.

scales, and their interaction affects the dynamics of both. When confined to a single triad of Fourier modes, the regular periodic motion of the quasigeostrophic (Euler) triad turn into complicated "chaotic" process for the shallow water triad. The QGS/Euler invariants loose their conservation property, and turn into slow "adiabatic invariants" of the shallow water. Such adiabatic invariants describe slow modulation of the 2D vorticity component of the shallow water system on long time scales.

We developed the shallow water modulation theory (Gurarie 1999) for a single Lorenz type triad, made of 3 vortical modes plus 6 gravity wave components. The present results would allow one to extend this "modulation theory" to yet larger truncated shallow water systems, based on its "integrable QGS-oscillators".

2 Triad oscillator

We write a 2D ideal Euler fluid in the vorticity form (1), and expand it in complex Fourier modes, $\omega = \sum_k z_k e^{ik\cdot x}$. The Fourier coefficients solve a coupled nonlinear system (2) with coefficients (3). Let us remark, that a similar system arises for quasigeostrophic equation for potential vorticity ω (in the shallow water approximation), but the Laplacian in the vorticity-stream relation being replaced by the Helmholtz operator: $\psi = (\lambda^2 - \Delta)^{-1}\omega$, $1/\lambda$ - Rossby deformation radius , so the new coefficients: $A_{pq} = p \wedge q \left(\frac{1}{\lambda^2+q^2} - \frac{1}{\lambda^2+p^2} \right)$.

Since real fields ω have complex conjugate coefficients: $z_{-k} = z_k^*$, products (2) could be recast in terms of balanced triangles $\{p+q+k=0\}$ as

$$\dot{z}_k + \sum_{p+q+k=0} A_{pq} z_q^* z_p^* = \ldots \tag{5}$$

A single triad $\{p;q;k\}$ forms a smallest 3D complex subsystem of (2),

$$\begin{aligned} \dot{z}_p &= A_{kq} z_k^* z_q^* \\ \dot{z}_k &= A_{qp} z_q^* z_p^* \\ \dot{z}_q &= A_{pk} z_p^* z_k^* \end{aligned} \tag{6}$$

that we shall call the **triad-oscillator**. We shall label wave-vectors $\{p;q;k\}$, so that $q \le p \le k$, hence signs of A-coefficients

$$\begin{aligned} A_{qp} &= -b;\ A_{pk} = -c \text{ - negative} \\ A_{kq} &= a = b + c \text{ - positive} \end{aligned}$$

Each triad oscillator is completely solvable in quadratures (Jacobi elliptic functions), explained below, and (2) could be viewed as coupled system of triad oscillators. Real coefficients $\{z_k = x_k\}$ correspond to cos-Fourier modes, hence even vorticity function, while imaginary ones give odd sin-Fourier expansion.

The complete system (2) has two conserved integrals:

$$\begin{aligned} \frac{1}{2}\int |\nabla\psi|^2 &= \frac{1}{2}\int \left(-\Delta^{-1}\omega\right)\cdot\omega^* = \sum \frac{1}{k^2}|\omega_k|^2 \text{ - energy} \\ \int |\omega|^2 &= \sum |\omega_k|^2 \text{ - enstrophy} \end{aligned}$$

We are interested in Galerkin-truncated fluid systems in Fourier space (2), with summation confined to a small coordinate lattice box: $\{k = (k_1; k_2) : |k_i| \le N\}$, and their "free nonlinear oscillations" in the absence of forcing and dissipation.

We start by reviewing the simplest case of a real triad.

2.1 Real triad

Real triad oscillator $\{x_p; x_k; x_q\}$, $p + k + q = 0$, corresponds to even cos-Fourier expansion

$$\zeta = x_p \cos p \cdot r + x_q \cos q \cdot r + x_k \cos k \cdot r$$

Its coefficients solve a nonlinear system,

$$\begin{aligned} \dot{x}_p &= A_{kq} x_k x_q = a x_k x_q \\ \dot{x}_q &= A_{pk} x_p x_k = -b x_p x_k \\ \dot{x}_k &= A_{qp} x_q x_p = -c x_q x_p \end{aligned} \tag{7}$$

that conserves two quadratic invariants (linear combinations of energy and enstrophy):

$$\xi = \frac{x_p^2}{a} + \frac{x_q^2}{b}; \quad \eta = \frac{x_p^2}{a} + \frac{x_k^2}{c} \tag{8}$$

Those yield upon substitution in the first equation (7), a single separable DE

$$\dot{x}_p = \pm \sqrt{abc \left(a\xi - x_p^2\right)\left(a\eta - x_p^2\right)}$$

and a closed form solution in the Jacobi elliptic functions of modulus $m = \min\left(\frac{\xi}{\eta}; \frac{\eta}{\xi}\right)$. Namely,

$$\begin{aligned} x_p &= \sqrt{bc\xi}\, \mathrm{cn}\left(\sqrt{\eta}\, t \,\Big|\, \frac{\xi}{\eta}\right) \\ x_q &= \sqrt{ac\eta}\, \mathrm{dn}\left(\sqrt{\eta}\, t \,\Big|\, \frac{\xi}{\eta}\right) \\ x_k &= \sqrt{ab\xi}\, \mathrm{sn}\left(\sqrt{\eta}\, t \,\Big|\, \frac{\xi}{\eta}\right) \end{aligned} \tag{9}$$

assuming[3] $\xi < \eta$. The motion is strictly periodic with period T defined by the complete elliptic integral of the first kind (see fig. 1)

$$T(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\cos^2\theta}}; \quad m = \min\left(\frac{\xi}{\eta}; \frac{\eta}{\xi}\right) \tag{10}$$

[3] The x_k and x_q -modes could change their Jacobi sn, dn -identities, depending on $\xi < \eta$ or $\xi > \eta$.

2.2 Complex triad (sextet)

Single complex triad $\{z_p, z_q, z_k\} = \{z_1, z_2, z_3\}$, (fig. 2) when expanded in 6 real modes $z_j = x_j + iy_j$, could be viewed as four coupled real oscillators

$$\begin{aligned} \dot{x}_1 &= a\,(x_3x_2 - y_3y_2)\,; \quad \dot{y}_1 = -a\,(x_3y_2 + y_3x_2) \\ \dot{x}_2 &= -c\,(x_1x_3 - y_1y_3)\,; \quad \dot{y}_2 = c\,(x_1y_3 + y_1x_3) \\ \dot{x}_3 &= -b\,(x_1x_2 - y_1y_2)\,; \quad \dot{y}_3 = b\,(x_1y_2 + y_1x_2) \end{aligned}$$

$I = \{x_1, x_2, x_3\}$; $II = \{x_1, y_2, y_3\}$; $III = \{y_1, x_2, y_3\}$; $IV = \{y_1, y_2, x_3\}$. The schematic arrangement of the sextet and its real triads is shown in fig. 3. Despite more complicated arrangement (4 coupled real triads) the system is still completely integrable and periodic in time. To show it we shall explicit complex form (6), and its 3 conserved integrals

$$\begin{aligned} \xi &= \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}; \quad \eta = \frac{|z_1|^2}{a} + \frac{|z_3|^2}{c} \\ \kappa &= \mathrm{Im}\,(z_1z_2z_3) = y_1x_2x_3 + y_2x_1x_3 + y_3x_2x_1 - y_1y_2y_3 \end{aligned} \tag{11}$$

The first two are similar to (8), the last one follows from

$$\frac{d}{dt}\,(z_1z_2z_3) = a\,|z_3z_2|^2 - b\,|z_1z_3|^2 - c\,|z_1z_2|^2 \text{ - real}$$

Let us remark, that function κ is Hamiltonian of system (6), with respect to the Poisson structure

$$\{f; g\} = f\left(\frac{2}{i}\left(a\overleftarrow{\partial}_1 \wedge \overrightarrow{\bar{\partial}}_1 - b\overleftarrow{\partial}_2 \wedge \overrightarrow{\bar{\partial}}_2 - c\overleftarrow{\partial}_3 \wedge \overrightarrow{\bar{\partial}}_3\right)\right)g \tag{12}$$

where $\overleftarrow{\partial}_m \wedge \overrightarrow{\bar{\partial}}_m$ abbreviates complex Jacobian operation

$$J_m\,(f; g) = \frac{2}{i}\left[(\partial_{z_m} f)\,(\partial_{\bar{z}_m} g) - (\partial_{\bar{z}_m} f)\,(\partial_{z_m} g)\right]$$

- arrows point to the function "subjected" to the corresponding derivative. Noteworthy, the real part of complex product $z_1z_2z_3$ determines the so called transfer term, that is rate at which any pair $\{z_j; z_m\}$ transfers energy to the third partner.

We recast system (6) in the polar form: $z_1 = r_1e^{i\theta_1}$, $z_2 = \ldots$, $z_3 = \ldots$,

$$\begin{aligned} \dot{r}_1 + ir_1\dot{\theta}_1 &= ar_3r_2e^{-i(\theta_1+\theta_2+\theta_3)} \\ \dot{r}_2 + ir_2\dot{\theta}_2 &= -br_3r_1e^{-i(\theta_1+\theta_2+\theta_3)} \\ \dot{r}_3 + ir_3\dot{\theta}_3 &= -cr_3r_2e^{-i(\theta_1+\theta_2+\theta_3)} \end{aligned} \tag{13}$$

and abbreviate total phase $\theta_1 + \theta_2 + \theta_3 = \theta$. All variables r_2, r_3 θ are expressed through a single polar radius $r = r_1$,

$$\begin{aligned} r_2 &= \pm\sqrt{b\,(\xi - r^2/a)};\; r_3 = \pm\sqrt{c\,(\eta - r^2/a)} \\ \sin\theta &= \frac{\kappa}{r_1r_2r_3} \end{aligned} \tag{14}$$

That along with the dynamic equation $r\dot{r} = ar_1r_2r_3\cos\theta = a\sqrt{(r_1r_2r_3)^2 - \kappa^2}$ allows one to integrate r as a function of time

$$\int_{r_0}^{r} \frac{r\,dr}{\sqrt{r^2\,(a\xi - r^2)\,(a\eta - r^2) - \kappa^2}} = \sqrt{bc}\,t$$

Thus we could express r^2 through the Weierstrass $\wp$-function, $\wp\,(t|g_2; g_3)$, whose parameters (invariants) $g_2;\ g_3$ depends on 3 conserved quantities $\xi; \eta; \kappa$. The entire 6D dynamics is periodic with time-period

$$T = 2\int_{e_3}^{e_2} \frac{du}{\sqrt{u\,(\xi - u)\,(\eta - u) - \kappa^2}} \tag{15}$$

depending on the roots of cubic polynomial

$$Q\,(u) = u\,(\xi - u)\,(\eta - u) - l^2 = (u - e_1)\,(u - e_2)\,(u - e_2)$$

Period T could be expressed through the elliptic integral of the first kind. Indeed, three roots $\{e_3 < e_2 < e_1\}$ link Weierstrass $\wp$ to Jacobi cnoidal harmonics

$$\mathrm{sn}\,(u|m) = \sqrt{\frac{e_1 - e_3}{\wp\,(t) - e_3}};\ \mathrm{cn}\,(u|m) = \sqrt{\frac{\wp\,(t) - e_1}{\wp\,(t) - e_3}};\ \mathrm{dn}\,(u|m) = \sqrt{\frac{\wp\,(t) - e_2}{\wp\,(t) - e_3}}$$

of modulus $m = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}$. This also results in time rescaling $t \to \tau = \sqrt{e_1 - e_3}\,t$, hence the Weierstrass period (15) expressed through the Jacobi period, complete elliptic integral (10), as $T = \frac{4K(m)}{\sqrt{e_1 - e_3}}$.

Once the first radial variable $r_1 = r$ is computed, two other moduli $\{r_2; r_3\}$, and three complex phases are computed explicitly in terms of r, via (14), and imaginary part of (13)

$$\theta_1 = -\int \frac{r_3r_2}{r_1}\sin\theta = -a\kappa\int_0^t \frac{1}{r^2}$$

$$\theta_2 = a\kappa\int_0^t \frac{1}{a\xi - r^2};\quad \theta_3 = a\kappa\int_0^t \frac{1}{a\eta - r^2}$$

Fig. 4 demonstrates solution-plots and phase-plots for each of 3 constituent pairs of the complex triad, all exhibit regular period patterns, albeit more complicated compared to the real triad (Jacobi) case.

3 Octaplet

Next we take an octaplet of Fourier modes, that fill four vertices: $\{w_1; w_1^*; w_2; w_2^*\}$, $w_j = u_j + iv_j$, and four midpoints $\{z_1; z_1^*; z_2; z_2^*\}$, $z_j = x_j + iy_j$, of the smallest coordinate lattice rectangle, symmetric about the origin (fig. 5). This system could be viewed as two coupled complex triads, or four coupled real triad oscillators below. The resulting complex system in variables: $\{z_1; z_2\}; \{w_1; w_2\}$ takes on the form

$$\begin{aligned} \dot{z}_1 &= \quad a w_1^* z_2^* \quad - a z_2 w_2 \\ \dot{z}_2 &= -b z_1^* w_1^* \quad + b z_1 w_2^* \\ \dot{w}_1 &= -c z_1^* z_2^* \\ \dot{w}_2 &= \qquad\qquad\quad c z_1 z_2^* \end{aligned} \tag{16}$$

so complex $\{z_1; z_2; w_1\}$-oscillator (first column in the r.h.s.) is coupled to $\{z_1; z_2^*; w_2^*\}$ (second column), via z_1- z_2 modes. The combined system has two real and two complex quadratic conserved integrals, derived similar to (11)

$$\begin{aligned} \xi &= \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b}; \quad \eta = \frac{|z_1|^2}{a} + \frac{|w_1|^2 + |w_2|^2}{c} \\ \zeta &= \frac{z_1^2}{a} - \frac{w_1^* w_2}{c}; \quad \psi = \frac{z_2^2}{a} + \frac{w_1^* w_2^*}{c} \end{aligned} \tag{17}$$

They allow to reduce the 8D dynamical system to 2D.

To implement this reduction explicitly we shall recast complex system (16) in the real form. Observe that the "corner variables" $\{u_j; v_j\}$ enter the system as sums or differences, so changing $\{u_j; v_j\}$ to capital variables

$$\begin{aligned} U_1 &= u_1 - u_2 = \mathrm{Re}\,(w_1 - w_2) \\ U_2 &= -(u_1 + u_2) = -\mathrm{Re}\,(w_1 + w_2) \\ V_1 &= v_2 - v_1 = \mathrm{Im}\,(w_2 - w_1) \\ V_2 &= -(v_1 + v_2) = -\mathrm{Im}\,(w_1 + w_2) \end{aligned} \tag{18}$$

we get the following 8D real system

$$\begin{aligned} \dot{x}_1 &= a\,(x_2 U_1 + y_2 V_1)\,; \quad \dot{U}_1 = -2c x_1 x_2 \\ \dot{y}_1 &= a\,(x_2 V_2 + y_2 U_2)\,; \quad \dot{V}_1 = -2c x_1 y_2 \\ \dot{x}_2 &= -b\,(x_1 U_1 + y_1 V_2)\,; \quad \dot{V}_2 = -2c y_1 x_2 \\ \dot{y}_2 &= -b\,(x_1 V_1 + y_1 U_2)\,; \quad \dot{U}_2 = -2c y_1 y_2 \end{aligned} \tag{19}$$

The latter could be viewed as four coupled real triads: $I = \{x_1; y_2; V_2\}$; $II = \{x_1; x_2; U_1\}$; $III = \{y_1; x_2; V_1\}$; $IV = \{y_1; y_2; U_2\}$, shown in fig. 5, thus x_1 couples I to II; x_2 : II to III, etc.

A convenient way to represent system (19) is by a combination of scalar and vector variables:

$$\begin{aligned} x &= x_1;\ y = y_1;\ z = (x_2; y_2) \\ u &= (U_1; V_1) = (u_1 - u_2; v_2 - v_1) \\ v &= (V_2; U_2) = -(v_2 + v_1; u_1 + u_2) \end{aligned} \tag{20}$$

Then (19) takes on the form

$$\begin{aligned}
\dot{x} &= a\,(z \cdot u) \\
\dot{y} &= \qquad\qquad a\,(z \cdot u) \\
\dot{u} &= -2cxz \\
\dot{v} &= \qquad\qquad -2cyz \\
\dot{z} &= -bxu \quad -byv
\end{aligned} \tag{21}$$

In such form the octet is broken into a pair of hybrid 5D-triads, that combine scalar and vector variables: $\{x; u; z\}$ (first column in the r.h.s.), and $\{y; v; z\}$ (second column), coupled via vector variable z. Before examining the entire octaplet we first look at the hybrid triad-oscillator $\{x; u; z\}$, obtained by suppressing y and v, the other case $x = 0$, $u = 0$ is treated similarly.

Remark 1 We could choose another (short) mid-side mode $z_q = (x_2; y_2)$, as scalar variables for hybrid representation. Then (20) is replaced with

$$\begin{aligned}
x = x_2;\ y = y_2;\ z = (x_1; y_1) \\
u = (U_1; V_2)\,;\ v = (V_1; U_2)
\end{aligned} \tag{22}$$

and (21) takes on a similar form but di®erent coe±cients and signs

$$\begin{aligned}
\dot{x} &= -b\,(z \cdot u) \\
\dot{y} &= \qquad\qquad -b\,(z \cdot u) \\
\dot{u} &= -2cxz \\
\dot{v} &= \qquad\qquad -2cyz \\
\dot{z} &= axu \quad +ayv
\end{aligned} \tag{23}$$

The reduced system $y = 0$, $v = 0$, would change accordingly. Indeed, combining vector and matrix variables:

$$X = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix};\ Y = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix};\ U = \begin{bmatrix} u_1 - u_2 & v_2 - v_1 \\ -v_2 - v_1 & -u_1 - u_2 \end{bmatrix}$$

we could recast octet in the triad form

$$\begin{aligned}
\dot{X} &= aUY \\
\dot{Y} &= -bU^tY \\
\dot{U} &= -2cXY^t
\end{aligned}$$

where U^t, Y^t denote the transposed matrices.

3.1 Hybrid triad

The 5D hybrid system

$$\begin{aligned}
\dot{x} &= a\,(z \cdot u) \\
\dot{u} &= -2cxz \\
\dot{z} &= -bxu
\end{aligned} \tag{24}$$

resembles simple triad oscillator (7), but for vector-quantities $\mathbf{u}$, $\mathbf{z}$. So we proceed the same way and produce a pair of invariants:

$$\xi = \frac{x^2}{a} + \frac{z^2}{b}; \quad \eta = \frac{x^2}{a} + \frac{u^2}{2c}$$

and another "product-type" conservation law

$$(\mathbf{z}\cdot\mathbf{u})^{\cdot} = -x\left(2cz^2 + bu^2\right) = -2bcx\left(\xi + \eta - \frac{2x^2}{a}\right)$$

The latter yields upon substitution in (24) a second order "Newton's equation" for a 1D-particle x in a potential force

$$\ddot{x} = -2abcx\left(\xi + \eta - \frac{2x^2}{a}\right) \tag{25}$$

Newton's equation (25) has conserved "particle-energy" integral with quartic potential well (fig. 6)

$$E = \frac{\dot{x}^2}{2} + abc\left[(\xi + \eta)\,x^2 - \frac{x^4}{a}\right] = \frac{\dot{x}^2}{2} + V(x) \tag{26}$$

In terms of the fluid system (24) it corresponds to a quartic conserved integral

$$E = 2bc\left[(\xi + \eta)\,x^2 - \frac{x^4}{a}\right] + a\,(\mathbf{z}\cdot\mathbf{u})^2$$

The quartic potential well has a well known phase-plot (fig. 6). The system will oscillate in the phase-space region between two separatrices, that correspond to critical value $E_{crit} = \frac{abc(\xi+\eta)^2}{2} = V_{max}$, since x^2 is always dominated by $a(\xi + \eta)$. Particle system (26) is solved by standard separation

$$\int \frac{dx}{\sqrt{2\left[E - abcx^2\left(\xi + \eta - \frac{x^2}{a}\right)\right]}} = t$$

For $E < E_{cr}$, quartic polynomial has real roots

$$x^2 = \lambda_{1;2} = \frac{a(\xi + \eta)}{2} \pm \sqrt{\left[\frac{a(\xi+\eta)}{2}\right]^2 - \frac{E}{bc}}; \quad \lambda_1 > \lambda_2$$

-themselves invariants of the system.

As the result we get once again the classical Jacobi description of the 5D-oscillator, with modulus and period expressed through the complete elliptic integral of the first kind

$$m = \frac{\lambda_2}{\lambda_1}; \quad T = \frac{4}{\sqrt{2bc\lambda_1}} K\left(\frac{\lambda_2}{\lambda_1}\right)$$

and elliptic (cnoidal) solution $x(t) = \sqrt{a\lambda_2}\,\mathrm{cn}\left(\sqrt{2bc\lambda_1}\,t|m\right)$. Once x is computed the other variables $z; u$ are also expressed through elliptic functions and integrals, via hyperbolic rotation

$$\begin{pmatrix} u \\ z \end{pmatrix} = \begin{bmatrix} \cosh\left(-\sqrt{2bc}\int_0^t x\right) & \sqrt{\frac{c}{2b}}\sinh\left(-\sqrt{2bc}\int_0^t x\right) \\ \sqrt{\frac{b}{8c}}\sinh\left(-\sqrt{2bc}\int_0^t x\right) & \cosh\left(-\sqrt{2bc}\int_0^t x\right) \end{bmatrix} \begin{pmatrix} u_0 \\ z_0 \end{pmatrix}$$

in terms of the initial state $(u_0; v_0)$.

Remark 2 For alternative choice of variables (22) hybrid triad has somewhat different invariants

$$\xi = \frac{x^2}{b} + \frac{z^2}{a}; \quad \eta = \frac{u^2}{2c} - \frac{x^2}{b}$$

hence the change of sign in the quartic potential for the x-particle (26)

$$V = abc\left[\frac{x^4}{b} - (\xi + \eta)\,x^2\right]$$

The rising quartic potential well $V(x)$ (fig. 7), has two types of periodic motions: oscillations within one of two wells in the phase-space region below the separatrix $E = 0$ (consistent with the dn-type behavior of the shortest mode x_q in a single triad (9)), and a single periodic path that takes particle from one well to the other for $E > 0$. In the former case, quartic equation $V(x) = -E < 0$, has two pairs of real roots: $\pm\sqrt{\lambda_1}; \pm\sqrt{\lambda_2}$, where

$$\lambda_{1;2} = \frac{c}{2}(\xi + \eta) \pm \sqrt{\left[\frac{c}{2}(\xi + \eta)\right]^2 - \frac{E}{2ab}}$$

Here the (Jacobi) modulus and period are given by

$$m = \frac{\lambda_2}{\lambda_1}; \quad T = \frac{2\sqrt{2}}{\sqrt{ab\lambda_1}}K\left(\frac{\lambda_2}{\lambda_1}\right)$$

with elliptic (cnoidal)

$$x(t) = \sqrt{c\lambda_2}\,\mathrm{dn}\left(\sqrt{2ab\lambda_1}\,t|m\right) \tag{27}$$

In the latter case, $V(x) = E > 0$ has a pair of roots $\pm\sqrt{\lambda}$,

$$\lambda = \frac{c}{2}(\xi + \eta) + \sqrt{\left[\frac{c}{2}(\xi + \eta)\right]^2 + \frac{E}{2ab}},$$

Hence $m = -1$, and period

$$T = \frac{2\sqrt{2}}{\sqrt{ab\lambda_1}}K(-1) = \frac{10}{3}\sqrt{\frac{2\pi}{ab\lambda_1}}\,\frac{\Gamma(1/4)}{\Gamma(-1/4)} \approx \frac{6.18}{\sqrt{ab\lambda_1}}$$

Once y is computed (27) the other variables u, v are also expressed through the Jacobi elliptic parametrization.

Fig. 9 shows a typical period pattern of the hybrid triad-oscillator.

3.2 Real form and particle reduction of octaplet

The full octet system (21) has 4 quadratic invariants

$$\begin{aligned} \xi &= \frac{x^2+y^2}{a} + \frac{z^2}{b}; \\ \eta_1 &= \frac{x^2}{a} + \frac{u^2}{2c}; \ \eta_2 = \frac{y^2}{a} + \frac{v^2}{2c}; \\ \kappa &= \frac{u \cdot v}{2c} + \frac{xy}{a} \end{aligned} \tag{28}$$

Those are related to the complex invariants (17) as follows

$$\begin{aligned} \xi &= \xi; \\ \eta_1 + \eta_2 &= \eta; \ \eta_1 - \eta_2 = 2\,\mathrm{Re}\,\zeta; \\ \kappa &= \mathrm{Im}\,\zeta \end{aligned}$$

As in the "hybrid-triad" case we produce two dynamic conservation laws

$$\begin{aligned} \frac{d}{dt}(u \cdot z) &= -2cxz^2 - b\left(xu^2 + yu \cdot v\right) \\ \frac{d}{dt}(v \cdot z) &= -2cyz^2 - b\left(xu \cdot v + yv^2\right) \end{aligned}$$

The latter yield upon substitution in (21) a second order reduced (Newtonian) system for $x; y$

$$\begin{aligned} \ddot{x} &= -2abc\left[(\xi + \eta_1)\,x + \kappa y - \frac{2}{a}\left(x^2+y^2\right)x\right] \\ \ddot{y} &= -2abc\left[\kappa x + (\xi + \eta_2)\,y - \frac{2}{a}\left(x^2+y^2\right)y\right] \end{aligned} \tag{29}$$

Thus $(x; y) = z_p$ behaves as a 2D-particle in the quartic potential well (fig. 8), depending on its 4 conserved integrals

$$V(x; y) = abc\left[(\xi + \eta_1)\,x^2 + (\xi + \eta_2)\,y^2 + 2\kappa xy - \frac{(x^2+y^2)^2}{a}\right] \tag{30}$$

Hence we get another conserved integral (particle-energy)

$$E = \frac{\dot{x}^2 + \dot{y}^2}{2} + V(x; y) = \frac{a^2}{2}\left[(u \cdot z)^2 + (v \cdot z)^2\right] + V(x; y) \tag{31}$$

Unlike the reduced "hybrid" case however, the particle-energy integral is no more independent of other invariants. Indeed, with a bit of algebra one could show

$$E = a^2bc\left(\frac{\xi^2}{4} + \frac{3\eta^2}{4} + |\zeta|^2 - |\psi|^2\right)$$

in terms of the complex invariants (17). Any further conserved integral seems unlikely, our solutions exhibit patently quasi-periodic pattern with two basic periods (fig.-s 10, 11).

Six quadratic conserved integrals (17) foliate the 8D-phase-space into 2D invariant compact surfaces, and reduce the 8D system to a pair of independent variables. One choice of such pair are polar radii $\rho_1 = |w_1|$; $\rho_2 = |w_2|$. The resulting 2D dynamical system has algebraic expressions in ρ_1, ρ_2 (in radicals), so we won't bring it here.

The exact nature of the foliated surfaces, and the ensuing dynamics (whether those are simple tori, or more complicated structures, quasi-periods etc.) remains open. Its analysis would require a proper generalization of the Jacobi elliptic theory to 2D algebraic surfaces, as opposed to algebraic curves - a topic in complex projective geometry.

Fig. 10 shows 4 solution-plots and the corresponding phase-plots of the octaplet ($x_1; y_1$- for p-mode, $x_2; y_2$- for q, etc.). Let us point out some features of these solutions.

- The longer of two mid-side modes $z_p = (x; y)$ performs a (quasi-periodic) bound motion of a 2D particle in a quartic potential (fig. 6) below the critical energy value, and symmetric about the origin

- The shorter mode z_q typically falls into one of two asymmetric wells of its rising quartic potential well (fig. 7)

- The frequency spectra of four modes (fig. 11) clearly show two basic frequencies $\omega \approx .155$, and $\Omega \approx .01$, along with their harmonics. We observe that p-mode has only odd multiples of ω, while k and q have all. Such structure is consistent with their Jacobi sn, cn, dn - identities, and the well-known Fourier series expansion (see e.g. Bateman et al 1953)

$$\mathrm{sn}\,(t|m) = \frac{\pi}{mK} \sum_{n=1} \frac{\sin\left[\frac{\pi}{2K}(2n-1)\,t\right]}{\sinh\left[\frac{\pi K'}{2K}(2n-1)\right]}$$

$$\mathrm{cn}\,(t|m) = \frac{\pi}{mK} \sum_{n=1} \frac{\cos\left[\frac{\pi}{2K}(2n-1)\,t\right]}{\cosh\left[\frac{\pi K'}{2K}(2n-1)\right]}$$

$$\mathrm{dn}\,(t|m) = \frac{\pi}{K} \left\{ \frac{1}{2} + \sum_{n=1} \frac{\cos\left[\frac{\pi}{K} n t\right]}{\cosh\left[\frac{\pi K'}{K} n\right]} \right\}$$

  where $K = K(m)$ and $K' = K(1-m)$ are the principal (real/ imaginary) periods of the Jacobi functions. Of course, the octaplet's x_p; x_k; x_q will not reduce to the exact Jacobi form, but they still seem to maintain some gross spectral features.

4 Forcing-dissipation

Here we shall briefly discuss the forced-dissipative real triad

$$\begin{aligned} \dot{x} + D_1 x &= ayz + f \\ \dot{y} + D_2 y &= -bzx \\ \dot{z} + D_3 z &= -cxy \end{aligned} \qquad (32)$$

Such system could represent a single zonal mode (x-variable), driven by a zonal force f, and dissipation $D = \nu k^2 + \mu$, made of viscous term νk^2, plus bottom drag μ. It gives a crude version of the zonal jet, interacting with cyclonic/ anticyclonic modes $\{y; z\}$, in the so called supercritical regime (Dolzhanski et al 1981). We shall study equilibria of (32) and their stability, then using invariants $\xi; \eta$ of section 2 produce an asymptotic solution of non-integrable system (32), valid in the stable zonal regime.

4.1 Equilibria

We rescale variables $x; y; z$ to make all coefficients equal: $a = b = c = 1$, and cast equilibrium system (32) in the matrix form

$$
\begin{gathered}
-D_1 x + yz + f = 0 \\
\begin{bmatrix} D_2 & x \\ x & D_3 \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{gathered}
$$

So x plays the role of an "eigenvalue", and $(y; z)$ - eigenvector of 2×2 matrix in the l.h.s.

$$
x = \pm \sqrt{D_2 D_3}; \quad (y; z) = \lambda \left(\mp \sqrt{D_3 / D_2}; 1 \right)
$$

Substitution in the first equation yields $\lambda^2 = \pm f \sqrt{\frac{D_2}{D_3}} - D_1 D_2$. The off-shot are two sets of equilibria, depending on the dimensionless Reynolds parameter, $\mathrm{Re} = \frac{f}{D_1 \sqrt{D_2 D_3}}$. Namely,

$$
\begin{aligned}
I &= (x; y; z) = \left(\frac{f}{D_1}; 0; 0 \right) && \text{if Re} \leq 1 \\
II_{\pm} &= \begin{cases} x = \sqrt{D_2 D_3} \\ (y; z) = \pm \sqrt{f - D_1 x} \left(\sqrt[4]{D_3 / D_2}; \sqrt[4]{D_2 / D_3} \right) \end{cases} && \text{if Re} > 1
\end{aligned} ; \qquad (33)
$$

Equilibrium I corresponds to a zonal flow, while $II_{\pm}$ describe zonal flow with cyclonic/ anti-cyclonic gyres.

Next we look at the linear stability of the Jacobian matrix

$$
L = \begin{bmatrix} -D_1 & z & y \\ -z & -D_2 & -x \\ -y & -x & -D_3 \end{bmatrix}
$$

It shows stable (zonal) equilibrium I for all $f \leq f^* = D_1 \sqrt{D_2 D_3}$ (i.e. $\mathrm{Re} \leq 1$), and subsequent bifurcation into a pair of stable nodes $II_{\pm}$ past the critical value f^*. Indeed, Jacobian matrix

$$
L|_I = \begin{bmatrix} -D_1 & 0 & 0 \\ 0 & -D_2 & -x \\ 0 & -x & -D_3 \end{bmatrix}
$$

-strictly negative as long as $x^2 < D_2 D_3$. Past critical value f, the 2×2 lower diagonal block of L becomes singular with eigenvalues $\{0; -(D_2+D_3)\}$, and eigenvectors: $(y; z)$ and $(-z; y)$, made of coordinates $\{y; z\}$ of equilibrium II (33), so

$$L|_{II} \sim \begin{bmatrix} \begin{matrix} -D_1 & \alpha \\ -\alpha & 0 \end{matrix} & \\ & -(D_2+D_3) \end{bmatrix}$$

with $\alpha = |(y; z)| = \sqrt{y^2+z^2}$.

4.2 Non-stationary forcing

Next we assume equal dissipation rates[4], and augment the constant force f_0 with non-stationary component $f_1(t)$, e.g. periodic $f_1 \cos \omega t$. The invariants: $\xi = x^2/a + y^2/b$, $\eta = x^2/a + z^2/c$, of section 2.1 are no more conserved here. But new variables $\{x; \xi; \eta\}$ obey a coupled system

$$\begin{aligned} \dot{x} + \gamma x &= \sqrt{bc\,(a\xi - x^2)\,(a\eta - x^2)} + f \\ \dot{\xi} + 2\gamma\xi &= \frac{f}{a} x \\ \dot{\eta} + 2\gamma\eta &= \frac{f}{a} x \end{aligned}$$

The difference $\xi - \eta$ solves a simple (homogeneous) decay law

$$\frac{d}{dt}(\xi - \eta) = -2\gamma\,(\xi - \eta)$$

so two invariants become asymptotically equal at large time $t > 1/2\gamma$, and the problem is reduced to a 2D system in $x; \xi$

$$\begin{aligned} \dot{x} + \gamma x &= \sqrt{bc}\left(a\xi - x^2\right) + f \\ \dot{\xi} + 2\gamma\xi &= \frac{f}{a} x \end{aligned} \tag{34}$$

The latter resembles a "van der Pole type" nonlinear oscillator. It exhibits two types of behavior: forced linear response, or blowup, depending on the mean forcing value f_0. In the former regime, the nonlinear term in the x-equation drops out, and we get the standard linear response solution

$$x(t) \approx \frac{f_0}{\gamma} + e^{-\gamma t} \star f_1(t) = x_0(t) + x_1(t) \tag{35}$$

with $\star$ indicating time convolution.

[4] In general dissipation coefficients, due to viscosity, depend on the wave-mode number, and could be different for $x; y; z$

To show the validity of asymptotic solution (35), we need to find the conditions for decay of variable $w = a\xi - x^2$. It obeys the coupled system

$$\begin{aligned} \dot{x} &= -\gamma x + \sqrt{bc}\,w + f \\ \dot{w} &= -2\left(\gamma + \sqrt{bc}\,x\right) w \end{aligned} \tag{36}$$

derived from (34).So w could be expressed in terms of $\int x dt$, as

$$w(t) = w_0 \exp\left[-2\left(\gamma + \frac{f_0\sqrt{bc}}{\gamma}\right)t + 2\int_0^t x_1 dt\right]$$

Assuming bounded values for integral $\int_0^t x_1 dt$ (e.g. period or quasi-periodic source f_1), w could either oscillate, or grow exponentially depending on sign, $\gamma + \frac{f_0\sqrt{bc}}{\gamma} \gtrless 0$. The transition occurs as f_0 reaches critical negative value $f_0^* = -\frac{\gamma^2}{\sqrt{bc}}$, which describes the bifurcation of the triad equilibria (zonal to non-zonal) for constant forcing f_0. Indeed, system (36) has two equilibria in the x,w-plane: $I = \left(\frac{f_0}{\gamma}, 0\right)$ and $II = \frac{1}{\sqrt{bc}}\left(-\gamma, f_0 + \gamma^2/\sqrt{bc}\right)$. As f_0 descends through the critical value f_0^* two equilibria merge, and change their types, according to

f_0	$> f_0^*$	$< f_0^*$
I	sink	saddle
II	sadlle	sink

The oscillation-to-growth transition for w coincides with the loss of stability of zonal equilibrium I. Thus we get an asymptotic solution (35) valid within the stability range of parameter f_0, i.e. stable zonal flow.

5 Conclusions

- We examined some low-mode truncated fluid systems: triad, sextet (vortex triad) and octaplet, produced their conserved integrals, and explicit solutions in elliptic (Jacobi, Weierstrass) functions. The former two are completely solvable and periodic, the latter is quasi-periodic.

- We studied the forced dissipative triad as a simple model of zonal flow, and its transition from "stable zonal" to "unstable zonal" regime. When constant f_0 is augmented with non-stationary (oscillating) force $f_1(t)$, we use invariants of section 2.1 to construct and asymptotic solution of the forced system, valid in the regime of stable zonal flow.

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6 References

Arnold, V., 1978, Mathematical methods of classical mechanics, Springer, NY

Kuznetsov, E. A., A.V. Mikhailov, 1980, Phys. Lett., 77A, 37-.

Marsden, J., Weinstein, A. 1983, Coadjoint orbits, vortices and Clebsch variables for incompressible °uids, Physica D, 7, 305-323.

Hasimoto, H., 1972 A soliton on a vortex ¯lament, J. Fluid Mech., 51, 477-85

Kuznetsov E. A., V. P. Ruban, 1999 Vortex line representation in the hydrodynamic type systems, preprint

Gurarie, D., 1995 Symmetries and conservation laws for two-dimensional hydrodynamics, Physica D, 86, 621-36

Dolzhanski, F., Gledzer, E. and A. Obukhov, 1981, Systemy gidrodynamicheskogo typa i ikh prilozheniya ("Systems of hydrodynamic type and their applications"), Moscow, Nauka (Russian)

Zeitlin V., 1991, Finite mode analogues of 2D ideal hydrodynamics: coadjoint orbits and local canonical structure, Physica D, 49, 353-362

Gurarie, D., 1999 , Long range dynamics of shallow water: renormalization and modulation, preprint

Bateman, H. and A. Erdelye, 1953, Higher transcendental functions, vol. II, McGrow-Hill.

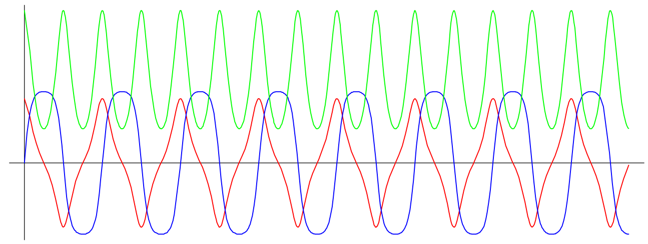

Figure 1: Real triad solution, made of Jacobi sn, cn, dn

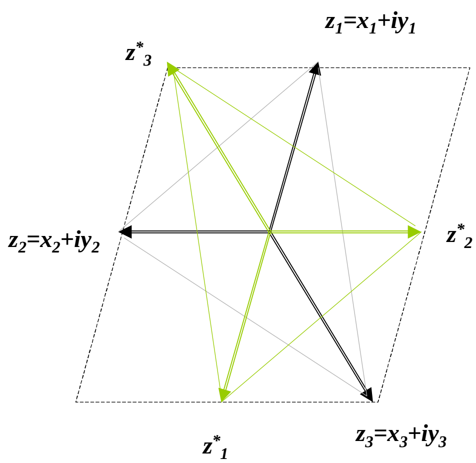

Figure 2: Complex triad considered as real sextet

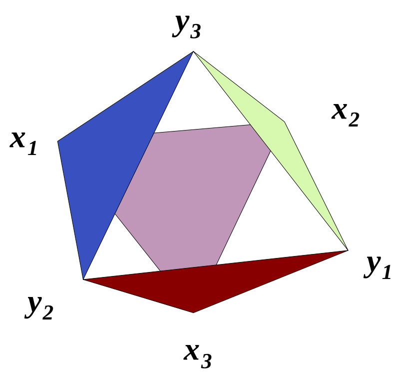

Figure 3: Complex triad: x and y modes at the vertices of octahedron, with shaded faces representing its real triad constituents

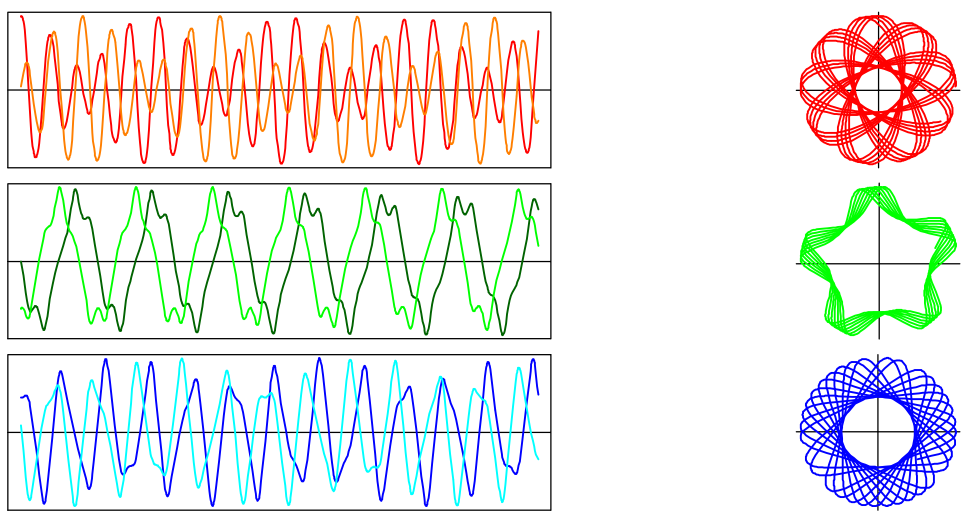

Figure 4: Solution- and phase-plots of three component pairs of the complex triad: $(x_p; y_p)$; $(x_q; y_q)$; $(x_k; y_k)$

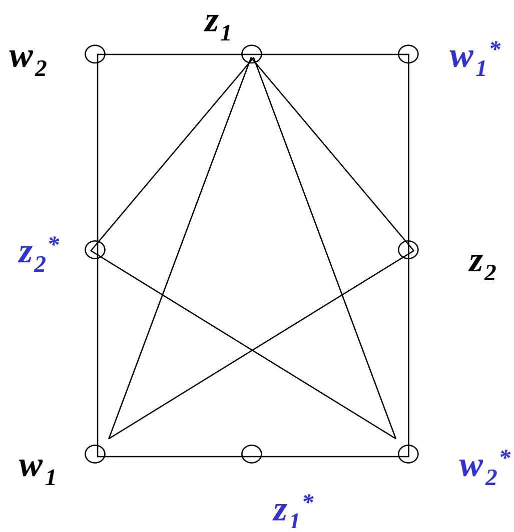

Figure 5: Octaplet: lattice (left), and constituent triads (right)

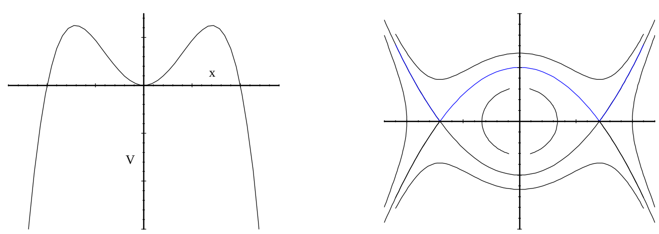

Figure 6: Quartic potential (left), and its phase-plot

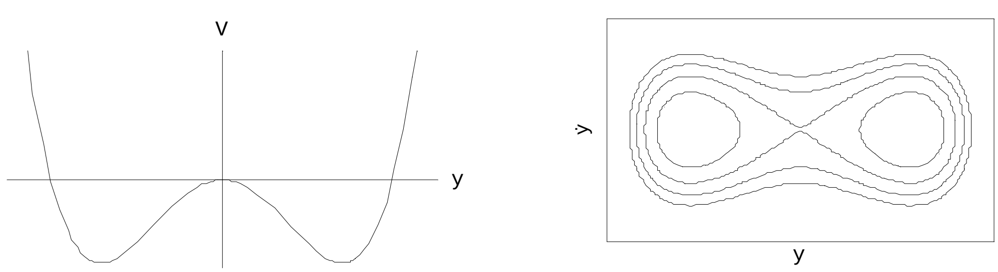

Figure 7: Rising quartic potential, and its phase-plot

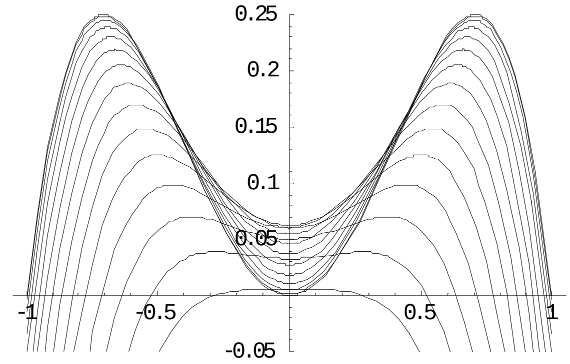

Figure 8: Quartic potential: frontal view in the $y = 0$ -plane

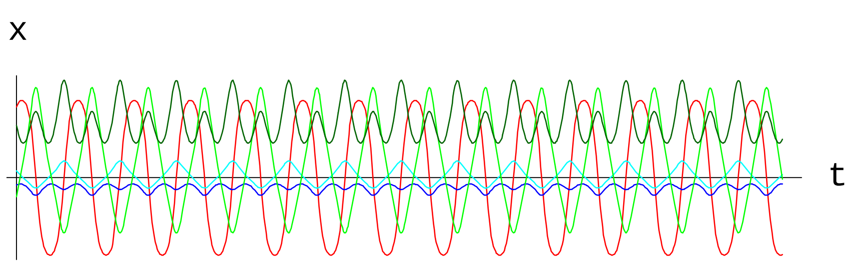

Figure 9: Typical plot of the 5D hybrid oscillator

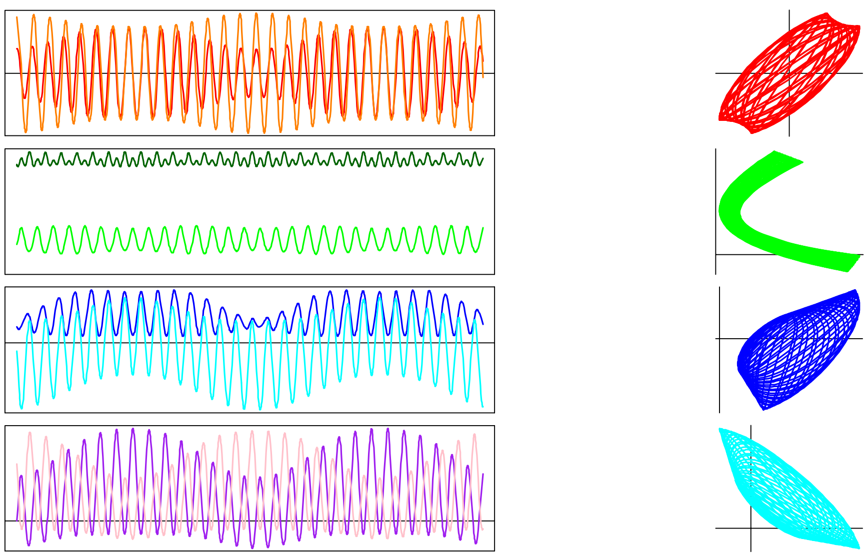

Figure 10: Solution- and phase-plots of the p, q, k and m -modes of the octaplet

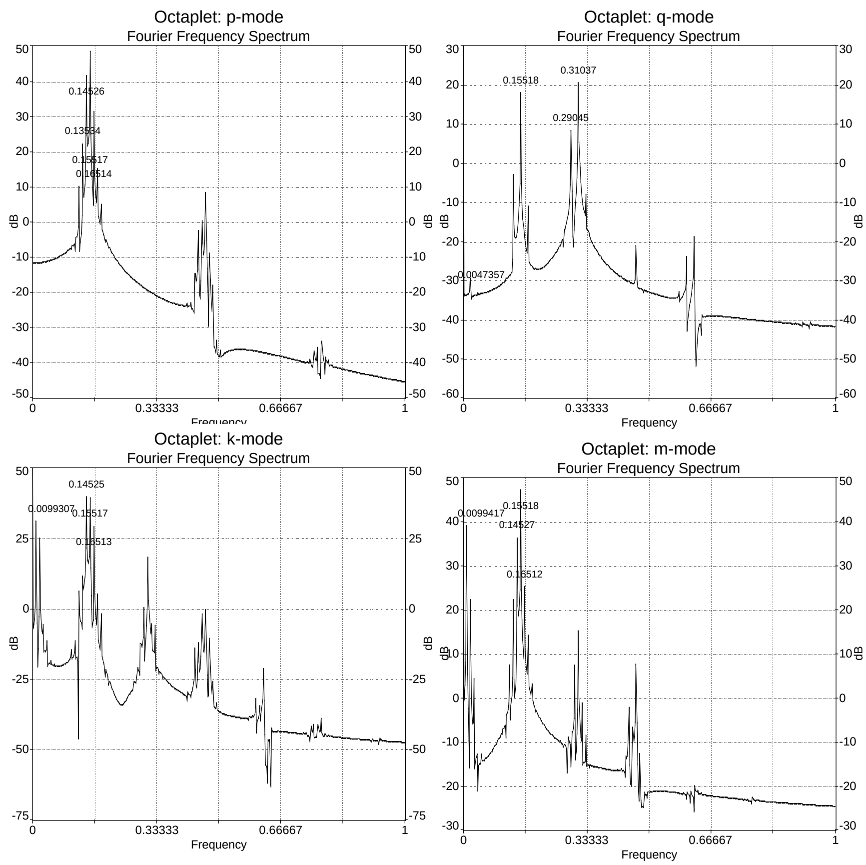

Figure 11: Fourier spectra of the x-modes of octaplet