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Asymptotic inverse spectral problem for anharmonic oscillators with odd potentials

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Abstract. The present paper continues our earlier work on asymptotic spectral problems for perturbations of the harmonic oscillator $L=A+B$ on $\mathbb{R}$, where $A=\frac{1}{2}(-\partial^2+x^2)$. The class of functions B is defined by their asymptotics at $\{\infty\}$

$$B(x)\sim|x|^{-\alpha}\times\text{trigonometric } V(x).$$

The even-potential case of the author's earlier paper, i.e. $V=\Sigma\, a_m \cos \omega_m x$, is now extended to a more difficult odd case, $V=\Sigma\, b_m \sin \omega_m x$. Precisely, the eigenvalues of L, $\lambda_k=k+\mu_k$, are shown to admit an asymptotic expansion

$$\mu_k\sim\text{constant}\times k^{-\gamma}\tilde{V}(\sqrt{2k})+\ldots \qquad \text{as } k\to\infty$$

where the exponent $\gamma=\frac{1}{2}\alpha+\frac{3}{4}$, and $\tilde{V}$ denotes the 'odd version' of the so-called 'Radon transform' of V.

As a consequence we get a unique and explicit solution of the asymptotic inverse spectral problem for odd perturbations of A, similar to the even case previously consider by the author.

The methods employed here combine some ideas of the n-sphere Schrödinger theory as well as the techniques developed by the author.

This paper is a sequel to earlier work [1] on asymptotic spectral problems for perturbations $L=A+B$ of the harmonic oscillator $A=\frac{1}{2}(-\partial^2+x^2)$ on $\mathbb{R}$.

Spectral problems (direct and inverse) in the context of perturbation theory usually refer to the relation: $B\leftrightarrow$eigenvalue of L. However exact eigenvalues are rarely available, so most often one has to contend with an approximate (asymptotic) expansion of $\{\lambda_k\}$. The problem then amounts to extracting sufficient asymptotic data from the sequence $\{\lambda_k\}$ in terms of B. Such data may hopefully lead to solution of the inverse problem, i.e. explicit reconstruction of B (more typically the isospectral class of B) from asymptotics of $\{\lambda_k\}$.

Spectral theory was best developed in the case of the classical Sturm–Liouville (SL) problem: $L=-\partial^2+V(x)$ on $[0, 1]$. Here the asymptotic expansion of a large eigenvalue, due to Borg [2], takes the form

$$\lambda_k\sim(\pi k)^2+b_0+(b_1/k^2)+\ldots \qquad \text{as } k\to\infty. \tag{1}$$

The main contribution, $(\pi k)^2$, comes from the principal part $A=-\partial^2$, while higher-order corrections: $b_0=\int V\,\mathrm{d}x$; b_1; ... depend on Fourier coefficients of V.

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Large families of isospectral potentials V are known to exist for SL operators with various types of boundary conditions [3–5]. Therefore unique determination of V (inverse problem) usually requires an infinite set of additional parameters, e.g. normalising constants.

Recently some aspects of the classical SL theory were extended to multidimensional problems, i.e. Schrödinger operators $-\Delta+V$ in various geometric settings: n-sphere [6–8]; n-torus [9]; negatively curved Riemannian manifolds [10]; as well as singular SL problems, like the harmonic oscillator [1, 11]. In particular McKean and Trubowitz [11] described the isospectral class of the harmonic oscillator $A=\frac{1}{2}(-\partial^2+x^2)$ on $\mathbb{R}$, i.e. all operators $L=A+B(x)$, whose spectrum is equal to the spectrum of $A=\{0, 1, 2, \ldots\}$.

It turns out that the isospectral class of the B is infinite dimensional, as in the classical SL case, but they all decay exponentially at ∞. Thus larger classes of potentials with an algebraic rate of decay, $B(x)=\mathrm{O}(|x|^{-\alpha})$, are excluded. Such classes were studied in [1]. Precisely, the class of perturbations $\{B\}$ was described in terms of their asymptotic behaviour at ∞,

$$B(x)\sim|x|^{-\alpha}V(x) \qquad \text{as } |x|\to\infty \tag{2}$$

with a trigonometric function $V(x)=\Sigma\, a_m \cos\omega_m x+b_m \sin\omega_m x$.

One can show that the operator B is small (compact) relative to A (see [12]), so eigenvalues fo L, $\lambda_k=k+\mu_k$, $k=0, 1, \ldots$, have the leading-order term, k, due to A, and small fluctuations resulting from B; $\mu_k\to 0$, as $k\to\infty$.

Whereas the standard spectral problem requires a complete determination of spectral shifts $\{\mu_k\}$ from B (or vice versa, B from $\{\mu_k\}$), we were mostly concerned with asymptotic aspects of the relation: $B\leftrightarrow\{\mu_k\}$.

By asymptotics of B we mean its expansion (2) at large x (exponent α and trigonometric part V), while asymptotics of μ_k obviously refer to behaviour at large k.

Specifically we showed that for even potential B, i.e. $V=\Sigma\, a_m \cos\omega_m x$, $\mu_k=\mathrm{O}(k^{-(\alpha/2)-(1/4)})$; furthermore we established the analogue of Borg's formula (1)

$$\mu_k\sim\text{constant}\times k^{-\gamma}\tilde{V}(\sqrt{2k})+\ldots \qquad \text{as } k\to\infty \tag{3}$$

where $\gamma=\frac{1}{2}\alpha+\frac{1}{4}$, and $\tilde{V}$ is the so-called 'Radon transform' of V,

$$\tilde{V}(x)=\sqrt{\frac{2}{\pi}}\sum\frac{a_m}{\sqrt{\omega_m}}\cos(\omega_m x-\tfrac{1}{4}\pi). \tag{4}$$

Though (3) gives only the leading term in the asymptotic expansion of μ_k, it has one important advantage compared with (1). Namely, the 'coefficient' $\tilde{V}(\sqrt{2k})$, which plays the role of Borg's b_0, is no longer constant but depends in a certain periodic (quasiperiodic) fashion on k. In particular, this enables one to uniquely solve the inverse spectral problem, i.e. to recover the asymptotic part of $B(x)$ (exponent α and trigonometric V) from the asymptotics of $\{\mu_k\}$.

The formula (3) also yeilds a continuous limiting distribution of properly modified average spectral fluctuations, Weinstein's so-called band-invariant β_0 (cf [6, 8, 10]). Both results were derived in [1].

Unfortunately the method of [1] was limited to the case of even potentials. Our goal in the present paper is to extend the results of [1] to odd potentials following some ideas of [8, 13], where a similar problem was studied for Schrödinger operators on the n-sphere.

Let us remark that in both cases, the n-sphere Schrödinger operator and the anharmonic oscillator, changing from even to odd potential has the effect of decreasing the (algebraic) rate of decay of fluctuations $\{\mu_k\}$, due to certain cancellations.

In the n-sphere problem [13] it takes the form

$$\mu_{kj} = O(1)\,(\text{even}) \qquad \mu_{kj} = O(k^{-2})\,(\text{odd})$$

μ_{kj} being the jth fluctuation in the kth cluster, while in our setting the 'even' estimate $\mu_k = O(k^{-(\alpha/2)-(1/4)})$ of [1] is replaced by the 'odd' $\mu_k = O(k^{-(\alpha/2)-(3/4)})$.

The main result of this paper is the 'odd' version of the asymptotic formula (3). To state it precisely we need to impose a technical condition on the trigonometric function V. Namely, Fourier coefficients $\{b_m\}$ of V are assumed to be summable with the weight $(\omega_m^\alpha + \omega_m^{1/2})$

$$\sum |b_m|(\omega_m^\alpha + \omega_m^{1/2}) < \infty . \tag{5}$$

Theorem. Let $L = A + B$ be the anharmonic oscillator with an odd potential $B(x) \sim |x|^{-\alpha} V(x)$, where a trigonometric function $V(x) = \Sigma\, b_m \sin \omega_m x$ satisfies (5). The kth spectral shift, μ_k, of L is asymptotic to

$$\mu_k \sim C_\alpha(V) k^{-\gamma} \tilde{V}(\sqrt{2k}) \qquad \text{as } k \to \infty \tag{6}$$

where the exponent $\gamma = \frac{1}{2}\alpha + \frac{3}{4}$; the coefficient

$$C_\alpha(V) = \Gamma(1-\alpha) \cos\left[\tfrac{1}{2}\pi(1-\alpha)\right] \sum b_m \omega_m^\alpha \tag{7}$$

and $\tilde{V}$ denotes the 'odd Radon transform' of V

$$\tilde{V}(x) = \sqrt{\frac{2}{\pi}} \sum b_m \sqrt{\omega_m} \sin(\omega_m x - \tfrac{1}{4}\pi). \tag{8}$$

Let us note the difference between 'even' (4) and 'odd' (8) versions of the Radon transform. They correspond to formal fractional derivative operations applied to V:

Even: $\quad V \to \sqrt{2/\pi}\,|\partial|^{-1/2}(\mathrm{I} - \partial/|\partial|)[V] \qquad \text{order} = -\frac{1}{2}$

Odd: $\quad V \to \sqrt{2/\pi}\,|\partial|^{1/2}(\mathrm{I} - \partial/|\partial|)[V] \qquad \text{order} = \frac{1}{2}$.

As in [1], the theorem yields a unique solution of the inverse problem: determination of the asymptotic part of B (exponent α; frequencies $\{\omega_m\}$; and Fourier coefficients $\{b_m\}$ of V) from asymptotics of $\{\mu_k\}$.

Namely, given an admissible sequence of spectral shifts $\{\mu_k \sim k^{-\gamma} F(\sqrt{2k})\}$ with yet unknown trigonometric function $F(x)$ we proceed as follows.

(1) Find $\gamma = \inf\{p\colon \lim_k k^p \mu_k = 0\}$, consequently $\alpha = 2\gamma - \frac{3}{2}$.

(2) Use the 'uniform distribution property' of the sequence $\{\sqrt{2k}\}$ modulo any $T > 0$ or a tuple $\{T_1; \ldots; T_n\}$ to recover the period (or quasiperiods) of $F(x)$. The reconstruction procedure essentially amounts to 'screening out' values of T; $(T_1 \ldots T_n)$ which result in 'small' oscillations

$$O(x; \varepsilon; T) = \sup\{|j^\gamma \mu_j - k^\gamma_{\mu k}|\}$$

where the supremum is taken over a subsequence $\{k\colon |x - \sqrt{2k}| < \varepsilon \bmod (T_1; \ldots; T_n)\}$ (see [1] for details).

(3) Once periods $\{T_1; \ldots; T_n; \ldots\}$ are found or presecribed we recover frequencies

$$\omega_m = 2\pi \sum \frac{m_i}{T_i}$$

where $m = (m_1; m_2 \ldots)$ is a tuple of integers, and Fourier coefficients $\{\tilde{b}_m\}$ of $F(x) = \Sigma\, \tilde{b}_m \sin(\omega_m \times -\frac{1}{4}\pi)$. The latter are given by

$$\tilde{b}_m = \sqrt{2} \lim_k \frac{1}{k} \sum_1^k j^\gamma \sin(\omega_m \sqrt{2j}). \tag{9}$$

Here we utilised once again the 'uniform distribution property' of $\{\sqrt{2k}\}$.

Finally the trigonometric function $V(x)$ is determined by inverting the Radon transform

$$F(x) \rightarrow V(x) = \sqrt{\frac{\pi}{2}} \sum \frac{\tilde{b}_m}{\sqrt{\omega_m}} \sin \omega_m x.$$

Let us note that the transform $V \rightarrow F = C_a(V)\tilde{V}$ is now quadratic in V, unlike the even case of [1], where $V \rightarrow F$ was linear. However, the Fourier coefficients of F are all of the type

$$\tilde{b}_m = \Phi(\boldsymbol{b}) b_m \sqrt{\omega_m}$$

with a fixed linear form

$$\Phi(\boldsymbol{b}) = \text{Constant} \times \sum b_m \omega_m^a \qquad \boldsymbol{b} = (\ldots b_m \ldots).$$

Hypothesis (5) guarantees its convergence! Coefficients $\{b_m\}$ are easily recovered now from $\{\tilde{b}_m\}$

$$b_m = \tilde{b}_m / \sqrt{\omega_m \Phi(\ldots \tilde{b}_m / \sqrt{\omega_m} \ldots)}.$$

Thus we completely recover a trigonometric part $V = \Sigma\, b_m \sin \omega_m x$ of B. As a corollary we obtain a characterisation of admissible sequences of spectral fluctuations $\{\mu_k\}_1^\infty$ for odd potentials and uniqueness of the solution of the inverse spectral problem, similar to the even case of [1]. In the rest of the paper we shall outline the proof of the theorem.

Let us observe that the averaging procedure of Weinstein [6], adopted in [1], fails in the odd case for a simple reason, the average operator

$$\bar{B} = \frac{1}{2\pi} \int_0^{2\pi} \exp(\mathrm{i}tA) B \exp(-\mathrm{i}tA) \mathrm{d}t = 0.$$

Indeed, computing matrix entries of $\bar{B}$ in the basis of eigenfunctions (Hermite functions) $\{\psi_k\}_1^\infty$ of A yields

$$\langle \bar{B}\psi_k; \psi_m \rangle = \int_0^{2\pi} \exp\,[\mathrm{i}(k-m)t]\, \mathrm{d}t \langle B\psi_k; \psi_m \rangle = 0 \qquad \text{for all } k, m.$$

The approach adopted in the present paper follows ideas of Guillemin [7, 13] based on the so-called 'return operator'

$$W = \exp[2\pi \mathrm{i}(A+B)] - \mathrm{I}. \tag{10}$$

The operator W closely resembles some known operators of classical and quantum mechanics; eg. the Poincaré map, the transfer matrix and the wave operator, and plays a similar role in our discussion.

The factor $2\pi i$ in the exponential reflects the periodicity of the unitary group $\{\exp(itA)\}_{t=0}^{2\pi}$, equivalent to 'integer eigenvalues' of the operator A, spec $A \subset \{1, 2, \ldots\}$.

The return-operator approach is more fundamental than the averaging method; it will be shown to yield both the even and odd cases. They correspond to the first- and second-order asymptotic approximations of W.

We expect that further development of the return-operator method could yield the generic (even + odd potential) case as well as higher-order asymptotics of fluctuations $\{\mu_k\}$ (cf [6, 8]).

Let us first explain heuristically the construction and expansion of W and its relation to the spectrum of L.

Denote by $B(t)$ the conjugate of B,

$$B(t) = \exp(itA)\, B \exp(-itA) \qquad 0 \leqslant t \leqslant 2\pi.$$

Using the Trotter product formula one can derive the following expansion for the unitary group $\exp[it(A+B)]$ (cf [13]):

$$\exp[it(A+B)] = \exp(itA)\left[I + i\int_0^t B(\tau)\, d\tau - \frac{1}{2}\left(\int_0^t B(\tau)\, d\tau\right)^2 + \frac{1}{2}\int_0^t \int_0^s [B(s); B(\tau)]\, d\tau\, ds + \ldots \right]. \tag{11}$$

Here [;] means the usual commutator $B_1B_2\text{-}B_2B_1$ of two operators.

To get (11) one takes the nth Trotter approximant $M_n = \{\exp[(it/n)A]\exp[(it/n)B]\}^n$ and conjugates all B factors with $\exp(itA)$.

$$M_n = \exp(itA) \prod_1^n \exp[(it/n)B(kt/n)]. \tag{12}$$

To simplify the notation let us denote $\varepsilon = 1/n$ and $B_k = B(kt/n)$. Expanding each factor of (12) as a Taylor series

$$\exp(i\varepsilon B_k) = I + i\varepsilon B_k - \tfrac{1}{2}\varepsilon^2 B_k^2 + \ldots \tag{13}$$

multiplying expansions (13) and collecting powers of ε, we get

$$M_n = \exp(itA)\left[I + i\varepsilon \sum_1^n B_k - \frac{\varepsilon^2}{2}\left(\sum_k B_k^2 + 2\sum_{k<j} B_kB_j\right) + O(\varepsilon^3)\right]. \tag{14}$$

It remains to rewrite the quadratic term as

$$\sum_k B_k^2 + 2\sum_{k<j} B_kB_j = \left(\sum_k B_k\right)^2 - \sum_{k<j}[B_j; B_k]$$

and take limit of the right-hand side of (14) as $n \to \infty$.

After (11) is established we aply it to an operator A with an 'integral spectrum', and set $t=2\pi$:

$$W=\exp[i2\pi(A+B)]-\mathrm{I}=i\left(\int_0^{2\pi} B(t)\,dt\right)-\frac{1}{2}\left(\int_0^{2\pi} B(t)\,dt\right)^2$$
$$+\frac{1}{2}\int_0^{2\pi}\int_0^t [B(t);B(\tau)]\,d\tau\,dt+\ldots \tag{15}$$

Let us now introduce the first- and second-order 'averages' of B

$$B_1=\frac{1}{2\pi}\int_0^{2\pi} B(t)\,dt \qquad B_2=\frac{1}{4\pi i}\int_0^{2\pi}\int_0^t [B(t);B(\tau)]\,d\tau\,dt$$

and rewrite (15) in terms of B_1 and B_2 as

$$W=\exp[2\pi i(A+B)]-\mathrm{I}=2\pi i\,B_1-2\pi^2B_1^2+2\pi i\,B_2+\ldots \tag{16}$$

Comparing spectra on both sides of (16), we obtain on the left-hand side

$$2\pi i\mu_k-\frac{1}{2!}(2\pi\mu_k)^2+\ldots$$

while the right-hand side is approximated either by eigenvalues of B_1 (even case), or in the odd case ($B_1=0$) by the eigenvalues of B_2.

The even-case approximation was justified in [1]. Namely, if $\bar{\mu}_k$ denotes the kth eigenvalue of B_1, then it was shown that

$$|\mu_k-\bar{\mu}_k|=O(k^{-\alpha-1/4}).$$

We need a similar result in the odd case.

Lemma. Let μ_k be the kth spectral fluctuation of the operator $L=A+B$ with odd B and let $\bar{\mu}_k$ be the kth eigenvalue of B_2. Then

$$|\mu_k-\bar{\mu}_k|\leqslant \text{constant}\times k^{-\alpha-3/4}.$$

It follows immediately from the lemma that the leading asymptotics (6) of $\{\mu_k\}$ are equal to asymptotics of $\{\bar{\mu}_k\}$.

Thus, to establish the theorem we can replace perturbation B in L with B_2, i.e. consider the operator $\tilde{L}=A+B_2$.

The advantage of the latter is that B_2 commutes with A(cf [1]).

Proposition 1. The following commutation relations always hold:

$$[A;B_1]=0 \qquad [A;B_2]=[B;B_1].$$

Proof. Both are verified straightforwardly by using the differentiation formula

$$B'(t)=\frac{d}{dt}[\exp(itA)B\exp(-itA)]=i[A;B(t)]$$

and integrating by parts. Then

$$[A;B_1]=\frac{1}{2\pi i}\int_0^{2\pi}\frac{d}{dt}B(t)\,dt=0$$

$$[A;B_2]=\frac{1}{4\pi}\int_0^{2\pi}\int_0^t([B'(t);B(s)]+[B(t);B'(s)])\,ds\,dt.$$

The second integral in the expression for $[A;\,B_2]$ yields

$$\frac{1}{4\pi}\int_0^{2\pi}[B(t);B(t)-B]=-\tfrac{1}{2}[B_1;B] \tag{17}$$

while the first, $\int_0^{2\pi}[B'(t);\,B^{(-1)}(t)]\,dt$, is integrated by parts

$$\frac{1}{4\pi}[B(t);B^{(-1)}(t)]\bigg|_0^{2\pi}-\frac{1}{4\pi}\int_0^{2\pi}[B(t);B(t)]=\tfrac{1}{2}[B;B_1] \tag{18}$$

Here $B^{(-1)}(t)$ means the antiderivative $\int_0^t B(s)\,ds$.

Combining (17) and (18) we prove proposition 1. □

In the odd case, i.e. $B_1=0$, we see that A commutes with B_2.

Now let us turn to the proof of the lemma. The key to the proof is to show that the operators $A+B$ and $A+B_2$ are 'almost unitarily equivalent' and to estimate the remainder.

Proposition 2. There exist a skew-symmetric operator Q and a unitary operator $U=e^Q$ such that:

(i) $A+B=U(A+B_2)U^{-1}$ + small remainder R;

(ii) remainder R satisfies the operator inequality

$$|R|=(R^*R)^{1/2}\leqslant \text{constant}\times A^{-\alpha-3/4} \tag{19}$$

in the sense of comparison of self-adjoint operators.

Proof. Operator Q is constructed explicitly. We introduce the following integrals:

$$Q_1=\frac{i}{2\pi}\int_0^{2\pi}(2\pi-t)B(t)\,dt \qquad Q_2=\frac{1}{4\pi}\int_0^{2\pi}(2\pi-t)\int_0^t[B(t);B(s)]\,ds\,dt$$

and verify the commutation relations

$$[A;Q_1]=B_1-B \qquad [A;Q_2]=B_2-\tfrac{1}{2}[Q_1;B] \tag{20}$$

as in proposition 1.

In the even case, one takes $Q=Q_1$ (see [1, 6]). In the odd case we set $Q=Q_1+Q_2$. It follows from (20) that Q satisfies

$$[A;Q]=B_2-B-R_1 \qquad \text{with remainder } R_1=\tfrac{1}{2}[Q_1;B]. \tag{21}$$

The rest of the argument essentially follows the exposition of [1, 6]. Namely, we 'exponentiate' the commutation relation (21)

$$Ae^Q-e^QA=B_2-B+\left(R_1+\sum_2^\infty\frac{1}{n!}\sum_0^{n-1}Q^{n-1-j}(B_2-B-R_1)Q^j\right)=B_2-B+R_2$$

then replace B_2-B with $e^QB_2-B\,e^Q$ + another remainder, and rewrite the whole expression as

$$Ae^Q-e^QA=e^QB_2-B\,e^Q+R.$$

This proves the first statement.

The new remainder

$$R = R_2 + \sum_1^\infty \frac{1}{n!}(Q^n B - B_2 Q^n) \tag{22}$$

is to be estimated in terms of the operator A.

To this end we shall exploit a form of symbolic calculus introduced in [1]. Our symbol classes S^m consist of functions $\sigma(x, \xi)$, which admit an asymptotic expansion in polar coordinates: $r = \sqrt{x^2 + \xi^2}$; $\theta = \cos^{-1} x/r$

$$\sigma(r, \theta) \sim \exp(\mathrm{i}\omega r)(c_0 r^m + c_1 r^{m-1} + \ldots) \qquad \text{as } r \to \infty. \tag{23}$$

Here the phase function $\omega = \omega(\theta)$ and coefficients $c_j = c_j(\theta)$ are assumed to depend smoothly on θ away from $\xi = 0$, i.e. $\theta = \pi/2$. Exponent m is called the (principal) order of σ.

To each symbol σ we assign a pseudodifferential (Fourier integral) operator $K = K_\sigma$ by the Weyl convention

$$\sigma \to K_\sigma(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma\left(\frac{x+y}{2}; \xi\right) \exp[\mathrm{i}\xi(x-y)]\, \mathrm{d}\xi.$$

One example of symbols in S^m is the potential $B(x) \in S^{-\alpha}$. In [1], we have also shown that operators B_1 and Q_1 belong in the class $S^{-\alpha-1/2}$.

Similar results hold for B_2, Q_2 and Q. All three can be shown to belong to $S^{-\alpha-3/2}$. The proof of this statement involves somewhat tedious calculations which are carried out for symbol of B_2 in the appendix.

All standard results of symbolic (Fourier integral) calculus extend to classes S^m, though not in a straightforward manner (see [1]). In particular, one has the product formula:

$$\text{order}\,(AB) \leqslant \text{order}\, A + \text{order}\, B$$

as well as conjugation $A \to A^*$; inversion, etc. As a result, one can show that the remainder R belongs to our classes and its order is less than or equal to 'order B' + 'order Q' $= -(2\alpha + \frac{3}{2})$.

To complete the proof of proposition 2 it remains to observe that operators of classes $S^m(-\infty < m < \infty)$ can be compared to fractional powers of the elliptic operator $A = \frac{1}{2}(\partial^2 + x^2 - 1)$, whose symbol $\sigma_A = \frac{1}{2} r^2$.

Precisely, $R \in S^{-\beta}$ implies $A^{\beta/2} R$ is bounded (see [1]), or

$$|R| \leqslant c A^{-\beta/2}.$$

In our case $\beta = 2\alpha + \frac{3}{2}$, whence follows (19). □

After proposition 2, we can easily prove the lemma. Indeed, the relation

$$\mathrm{e}^Q (A + B) \mathrm{e}^{-Q} = A + \bar{B} + R$$

along with the estimates (19) of R, yield

$$k + \bar{\mu}_k - ck^{-\alpha-3/4} \leqslant k + \mu_k \leqslant k + \bar{\mu}_k + ck^{-\alpha-3/4}$$

or

$$|k^{(\alpha/2)+(3/4)}(\mu_k - \bar{\mu}_k)| \leqslant ck^{-\alpha/2}.$$ □

The lemma reduces the problem of calculating $\{\mu_k\}$ to the 'average' operator $\bar{L}=A+B_2$. Due to commutativity, $AB_2=B_2A$, spectral fluctuations of $\bar{L}$ are nothing but eigenvalues of B_2.

To complete the proof of the theorem we once again use the results of the symbolic calculus of [1]. Namely,

$$\text{symb } B_2=c_a r^{-[\alpha+3/2)]}\tilde{V}(r). \tag{24}$$

The principal symbol of B_2 is computed in the appendix.

It follows from (24) that the operator B_2 is aproximated (modulo lower-order error) by a 'function of A'

$$B_2\approx c_a(2A)^{-\gamma}\tilde{V}(\sqrt{2A}).$$

Hence the kth fluctuation $\mu_k\approx k$th eigenvalue of B_2 is approximated by

$$\mu_k\approx c_a(2k)^{-\gamma}\tilde{V}(\sqrt{2k}) \qquad \gamma=\tfrac{1}{2}\alpha+\tfrac{3}{4}$$

as was claimed in the theorem.

Remark. As in the even case ([1]), the theorem yields a continuous limiting distribution for properly modified 'average' spectral fluctuations.

The latter can be described, following [1, 6], by a sequence of discrete measures on $\mathbb{R}$

$$\mathrm{d}\rho_k(t)=\frac{1}{k}\sum_{1}^{k}\delta(t-j^{\gamma}\mu_j) \qquad \gamma=\tfrac{1}{2}\alpha+\tfrac{3}{4}. \tag{25}$$

The weight factors $\{j^{\gamma}\}$ in (25) take into account the algebraic rate of decay of $\mu_j=\mathrm{O}(j^{-\gamma})$.

Then using the equidistribution property of the sequence $\{\sqrt{k}\}_1^{\infty}$ (modulo T or $\{T_1; T_2; \ldots; T_m; \ldots\}$) (see [1]) one can easily derive the following analogue of Weinstein's n-sphere Schrödinger result [6]: the sequence of measures $\{\mathrm{d}\rho_k\}$ converges to a continuous measure $\beta(t)\,\mathrm{d}t$ on $\mathbb{R}$, whose density $\beta(t)$ is equal to the distribution function of the Radon transform $\tilde{V}$ of V.

Let us also observe that by analogy with the n-sphere case (see [1, 6–8, 13]), sequence $\{\mathrm{d}\rho_k\}$ admits an asymptotic expansion in powers of k^{-1}

$$\mathrm{d}\rho_k\sim\beta+\frac{\beta_1}{k}+\frac{\beta_2}{k^2}+\ldots$$

with distributional coefficients $\beta=$ 'Radon V', β_1, β_2; etc. The latter represent certain spectral invariants of L, which Weinstein [6] called *band-invariants*. However, in both cases, n-sphere Schrödinger and anharmonic oscillators, the higher-order band-invariants $\beta_1, \beta_2, \ldots$ become increasingly more difficult to calculate explicitly.

Summary. (1) Asymptotic spectral problems (direct and inverse) arise naturally in any setting where the exact eigendata are not available, and the input ('geometric data') can be defined by certain asymptotics. One such example is perturbations of the harmonic oscillator $L=A+B(x)$, with potentials $B\sim|x|^{-\alpha}V(x)$ at $\{\infty\}$.

(2) The main result of this paper (the theorem) gives an explicit relation between asymptotics of the input data B and those of eigenvalues $\{\lambda_k\}$ of L, in the case of the odd potential B, which is technically more difficult than the even case, studied in [1].

(3) Unlike all known examples of the exact inverse problem (in 1D), which typically have large isospectral classes (cf [4, 11]), our asymptotic problems exhibit uniqueness.

It would be interesting to extend this study to other classes of asymptotic potential data, e.g. functions B which are superpositions of non-linear Fourier-dispersion modes, and to explore possible uniqueness or non-uniqueness of the corresponding inverse problems.

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Appendix. Principal symbol of the second average operator B_2

Recall the definition of B_2

$$B_2 = \frac{1}{4\pi i}\int_0^{2\pi}\int_0^t [B(t); B(\tau)]\,d\tau\,dt.$$

Recall also that $B(t) = \exp(itA)\ B\ \exp(-itA)$ is obtained by conjugating B with a unitary group generated by A. Two basic results of the symbolic calculus of [1] will be utilised in the derivation of symbol (B_2).

(i) The first result is the conjugation with Fourier integral operators (a version of the so-called Egorov theorem of pseudodifferential calculus):

$$\text{symbol } \sigma_{B(t)} = \sigma_B \circ \exp(tH)$$

where $H = x\partial_\xi - \xi\partial_x$ is the Hamiltonian vector field of symbol $\sigma_A = \frac{1}{2}(x^2+\xi^2) = \frac{1}{2}r^2$ (Hamiltonian of the classical oscillator).

Observe that the corresponding Hamiltonian flow $\exp(tH)$ consists of rotations $\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$ in the (x, ξ) phase plane.

(ii) The second result is the commutator formula

$$\sigma_{[B_1;B_2]} = i\{\sigma_{B_1}; \sigma_{B_2}\}.$$

Here $[B_1; B_2]$ means the commutator of two operators $B_1B_2 - B_2B_1$, and $\{f; g\}$ is the Poisson bracket of two functions on the (x, ξ) plane,

$$\{f; g\} = \partial_\xi f\,\partial_x g - \partial_\xi g\,\partial_x f.$$

Applying (i) and (ii) to $\sigma = \sigma_{B_2}$ we get

$$\sigma = \frac{1}{4\pi}\int_0^{2\pi}\int_0^t \{B\circ\exp(tH); B\circ\exp(\tau H)\}\,d\tau\,dt. \tag{A1}$$

Let us calculate the Poisson bracket in (A1)

$$\{;\} = B'(x\cos t + \xi\sin t)B'(x\cos\tau + \xi\sin\tau)\sin(t-\tau) \tag{A2}$$

where B' means the derivative dB/dx.

Next we express all functions in (A2) in polar coordinates (r, θ) and use the asymptotic expansion of $B(x)$ at ∞,

$$B'(x) \sim -\alpha x^{-\alpha-1}V(x) + x^{-\alpha}V'(x). \tag{A3}$$

Keeping the highest-order second term of (A3) we rewrite (A1) as

$$\sigma(r, \theta) = \frac{1}{r^{2\alpha}} \int_0^{2\pi} \int_0^t \frac{V'(r\cos(t-\theta))}{|\cos(t-\theta)|^\alpha} \frac{V'(r\cos(\tau-\theta))}{|\cos(\tau-\theta)|^\alpha} \sin(t-\tau)\, d\tau\, dt.$$

After the change of variables $(t, \tau) \to (t-\theta; \tau-\theta)$ this becomes

$$\sigma = \frac{1}{r^{2\alpha}} \int_{-\theta}^{2\pi-\theta} \int_{-\theta}^t \frac{V'(r\cos t)}{|\cos t|^\alpha} \frac{V'(r\cos\tau)}{|\cos\tau|^\alpha} \sin(t-\tau)\, d\tau\, dt \tag{A4}$$

The integrand in (A4) breaks into the sum of terms $\Sigma\, T_{lm}$, where

$$T = T_{lm} = b_l b_m \omega_l \omega_m \frac{\cos(\omega_m r\cos t)}{|\cos t|^\alpha} \frac{\cos(\omega_l r\cos\tau)}{|\cos\tau|^\alpha} \sin(t-\tau). \tag{A5}$$

We shall evaluate the principal asymptotic contribution of each term T_{lm} to the integral $\int_0^{2\pi}\int_0^t \sigma\, d\tau\, dt$, using the stationary phase method, and then combine all contributions together.

It is convenient to divide the range of variable t into intervals $[0; \pi/2]$, $[\pi/2; \pi]$, ... and denote the corresponding integrals

$$I_1 = \int_0^{\pi/2} \int \dots \qquad I_2 = \int_{\pi/2}^{\pi} \int \dots \qquad I_3 = \int_{\pi}^{3\pi/2} \int \dots \qquad I_4 = \int_{3\pi/2}^{2\pi} \int \dots .$$

Let us observe that integrals of the type

$$I(r) = \int \frac{\cos(\omega\, r\cos\tau)}{|\cos\tau|^\alpha} \sin(t-\tau)$$

which appear in (A4), have two kinds of principal contributing points: critical points $\tau = 0; \pi$ and singular points $\tau = \frac{1}{2}\pi; \frac{3}{2}\pi$.

Their contributions to the large-r asymptotics of $I(r)$ are

$$\begin{aligned}
&\text{(i)} && \text{critial } \tau = 0\text{:} \int^0 \dots \sim \sqrt{\frac{\pi}{2\omega r}} \cos(\omega r - \pi/4)\sin t \\
&\text{(ii)} && \text{critical } \tau = \pi\text{:} \int^\pi \dots \sim \sqrt{\frac{\pi}{2\omega r}} \cos(\omega r + \pi/4)(-\sin t) \\
&\text{(iii)} && \text{singular } \tau = \pi/2\text{:} \int^{\pi/2} \dots \sim c_\alpha(\omega r)^{\alpha-1}(-\cos t) \\
&\text{(iv)} && \text{singular } \tau = 3\pi/2\text{:} \int^{3\pi/2} \dots \sim c_\alpha(\omega r)^{\alpha-1}\cos t.
\end{aligned} \tag{A6}$$

Here

$$c_\alpha = \int_0^\infty \cos z \frac{dz}{z^\alpha} = \cos[(\pi/2)(1-\alpha)]\Gamma(1-\alpha)$$

is a constant.

Let us now calculate the principal asymptotics of the inner integral $I(t, r) = \int_{-\theta}^{t} \ldots$ in (A4) for each term $T = T_{lm}$ (A5) in all four different ranges of t.

For convenience we present the result in table 1.

Table 1. The principal asymptotics the integral with respect to τ in (A4).

Range	Contributing points	Asymptotics of $I(t, r)$ as $r \to \infty$
$0 \leq t < \pi/2$	critical: $\tau = 0$	$\sqrt{\frac{\pi}{2\omega r}} \cos(\omega r - \pi/4) \sin t$
$\pi/2 \leq t < \pi$	critical: $\tau = 0$	$\sqrt{\frac{\pi}{2\omega r}} \cos(\omega r - \pi/4) \sin t - c_\alpha (\omega r)^{\alpha-1} \cos t$
	singular: $\tau = \pi/2$	
$\pi \leq t < 3\pi/2$	critical: $\tau = 0$; π	$\sqrt{\frac{\pi}{2\omega r}} [\cos(\omega r - \pi/4) - \cos(\omega r + \pi/4)] \sin t - c_\alpha (\omega r)^{\alpha-1} \cos t$
	singular: $\tau = \pi/2$	
$3\pi/2 \leq t \leq 2\pi$	critical: $\tau = 0$; π	$\sqrt{\frac{\pi}{2\omega r}} [\cos(\omega r - \pi/4) - \cos(\omega r + \pi/4)]$
	singular: $\tau = \pi/2$; $3\pi/2$	

Let us remark that contributions of the two singular points in the fourth line of table 1 cancel each other, whereas the contributions of the two critical points (bracketed term in the third and fourth lines) combine to $\sqrt{2} \sin \omega r$.

Now we perform the outer integration in (A4) and substitute asymptotics of the table 1 for an appropriate value of the inner integral. This yields

$$I_1 \sim \sqrt{\frac{\pi}{2\omega}} \frac{\cos(\omega r - \pi/4)}{r^{2\alpha+1/2}} \int_{\mathbf{0}}^{\pi/2} \frac{\cos(\omega' r \cos t)}{|\cos t|^\alpha} \sin t \, dt$$

$$I_2 \sim \sqrt{\frac{\pi}{2\omega}} \frac{\cos(\omega r - \pi/4)}{r^{2\alpha+1/2}} \int_{\boldsymbol{\pi/2}}^{\pi} \frac{\cos(\omega' r \cos t)}{|\cos t|^\alpha} \sin t \, dt - c_\alpha \frac{\omega^{\alpha-1}}{r^{\alpha+1}} \int_{\boldsymbol{\pi/2}}^{\pi} \frac{\cos(\omega' r \cos t)}{|\cos t|^\alpha} \cos t \, dt$$

$$I_3 \sim \sqrt{\frac{\pi}{2\omega}} \frac{\sqrt{2} \sin \omega r}{r^{2\alpha+1/2}} \int_{\pi}^{\boldsymbol{3\pi/2}} \frac{\cos(\omega' r \cos t)}{|\cos t|^\alpha} \sin t \, dt - c_\alpha \frac{\omega^{\alpha-1}}{r^{\alpha+1}} \int_{\pi}^{3\pi/2} \frac{\cos(\omega' r \cos t)}{|\cos t|^\alpha} \cos t \, dt$$

$$I_4 \sim \sqrt{\frac{\pi}{2\omega}} \frac{\sqrt{2} \sin \omega r}{r^{2\alpha+1/2}} \int_{\boldsymbol{3\pi/2}}^{2\pi} \frac{\cos(\omega' r \cos t)}{|\cos t|^\alpha} \sin t \, dt.$$

In the above formulae for I_1–I_4 the 'principal' contributing endpoint in the sense of asymptotics (A6) for every integral has been set in bold type; the other end can be disregarded (gives lower-order asymptotics) due to the factor $\sin t$ (or $\cos t$) vanishing there.

Let us remark that two types of integrals appear in the formula for I_1–I_4: that for the contributing 'singular' point

$$F_{\sin}=\int \frac{\cos(\omega' r\cos t)}{|\cos t|^{\alpha}}\sin \mathrm{d}t \sim c_\alpha(\omega' r)^{\alpha-1}\sin(\pm\pi/2)$$

and that for the contributing 'critical' point

$$F_{\cos}=\int \frac{\cos(\omega' r\cos t)}{|\cos t|^{\alpha}}\cos t\, \mathrm{d}t \sim \sqrt{\frac{\pi}{2\omega' r}}\cos(\omega' r\mp\pi/4).$$

Summing the four integrals $I_1+\ldots+I_4$, we get

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\frac{c_\alpha}{r^{\alpha+3/2}}\Bigg(&\frac{\cos(\omega r-\pi/4)}{\sqrt{\omega}}(\omega')^{\alpha-1}\\ &+\frac{\cos(\omega r-\pi/4)}{\sqrt{\omega}}\omega^{\alpha-1}+\frac{\cos(\omega' r+\pi/4)}{\sqrt{\omega'}}\\ &-\frac{\sqrt{2}\sin\omega r}{\sqrt{\omega}}(\omega')^{\alpha-1}+\omega^{\alpha-1}\frac{\cos(\omega' r+\pi/4)}{\sqrt{\omega'}}\\ &-\frac{\sqrt{2}\sin\omega r}{\sqrt{\omega}}(\omega')^{\alpha-1}\Bigg).\end{aligned} \tag{A7}$$

where the ith line of (A7) represents the contribution of I_i.

Formula (A7) gives a contribution of an individual term $T=T_{ml}(\omega=\omega_m,\ \omega'=\omega_l)$. Next we sum all the terms

$$\sigma\sim\sqrt{\pi/2}\,\frac{2c_\alpha}{r^{\alpha+3/2}}\sum_{l,m} b_m b_l\omega_m\omega_l\frac{[\cos(\omega_m r-\pi/4)+\cos(\omega_m r+\pi/4)-\sqrt{2}\sin\omega_m r]}{\sqrt{\omega_m}}\,\omega_l^{\alpha-1}.$$

Simplifying the trigonometric expression in square brackets to $\sqrt{2}\,\sin(\omega r-\pi/4)$, we finally rewrite the answer as

$$\sigma\sim\sqrt{\frac{\pi}{2}}\frac{4c_\alpha}{r^{\alpha+3/2}}\left(\sum b_m\omega_m^{\alpha}\right)\sum\sqrt{\omega_m}\,b_m\sin(\omega_m r-\pi/4)$$

which gives $r^{-\alpha-3/2}C_\alpha\tilde{V}(r)$, with constant $C_\alpha=C_\alpha(V)$ of the theorem.

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