Inverse Problems 6 (1990) 371–378. Printed in the UK

2-sphere Schrödinger operators with odd potentials

David Gurarie

Case Institute of Technology, Case Western Reserve University, Cleveland, OH 44106, USA

Received 4 August 1989, in final form 9 January 1990

Abstract. The paper extends our earlier results on spectral theory of the 2-sphere Schrödinger operators $H = -\Delta + V$ with even zonal potentials to a more difficult odd-potential case. We establish local spectral rigidity for odd zonal potentials and give the explicit solution of the inverse spectral problem.

We are interested in the spectral theory of Schrödinger operators $H = -\Delta + V$ on the 2-sphere S_2 with *odd zonal (axisymmetric) potentials* $V = V(x) = -V(-x)$, i.e. functions depending on the first coordinate of the vector $\boldsymbol{x} = (x; \ldots)$ in S_2. In particular, we seek a solution of the following inverse problem: determination of V by the spectrum of H, and the related *problem of spectral rigidity*: uniqueness of the map $V \to \mathrm{Spec}(H_V)$, modulo natural symmetries (rotations).

The spectrum of the S_2 Laplacian $-\Delta$ is well known to consist of eigenvalues $\{\lambda_k = k(k+1)\}_{k=1}^{\infty}$ of increasing multiplicity, $\sharp\lambda_k = 2k+1 = \mathrm{O}(k^{n-1})$. Adding a potential V destroys (or lowers) the underlying symmetry of $-\Delta$, so the multiple eigen λ_k splits into a cluster of simple (or less degenerate) eigens,

$$\lambda_{km} = k(k+1) + \mu_{km} \qquad |m| \leqslant k. \tag{1}$$

Spectral shifts $\{\mu_{km}\}$ make up the kth cluster of Spec(H). To study their asymptotics Weinstein [14] introduced a sequence of discrete (cluster-distribution) measures

$$\mathrm{d}\rho_k(\lambda) = \frac{1}{2k+1} \sum_{m=-k}^{k} \delta(\lambda - \mu_{km}) \qquad \text{on } \mathbb{R}. \tag{2}$$

It turned out that the sequence $\{\mathrm{d}\rho_k\}$ converges to a continuous limit, $\beta_0(\lambda)\mathrm{d}\lambda$, whose density is directly linked to V. Namely, β_0 is equal to the distribution function of the so-called *Radon transform* $\tilde{V}$ of V.

The distribution function β_0 of $\tilde{V}$ represents a new type of spectral invariant, called the *band invariant* [14]. It yields some valuable spectral information about H (cf [3, 4, 8, 12, 13]), but falls short of determining $\tilde{V}$, and consequently V itself. So the inverse and rigidity problems on S_n, unlike many other geometric settings (cf [2, 5]), remain largely open. Partial results are known for special classes of potentials: low-degree spherical harmonics [6], and even zonal potentials on S_2, studied in the recent work of the author [8].

The main idea of [8] for solving both problems was to replace asymptotics of cluster distribution measures $\mathrm{d}\rho_k$ (2) by asymptotics of individual spectral shifts $\{\mu_{km}\}$. An earlier attempt along these lines was made in [1].

An essential feature of zonal Schrödinger operators, exploited in [8], was an auxiliary symmetry given by the *angular momentum operator* $M = \mathrm{i}\partial_\theta$, that commutes with H. So two operators $\{H; M\}$, have a joint eigenfunction expansion:

$$H\psi_{km} = \lambda_{km}\psi_{km} \qquad M\psi_{km} = m\psi_{km}. \tag{3}$$

Hence, spectral shifts $\{\mu_{km}\}$ acquire a bigraded structure, with $\{k\}$ labelling the cluster number, and $\{m\}$ the angular momentum number.

In this setting we derived an asymptotic formula for $\{\mu_{km}\}$

$$\mu_{km} = \tilde{V}(m/k) + O(k^{-2}) \qquad \text{for } -k \leqslant m \leqslant k \tag{4}$$

in terms of the Radon transform $\tilde{V}$ of V, as a one-variable function† on [0; 1].

The derivation of [8] consisted of three steps:

(i) the (Weinstein) averaging procedure;
(ii) zonal reduction from the 2-sphere to a sequence of one-dimensional problems;
(iii) symbolic calculus of singular degenerate Legendre-type pseudodifferential operators (ψDOs).

The key idea of step (i) was to replace the perturbation V, or a closely related ψDO

$$B = \sqrt{H} - \sqrt{-\Delta} \approx \tfrac{1}{2}\Delta^{-1/2}V + \ldots$$

by the average of its conjugates $\{B(t) = e^{-itA}Be^{itA}\}$, where $A = \sqrt{-\Delta}$, i.e.

$$B \to \bar{B} = \bar{B} = \frac{1}{2\pi}\int_0^{2\pi} B(t)\,dt. \tag{5}$$

Obviously, $\bar{B}$ commutes with A, and one can show [4, 6–10, 12–14] that $A + B$ is almost unitarily equivalent to $A + \bar{B}$. The advantage of passing from B to $\bar{B}$ is the commutation relation $[A; \bar{B}] = 0$, so A and $\bar{B}$ can be simultaneously diagonalised.

If Spec(A) were simple (multiplicity free), then the commutation relation would yield $\bar{B} = f(A)$ for some function f, and such an f could be approximately computed by the standard techniques of the symbolic (ψDO) calculus on S_n. This in turn would yield spectral shifts $\{\mu\}$ of $A + B$, directly as $\mu_k \approx f(k)$. Unfortunately, S_n Laplacians have highly degenerate spectra, so the argument would not apply to the Schrödinger operator H as a whole.

Therefore step (ii) (zonal reduction) was used in [8] to break the S_2 Schrödinger problem into a sequence of one-dimensional reduced problems for perturbations of associated Legendre operators on [0; 1].

Namely, the L^2 space on S_2 is decomposed into the direct sum of eigensubspaces of M, $\mathfrak{H}_m = \{u\colon u = e^{im\theta}f(x);\ \text{with } f \in L^2[-1; 1]\} = \{u\colon M[u] = mu\}$. Subspaces $\{\mathfrak{H}_m\}$ are invariant under the Laplacian and all zonal Schrödinger operators. Their restrictions on $\mathfrak{H}_m$ are well known to be:

$\Delta|_{\mathfrak{H}_m} = L_m = \partial(1 - x^2)\partial - m^2/(1 - x^2)$: the mth associated Legendre operator;

$H|_{\mathfrak{H}_m} = H_m = L_m + V$: the perturbation of L_m with potential $V(x)$ on $[-1; 1]$.

$\mathfrak{H}_m$-zonal reduction was exploited in our derivation of (4), by interpreting $\{\mu_{km}\}$ as spectral shifts of reduced operators H_m,

$$\mathrm{Spec}(H_m) = \{k(k+1) + \mu_{km}\}_{k=m}^{\infty}.$$

The last step (iii) was to transplant (reduce) the calculus of operators and symbols on S_2 to the calculus of associated Legendre operators and their perturbations on [0; 1].

Asymptotics (4) yields a function $\tilde{V}$ at all rationals, hence everywhere on [0; 1]. Once $\tilde{V}$ is found the standard Radon inversion, $\tilde{V} \to V$, leads to a unique and explicit solution of the *joint inverse spectral problem* (3).

†Let us remark that the Radon transform of a zonal function is also zonal, i.e. depends on a single variable.

Turning to Spec(H) alone†, we combined (4) with a form of 'rigidity principle for almost arithmetic sequences' (theorem 2 of [8]), and thus were able to show *local spectral rigidity* for generic even zonal Schrödinger operators on S_2.

The odd-potential case differs from the even in many respects‡, and poses certain difficulties. Indeed, the basic Weinstein averaging (5) fails in the odd case and must be replaced by more sophisticated Guillemin's average [4] (see also [12, 13]). This in turn entails some other significant changes in symbolic calculus and the form of final results. Thus a simple Radon transforms $\tilde{V} = \mathfrak{R}(V)$ of the even case gets replaced by a more complicated transform $\mathfrak{I}(V^2)$, as we shall demonstrate below (theorem 1).

The main goal of the paper is to outline the odd-potential case. For the sake of presentation we restrict ourselves to S_2. The forthcoming work [10] will treat the n-dimensional case, and, more significantly, the case of general 'even + odd' potentials on S_n. The latter would require not only some major review of averaging methods, but also a new (non-standard) form of symbolic calculus on S_n due to Uribe [12]. The details will appear in [10].

To state the main result of the work we first recall the definition of the *Radon transform* $\mathfrak{R}$ on S_n, then compute it explicitly for zonal functions F. By definition, $\mathfrak{R}$ maps functions/symbols F on S_n into functions on the cosphere bundle $S^*(S_n) = \{(x; \xi)$: $x \in S_n$; $\xi \in T_x^*$; $|\xi| = 1\}$, or more precisely, on the *space* Ω *of geodesics/bicharacteristics* (great circles) $\{C = C_{x\xi}$, a circle through $\{x\}$ in the direction ξ, $\Omega \simeq S^*(S_n)/\mathrm{SO}(2)$. Namely,

$$\mathfrak{R}\colon F(x) \to \tilde{F}(C) = \tilde{F}(x; \xi) = \frac{1}{2\pi}\int_C F\,\mathrm{d}s \qquad C = C_{x\xi} \in \Omega.$$

Another expression for $\mathfrak{R}$ is obtained by composing F with the *geodesic flow* $\exp(t\Xi)$, i.e. the bicharacteristic flow of symbol $\sigma_A = |\xi|$. Here $\Xi = \xi' \cdot \partial_x$ denotes the *Hamiltonian vector field* of $\sigma_A = |\xi|$. Thus

$$\tilde{F} = \frac{1}{2\pi}\int_0^{2\pi} [F \circ \exp(t\Xi)](x, \xi))\,\mathrm{d}t.$$

Here F is thought of as a function on $S^*(S_2)$ (i.e. symbol), or as a pull back of a S_2 function onto $S^*(S_2)$, and the geodesic flow $\exp(t\Xi)$ acts both on $S^*(S_2)$ and on the space Ω.

Obviously, the Radon transform takes even functions/symbols into even, and annihilates odd. Since on S_2 each circle C is identified with its (north/south) pole, $\mathfrak{R}$ can be thought of as a transformation of S_2 functions into themselves.

For zonal functions $F = F(x)$, $x = \cos\phi \in [-1; 1]$, $\tilde{F}$ is also zonal, and as a function of the 'north pole' coordinate $\rho = \rho(C)$, it becomes [8]:

$$\mathfrak{R}\colon F(x) \to \tilde{F}(\rho) = \frac{2}{\pi}\int_0^1 F(x\sqrt{1-\rho^2})\,\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{2}{\pi}\int_0^{\sqrt{(1-\rho^2)}} F(x)\,\frac{\mathrm{d}x}{\sqrt{1-\rho^2-x^2}}. \tag{6}$$

Our main result here is the following theorem 1.

†The reader will notice a subtle difference between Spec(H), as a union of unstructured clusters $\{\mu_{km}$: $-k \leqslant m \leqslant k\}$, and the bigraded structure of the joint $(H; M)$ spectrum (3), where m corresponds to a particular value of the angular momentum M.

‡For example, spectral shifts $\mu_{km} = \mathrm{O}(k^{-2})$ for odd V compared with O(1) for even V (see [4]).

Theorem 1. Spectral shifts $\{\mu_{km}\}$ of the joint spectrum (3) of operators $H = -\Delta + V$, and $M = i\partial_\theta$ on S_2, with an odd zonal potential V, admit the following asymptotics:

$$\mu_{km} = k^{-2}U(m/k) + O(k^{-3}) \qquad \text{as } k \to \infty \tag{7}$$

uniformly in $-k \leqslant m \leqslant k$. Here the function $U(\rho)$ on [0; 1] is obtained from $V(x)$ by the transform

$$\mathfrak{I} = \frac{1}{2}\left(\frac{\rho^2}{1-\rho^2}\,\mathfrak{R}x\,\frac{d}{dx} - \mathfrak{R}\right)$$

applied to V^2, i.e.

$$U(\rho) = \frac{1}{\pi}\int_0^{\sqrt{1-\rho^2}}\left(\frac{\rho^2}{1-\rho^2}\,xF'(x) - F\right)\frac{dx}{\sqrt{1-\rho^2-x^2}} \qquad \text{with } F = V^2(x). \tag{8}$$

Theorem 1 represents the odd version of the main result (4) of [8], and the function $U = \mathfrak{I}(V^2)$ in (8), plays now the role of the Radon transform $\tilde{V}$ in (4).

As a corollary of theorem 1 we immediately get the solution of the inverse problem for the joint spectrum. Indeed, asymptotics (7) of the spectral shifts yield values of function U at any $\rho \in [0; 1]$:

$$U(\rho) = \lim_{m/k \to \rho} k^2 \mu_{km}.$$

Now it remains to invert transform $\mathfrak{I}$ to recover function V^2. Changing variables: $x \to s = x^2$ for V and $\rho \to t = 1 - \rho^2$ for U, the Radon transform becomes a ψDO $\mathfrak{R} = \partial_s^{-1/2}s^{-1/2}$ of order $-\frac{1}{2}$. Consequently operation $\mathfrak{I}$ of theorem 1 takes the form

$$\mathfrak{I} = \frac{1-t}{2t}\,|\partial|^{1/2}\sqrt{t}\,\mathrm{sgn}\,\partial - |\partial|^{-1/2}t^{-1/2} \approx t^{-1/2}[(1-t)\partial^{1/2} - \partial^{-1/2}] + \ldots \tag{9}$$

applied to $F = V^2 \circ \sqrt{t}$. In this form operation (9) can be shown to be invertible, whence we recover the function V^2 on [0; 1]. Of course, V^2 does not yet determine V itself. But for generic V (e.g. functions with simple zeroes on [0; 1]), V is uniquely determined by the choice of its sign at any particular point $x_0 \in [0; 1]$. Thus from asymptotics of the joint $(H; M)$ spectrum we are able to recover $\pm V$.

The sign ambiguity $\pm V$ is to be expected, as two Schrödinger operators $-\Delta \pm V$, with odd V, are unitarily equivalent by the change of variable, $x \to -x$, on S_2.

Thus we get a unique and explicit solution of the inverse problem for the *joint spectrum* (3) of H and M.

As for the inverse problem for the operator H alone, one can proceed as in [8], namely to combine solutions of the *joint problem* (3) and the rigidity result of [8] for almost arithmetic sequences. This would yield the local spectral rigidity for *odd generic potentials.*

Corollary. If two generic (in the sense of [8]) functions V_1 and V_2 are sufficiently close, and the spectra of two Schrödinger operators $H_j = -\Delta + V_j$ $(j = 1, 2)$ are equal, then $V_1 = V_2$†.

In the rest of the paper we shall outline the proof of theorem 1. The proof of the corollary is based on the above theorem 1 and on the rigidity theorem 2 of [8]. It closely follows the argument of [8] and hence will be omitted.

†There is no SO(3) rotational ambiguity for zonal functions as we fixed the axis of rotation!

In what follows it will be convenient to work with square roots (pseudodifferential operators) $A = \sqrt{-\Delta + \frac{1}{4}}$ and $\sqrt{H} = A + B$, instead of Δ and H. Here $B = \sqrt{-\Delta + V} - A \approx \frac{1}{2}A^{-1}V + \ldots$, and A and B have orders 1 and -1.

Notice that operator A has ψDOs integral spectrum hence generates a periodic unitary group $\{e^{itA}: 0 \leqslant t \leqslant 2\pi\}$.

As we have mentioned earlier, the Weinstein averaging (5) fails in the odd case and needs some modification. Here we shall adopt Guillemin's *return operator approach* [4]. The return operator is defined as

$$W = e^{i2\pi(A+B)} - I$$

and is expanded into a sereis of decreasing-order terms,

$$W = 2\pi i\bar{B} - 2\pi^2\bar{B} - 2\pi^2\bar{B}^2 + 2\pi i\bar{\bar{B}} + \ldots \tag{10}$$

the first three of which are given by the first (Weinstein) average $\bar{B}$, and the second (Guillemin) average is

$$\bar{\bar{B}} = \frac{1}{4\pi i}\int_0^{2\pi}\int_0^t [B(t); B(s)]\,ds\,dt. \tag{11}$$

For odd V the first two terms of (10) vanish, so $W = 2\pi i\bar{\bar{B}}$. As in the even case it can be shown that operator $\bar{\bar{B}}$ commutes with A, modulo lower-order terms and the pair $A + B$ and $A + \bar{\bar{B}}$ are almost unitarily equivalent [12, 13].

Furthermore, the eigenvalues of W are naturally related to spectral shifts $\{\mu = \mu_{km}\}$ of operator $A + B$,

$$\lambda(W) = e^{2\pi i\mu} - 1 = 2\pi i\mu - 2\pi^2\mu^2 + \ldots .$$

Obviously, any expansion of $\lambda(W)$ in powers of k^{-1},

$$\lambda_{km}(W) \sim k^{-1}a + k^{-2}b + k^{-3}c + \ldots \tag{12}$$

results in an appropriate expansion of $\{\mu_{km}\}$, and vice versa.

We shall show that for odd V (12) starts from the third order, then using symbolic calculus and zonal reduction we shall express the coefficients $c = c_{km}$ in terms of V.

To this end we need to expand a ψDO B in a decreasing-order series, and get the corresponding expansion of $W = W_1 + W_2 + W_3 + \ldots$.

Writing $B = B_1 + B_2 + B_3 + \ldots$, the first three terms are computed explicitly (see [10, 12, 13]):

$$B_1 = \tfrac{1}{2}A^{-1}V \qquad B_2 = -\tfrac{1}{4}A^{-2}\mathrm{ad}_A(V) \qquad B_3 = \tfrac{1}{8}(A^{-3}\mathrm{ad}_A{}^2(V) - A^{-3}V^2) \tag{13}$$

Here $\mathrm{ad}_A(\ldots)$ denotes the commutation operation $[A; X] = AX - XA$.

Then the corresponding expansion of W takes the form [12, 13]:

$$W = \tfrac{1}{2}A^{-1}\bar{V} + \tfrac{1}{4}A^{-3}(\bar{V}^2 - \bar{V^2}) + \tfrac{1}{4}A^{-2}\left([\bar{V}; Q] + \frac{1}{4\pi i}\int_0^{2\pi}\int_0^t [V(t); V(s)]\,ds\,dt\right) + \ldots \tag{14}$$

where $V(t)$ denotes the conjugation $(e^{-itA}Ve^{itA})$ of a multiplication operator V, $\bar{V}$ its average, while

$$Q = \frac{1}{2\pi i}\int_0^{2\pi}\int_0^t V(t)\,ds\,dt.$$

For odd potentials V some terms of (14) vanish, so it simplifies to

$$W = \tfrac{1}{4}A^{-3}\bar{V}^2 + \frac{1}{16\pi i}A^{-2}\int\int [V(t); V(s)]\,ds\,dt + \dots . \tag{15}$$

We need to compute the principal symbol of a ψDO W (15) of order -3. By the standard (product, commutator, conjugation) rules of symbolic calculus, it follows (cf [12, 13]) that

$$\sigma_W = \frac{1}{|\xi|^3}\left\{\frac{1}{4}\int_0^{2\pi}\sigma_{V^2(t)}\,dt - \frac{1}{8}\int_0^{2\pi}\int_0^t \{\sigma_{V(t)}; \sigma_{V(s)}\}\,ds\,dt\right\}. \tag{16}$$

Here σ_V, σ_{V^2} are ψDO symbols of operators $V(t)$, $V^2(t)$, and $\{\dots;\dots\}$ denotes the Poisson bracket of a pair of symbols on the phase space $T^*(S_2)$, homogeneous in the momentum variable ξ and restricted on the cosphere bundle $S^*(S_2)$.

The symbol of operator $V(t)$ is obtained by composing the symbol/function V on S_2 with the Hamiltonian (geodesic) flow, $\exp(t\Xi)$, of $\sigma_A = |\xi|$ (Egorov's theorem). In other words, $\sigma_{V(t)}(x;\xi) = V\circ\exp(t\Xi)$, is obtained by evaluating the function V along the geodesics (great circle) $C = C_{x\xi}$. So by some abuse of notation we shall call $V(t)$ three different but closely related objects: the operator $V(t) = e^{-itA}Ve^{itA}$, its symbol $V\circ\exp(t\Xi)$, and the function V evaluated along C, as a function of the flow parameter (angle) t.

With this convention in mind the Poisson bracket $\{V(t); V(s)\}$ is computed in terms of the tangential and normal derivatives of V along C at points $\{t\}$ and $\{s\}$, $(\partial_u V)(t)$; $(\partial_n V)(t)$. Namely (see [10] for details),

$$\{V(t); V(s)\} = (t-s)(\partial_u V)(t)(\partial_u V)(s) + \sin(t-s)(\partial_n V)(t)(\partial_n V)(s). \tag{17}$$

Substituting (17) in (16) and carrying over integration of the tangential part we end up with the following expression of σ_W:

$$\sigma_W = \frac{1}{4}\int_0^{2\pi} V^2(t)\,dt - \frac{1}{8}\int_0^{2\pi}\int_0^t \sin(t-s)(\partial_n V)(t)(\partial_n V)(s)\,ds\,dt. \tag{18}$$

Formula (18) explicates the previously known version (17) of [13], and has many applications, e.g. it yields the second Weinstein band invariant β_2 in the odd case, in terms of the distribution function of σ_W.

For the purpose of the inverse problem, however, we need more detailed information, namely asymptotic values of the spectral shifts $\{\mu_{km}\}$, rather than their distribution measures (2). If Spec(A) were multiplicity free this would amount to a straightforward symbolic calculation with (18), namely $\mu_k \approx \sigma_w|_{|\xi|=k}$ (cf [6, 7, 9]). But, as we mentioned, spectra of operators Δ and A are highly degenerate for all spheres S_n, $n \geqslant 2$.

At this point we invoke the zonal reduction/extension procedures of [8]. The key observation made in [8] was to show that all the above constructions, expansions and estimates with operators $\sqrt{-\Delta + V} = A + B$ on $L^2(S_n)$, (including averages (5) and (11), the return operator W and their symbols), can be transferred from S_n to the reduced operators and symbols on $[-1;1]$ (cf [11]):

$$L_m = \Delta|_{\mathfrak{H}_m} \qquad H_m = H|_{\mathfrak{H}_m} \qquad A_m = \sqrt{L_m} \qquad B = \sqrt{H_m} - A_m \qquad \text{etc.}$$

The resulting ψDOs have singular degenerate symbols defined on $[-1;1]\times\mathbb{R}$-phase space of the type:

$$\sigma_{A_m} = a = \left((1-x^2)\xi^2 + \frac{m^2}{(1-x^2)}\right)^{1/2} \qquad \sigma_B \sim \frac{V}{2a} + \dots .$$

Consequently, the symbol $|\xi| = \sigma_{\sqrt{-\Delta}}$ is replaced everywhere in formulae (16)–(18) by the symbol $a(x, \xi)$ of $\sqrt{L_m}$.

Thus to find the symbol of the reduced return operator W_m, we replace $|\xi|$ with a in (16), and write

$$\sigma_{W_m} = \frac{1}{a^3}\left\{\frac{1}{4}\int V^2 \mathrm{d}t - \frac{1}{8}\int\int \sin(t-s)(\partial_n V)(t)(\partial_n V)(s)\,\mathrm{d}s\,\mathrm{d}t\right\} = a^{-3}U\left(\frac{m}{a}\right). \tag{19}$$

The function $U(x)$ in the RHS of (19) will be computed below in terms of the potential V.

As its 'full S_2 precursor', the reduced operator W_m commutes with A_m (modulo A_m^{-4}), but this time both L_m and $A_m = L_m^{1/2}$ have simple (multiplicity-free) spectra. Hence the eigenvalues of W, consequently, the spectral shifts $\{\mu_{km}\}_{k=0}^{\infty}$ of $A_m + B$ are immediately read from σ_w by evaluating the function U at the level set $a(x;\xi) = k$, i.e. $\mu_{km} = k^{-2}U(m/k) + \mathrm{O}(k^{-3})$, as claimed in theorem 1.

It remains to compute the function U of (19). We expect the average reduced symbol σ_{W_m} to be a function of $\{a\}$ alone, because of the commutation relation $[A_m; W] = 0$. However, to get the specific form of U in the RHS of (19), one needs to carry out two integrations (single and double) along trajectories of the reduced Hamiltonian flow, i.e. the flow of symbol $a = \sqrt{p\xi^2 + q}$, where $p = (1 - x^2)$, $q = m^2/(1 - x^2)$ (see [8]).

We start with the single integral. Fixing the energy level $a = k$, the flow parameter t is found in terms of the x variable,

$$\mathrm{d}t = \frac{\mathrm{d}x}{\sqrt{1 - (m/k)^2 - x^2}}.$$

Then substitution of $\mathrm{d}t$ in the first integral of (19) yields (modulo lower-order errors) the Radon transform (6) of V^2, evaluated at $\{m/k\}$, $\tilde{V}^2(m/k)$.

The final step involves evaluation of the double-integral term of (19) for zonal odd V. We denote by ϕ the angle between the polar axis of the circle C and the first coordinate axis, and then compute the normal derivatives of V along C

$$(\partial_n V)(t) = \sin\phi\; V'(\cos\phi\cos t)$$

where V' denotes the ordinary derivative of V on $[0; 1]$. Similar formulae hold for $(\partial_n V)(s)$.

Substitution of $\{\partial_n V\}$ into (19), followed by integration by parts, by change of variable $x = \cos\phi\cos t = \rho\cos t$, $y = \cos\phi\cos s = \rho\cos s$, where $\rho = \rho(C)$ is the 'north pole parameter' of C, and a few more tedious calculations, yield the following expression of the double integral (see [10]):

$$\int\int \ldots \mathrm{d}s\,\mathrm{d}t = \frac{\rho^2}{\sqrt{1-\rho^2}}\int_0^1 \left(y\frac{\mathrm{d}}{\mathrm{d}y}V^2\right)(y\sqrt{1-\rho^2})\frac{\mathrm{d}y}{\sqrt{1-y^2}} = \frac{\rho^2}{1-\rho^2}\,\mathfrak{R}\left(x\frac{\mathrm{d}}{\mathrm{d}x}V^2\right).$$

Combining the two integrals we get the function U of theorem 1,

$$U(\rho) = \left(\mathfrak{R} - \frac{\rho^2}{1-\rho^2}\,\mathfrak{R}\,x\frac{\mathrm{d}}{\mathrm{d}x}\right)(V^2)$$

and thus complete the proof.

Acknowledgments

We would like to acknowledge the referee's comments and corrections. This research was partly supported by the US NSF Grant DMS-8620231.

References

[1] Colin de Verdier Y 1980 Spectre conjoint d'operatuers pseudodifferentials qui commutent *Math. Z* **171** 51–73

[2] Eskin G, Ralston J and Trubowitz E 1984 The multidimensional inverse spectral problem with a periodic potential *Comtemp. Math.* **27**

[3] Guillemin V 1978 Some spectral results for the Laplacian on the n-sphere *Adv. Math.* **27** 273–86

[4] Guillemin V 1981 Spectral theory on S_2: some open questions *Adv. Math.* **42** 283–2

[5] Guillemin V and Kazdan D 1980 Some inverse spectral results for negatively curved manifolds *Proc. Symp. Pure Appl. Math.* **36** (Providence, RI: American Mathematical Society) pp 153–80

[6] Gurarie D 1987 Asymptotic inverse spectral problem for anharmonic oscillators *Commun. Math. Phys.* **112** 491–502

[7] Gurarie D 1988 Averaging methods in spectral theory of Schrödinger operators *Maximal Principles and Eigenvalue Problems in PDE* ed P W Schaefer (*Research Notes in Mathematics 175*) (London: Pitman) pp 167–77

[8] Gurarie D 1989 Inverse spectral problem for the 2-sphere Schroedinger operators with zonal potentials *Lett. Math. Phys.* **16** 313–23

[9] Gurarie D 1989 Asymptotic inverse spectral problem for anharmonic oscillators with odd potentials *Inverse Problems* **5** 293–306

[10] Gurarie D 1990 Zonal Schrödinger operators on the n-sphere: inverse spectral problem and rigidity *Commun. Math. Phys.* in press

[11] Karasev V M and Maslov V P Asymptotic and geometric quantization *Russian Math. Surveys* **39**(6) 115–73

[12] Uribe A 1984 A symbol calculus for a class of pseudodifferential operators on S_n *J. Funct. Anal.* **59** 535–56

[13] Uribe A 1985 Band invariants and closed trajectories on S_n *Adv. Math.* **58** 3 285–99

[14] Weinstein A 1977 Asymptotics of eigenvalue clusters for the Laplacian plus a potential *Duke Math. J.* **44** 883–92