D. GURARIE

Chemotaxis

$u(x,t)$ – bacteria / density

$c(x,t)$ – chemical / . . .

1) **Model:**

logistic growth

$$\begin{cases} u_t = r\,u(1-u/N) - \nabla\cdot(\chi\,\nabla c\,u) + D_u \nabla^2 u \\ c_t = \alpha u - \beta c + D_c \nabla^2 c \end{cases}$$

chemo advection

production decay

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2) **Equilibria** : (symmetric)

(u^*, c^*)	$(0,0)$	$(N, \frac{\alpha}{\beta} N)$
J	$\begin{bmatrix} r & 0 \\ \alpha & -\beta \end{bmatrix}$	$\begin{bmatrix} -r & 0 \\ \alpha & -\beta \end{bmatrix}$
	saddle	sink

3) **Linearized problem:** diff. operator

$$L = \begin{bmatrix} D_u & -\chi N \\ 0 & D_c \end{bmatrix} \nabla^2 + J$$

For Laplace eigenvalue μ use auxiliary

$$B(\mu, J) = -\mu \begin{bmatrix} D_u & -\chi N \\ 0 & D_c \end{bmatrix} + J$$

Ask for EV data : $\begin{pmatrix} \lambda^{\pm}(\mu, J) \\ X^{\pm}(\mu, J) \end{pmatrix}$ – values – vectors

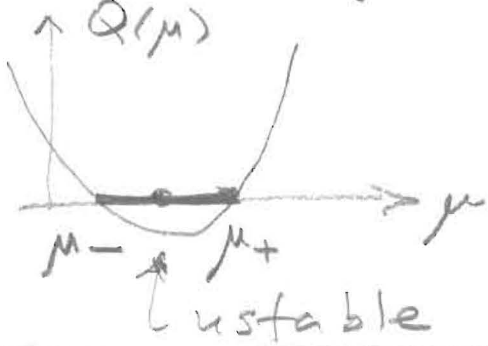
4) Analysis:

$$\mathrm{tr}\, B(\mu) = \mathrm{tr}\, J - \mu (D_u + D_c) < 0$$

$$Q(\mu) = \det B(\mu) = \underbrace{D_u D_c}_{a_0} \mu^2 + \underbrace{(r D_u + \beta D_u - \alpha \chi N)}_{a_1} \mu + \underbrace{r\beta}_{a_2}$$

Conditions for instability

$$\begin{cases} a_1 < 0 \\ a_1^2 - 4 a_0 a_2 \geq 0 \end{cases}$$



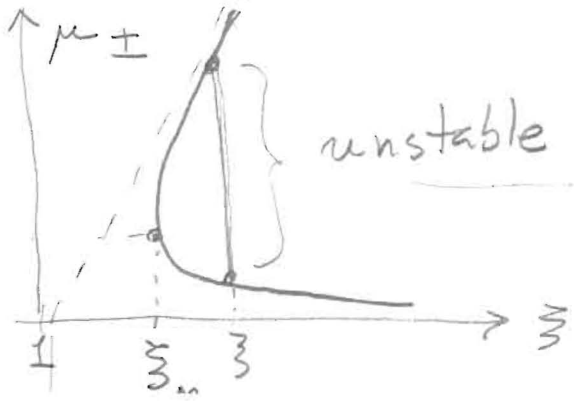
Essential parameters (d-less):

$$\boxed{\xi = \frac{\alpha \chi N}{r D_c + \beta D_u} > 1; \quad \eta = \frac{r D_c - \beta D_u}{r D_c + \beta D_u} < 1}$$

Marginal stabily = roots $Q(\mu) = 0$

$$\mu_\pm = \underbrace{\left(\frac{r D_c + \beta D_u}{2 D_u D_c}\right)}_{} \left[(\xi - 1) \pm \sqrt{(\xi-1)^2 - (1-\eta)} \right]$$

Length scales: $\left(\frac{1}{r_u^2} + \frac{1}{r_c^2}\right)$



Unlike Turing RD system

chemotaxis does not require widely different diffusivities D_u/D_c to generated patterns.

Rescaled chemotaxis:

$$\begin{cases} t \to rt \\ x \to x/\ell_u \quad \left(\ell_u = \sqrt{\frac{2D_u}{r}} = u\text{-scale}\right) \\ u \to u/N; \quad c \to c/N \end{cases} \quad \begin{cases} \chi \to \frac{\chi N}{\ell_u^2}; \\ d = 2D_c/D_u - \left(\begin{matrix}\text{d-less} \\ \text{diffusivity}\end{matrix}\right) \\ \alpha \to \alpha/r; \quad \beta \to \beta/r \end{cases}$$

$$(CT)\begin{cases} u_t = (1-u)u - \chi \nabla\cdot(u\nabla c) + \frac{1}{2}\Delta u \\ c_t = \alpha u - \beta c + d\,\Delta c \end{cases}$$

New parameters: $\xi = \frac{\alpha\chi}{d+\beta}$; $\eta = \left(\frac{d-\beta}{d+\beta}\right)^2$

Bifurcation: $\boxed{\xi = \frac{\alpha\chi}{d+\beta} \overset{?}{\geq} 1 + \sqrt{1-\eta^2}}$

Wave # range (for x):

$$\boxed{\mu_\pm = \frac{\alpha\chi}{d+\beta}\left[(\xi-1) \pm \sqrt{(\xi-1)^2 - (1-\eta)}\right]}$$

Case: $\boxed{d=1;\ \beta=1} \Rightarrow \boxed{\xi = \frac{\alpha\chi}{2} > 2} \Rightarrow \alpha\chi \geq 4$

Wave # range: $\mu_\pm \overset{\eta=0}{=} \{0, 2\xi\} = \{0, \alpha\chi\}$

For pattern bifurcation in 1D interval $[0,\ell]$ need

$$\boxed{\left(\frac{\pi k}{\ell}\right)^2 \leq \alpha\chi} \iff \boxed{\ell/k \geq \frac{\pi}{\sqrt{\alpha\chi}}}$$