DG.

Turing-Hopf instability

("symmetry-breaking bifurcation")

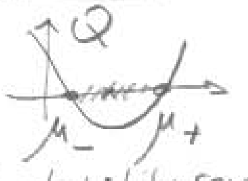
1. Linearized double-diffusive system (SRA-LRI) $\circlearrowright u \underset{\dashleftarrow}{\longrightarrow} v$ with $J=\begin{bmatrix} a & -b \\ c & -d \end{bmatrix}$

Op-tor: $L = \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} \Delta + \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}$

2. Dispersion (marginal stability) curve

$Q(\mu) = \det\left[J - \mu \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}\right]$; μ = -Laplace eigen

$\boxed{Q = \underbrace{D_1 D_2}_{a_0} \mu^2 + a_1 \mu + a_2}$; $a_2 = \det J$



instability range

Turing conditions:

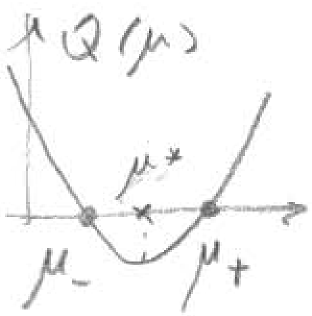
(T1) $a_1 = dD_1 - aD_2 < 0 \Rightarrow \boxed{aD_2 \geq dD_1}$

(T2) $\text{discr.}(Q) = \Delta = a_1^2 - 4a_0 a_2 > 0 \Rightarrow \boxed{(aD_2 + dD_1)^2 \geq 4 D_1 D_2 bc}$

3. Length scales for SRA(u) & LRI(v).

(u) $l_1 = \sqrt{2D_1/a}$	(T1) $l_1 < l_2$
(v) $l_2 = \sqrt{2D_2/d}$	(T2) $\frac{1}{l_1^2} + 1/l_2^2 \geq \sqrt{\frac{bc}{D_1 D_2}}$

4. Domain size for instability / pattern (?)



$$\mu = \mu_k = \left(\frac{\pi k}{l}\right)^2 \text{ should be } \mu_- < \mu < \mu_+$$

$$\mu \approx \mu^* = \frac{\mu_+ + \mu_-}{2} = \frac{aD_1 - dD_2}{2D_1 D_2} = \frac{1}{l_1^2} - \frac{1}{l_2^2}$$

$$\mu_\pm = \mu^* \pm \frac{\Delta}{2D_1 D_2}$$

$\Rightarrow$ Domain size (1D)

$$l \approx \frac{\pi k\, l_1}{\sqrt{1 - (l_1/l_2)^2}}$$

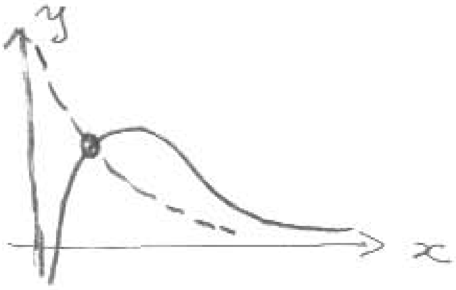
For $\mu_- < \left(\frac{\pi k}{l}\right)^2 < \mu_+$ get "pattern" with wave # $= k$

Brusselator (I. Prigogine)

$A \to X$
$2X + Y \to 3X$
$B + X \to Y + D$ } products
$X \to E$

Dynamic variables
$[X] = x(t)$
$[Y] = y(t)$

DS $\begin{cases} \dot{x} = a + x(xy - b - 1) \\ \dot{y} = (b - xy)x \end{cases}$



Equilibrium: $\begin{cases} x^* = a \\ y^* = b/a \end{cases}$

$J = \begin{bmatrix} -1+b & a^2 \\ -b & -a^2 \end{bmatrix}$ $\quad tr = b - (a^2 + 1) \quad det = a^2 > 0 \Rightarrow$ Bifurc. $b^* = 1 + a^2$ Hopf.

For Turing patterns use

$\boxed{\begin{matrix} a = 1 \\ b^* = 2 \end{matrix}}$ $J = \begin{bmatrix} b-1 & 1 \\ -b & 1 \end{bmatrix}$; Diffusive scales:
$l_1 = \sqrt{2D_1/(b-1)} < l_2 = \sqrt{2D_2}$

Take: $\begin{cases} D_1 = 1 \\ D_2 = \frac{1}{b-1} \end{cases} \Rightarrow$ $l_1 = \sqrt{2/(b-1)} < \sqrt{2D_2} = l_2$; $1 < b < 2 = b^*$
$\Delta = \sqrt{\frac{b-1}{2} + \frac{1}{D_2}\left(\frac{3}{2} - b\right)} \Rightarrow D_2 > \frac{2b-1}{b-1}$

Size bifurcation (in 1D):
russelator patterns for $a = 1$; $b = 1.9$; $D_1 = 1$; $D_2 = 10$
give limits for $1.2 = \frac{1}{\sqrt{\mu_-}} > \frac{l}{\pi k} > \frac{1}{\sqrt{\mu_+}} = .65$