1.30 Laplace transform method for continuous Euler renewal equations

$f(a) = l(a)m(a)$

$$b(t) = \int_0^t b(t-a) f(a)\, da + g(t) \quad \circledast$$

net maternity function, the probability density that an individual both survives to age a and give birth then

a)

$\tilde{b}(s) = \int_0^\infty b(t) e^{-st} dt$, $\quad \tilde{g}(s) = \int_0^\infty g(t) e^{-st} dt$.

$\int_0^t b(t-a) f(a)\, da = (b*f)(t)$ and $\mathcal{L}\{b*f(t)\} = \mathcal{L}\{b(t)\} \cdot \mathcal{L}\{f(t)\}$

$= \tilde{b}(s) \tilde{f}(s)$

Apply Laplace transform to $\circledast$:

$$\tilde{b}(s) = \tilde{b}(s) \tilde{f}(s) + \tilde{g}(s)$$

$$\Rightarrow \tilde{b}(s) = \frac{\tilde{g}(s)}{1-\tilde{f}(s)}$$

② $s \in \mathbb{C}$ ① $s \in \mathbb{R}$ decreasing strictly, so there is only one real ro[ot] for $\tilde{f}(s) = 1$

b). ~~let~~ $s = S_r + S_I i$, $S_r, S_I \in \mathbb{R}$.

Then $\tilde{f}(s) = \int_0^\infty f(a) e^{-sa} da = \int_0^\infty f(a) e^{-(S_r + iS_I)a} da$

$$= \int_0^\infty f(a) e^{-S_r a} \cos(S_I a)\, da + i \int_0^\infty f(a) e^{-S_r a} \sin(S_I a)$$

~~when $S_I = 0$, $\sin(S_I a) = 0$, and S_r solves $\int_0^\infty f(a) e^{-S_r a} da = 1$~~

Consider $\tilde{f}(S) = \int_0^\infty f(a) e^{-Sa} da$, $S \in$

$\tilde{f}(0) = \int_0^\infty f(a)\, da \geq 1$

$$\mathrm{Re}\, \tilde{f}(s) = \int_0^\infty f(a) e^{-S_r a} \cos S_I a\, da = 1$$

$\tilde{f}(s) = 1 \quad \forall s \in \mathbb{C}$

$$\Rightarrow 1 = |\tilde{f}(s)| \leq \int_0^\infty f(a) e^{-S_r a} da = \tilde{f}(S_r), \ S_r \text{ is real.}$$

when $s \to \infty$

$\tilde{f}(s) \leq \int_0^\beta e^{-sa} da$

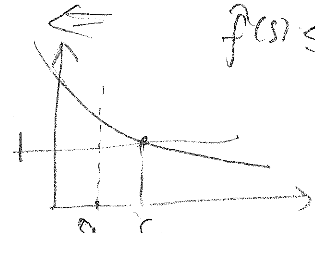
$= -\frac{1}{s} e^{-sa}$

$\to 0$

$\tilde{f}(S_1) =$

and $\tilde{f}(S_r)$ decrease monotonously

From the right, we see that $S_r < S_1$



the characteristic equation has one real root λ_1, and that all the other roots have real part less than λ_1.

c). $\tilde{b}(s) = \dfrac{\tilde{g}(s)}{1-\tilde{f}(s)} = \dfrac{\tilde{g}(s)}{\prod_{i=0}^{\infty}(s-s_i)}$, $s_i \neq s_j$, $i \neq j$

$= \sum_{k=0}^{\infty} \dfrac{A_k}{s-s_k}$, where A_k is the residual on s_k and $A_k = \dfrac{g(s_k)}{f'(s_k)}$

$$b(t) = \mathcal{L}^{-1}\{\tilde{b}(s)\} = \mathcal{L}^{-1}\left\{\frac{\tilde{g}(s)}{1-\tilde{f}(s)}\right\}$$

$$= \mathcal{L}^{-1}\left(\sum_{k=0}^{\infty} \frac{1}{s-s_k} \cdot \frac{g(s_k)}{f'(s_k)}\right)$$

$$= \sum_{k=0}^{\infty} \frac{g(s_k)}{f'(s_k)} \mathcal{L}^{-1}\left\{\frac{1}{s-s_k}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{g(s_k)}{f'(s_k)} e^{s_k t}$$