Math. 445. Potential theory of surface sources.

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1 Surface potentials.

Charges, or dipoles distributed over surface Γ with density q (for charges) and μ (dipoles) create potentials

$$u(x) = \iint_{\Gamma} \frac{q(\xi)}{|x-\xi|} d\xi; \quad v(x) = \iint_{\Gamma} \partial_{n_\xi} \left(\frac{1}{|x-\xi|} \right) \mu(\xi)\, d\xi$$

(both multiplied with factor $\frac{1}{4\pi}$). Here variable ξ varies over surface and n_x, n_ξ denote normals. We ask for limits of two potentials and their normal derivatives on surface Γ: $\lim_{x \to \pm\Gamma} \{u(x); \partial_n u(x); v(x); \partial_n v(x)\}$.

1.1 Planar boundary

We fix horizontal (boundary) variable x and vertical (normal) y, and write the corresponding single layer potential

$$u(x,y) = \iint \frac{q(\xi)}{\sqrt{|x-\xi|^2 + y^2}} d\xi \to \iint \frac{q(\xi)}{|x-\xi|} d\xi; \quad \text{as } y \to 0$$

Function $u(x,y)$ has continuous limit (even in y), while its normal derivative

$$\partial_y u = -\iint \frac{y}{\left(|x-\xi|^2 + y^2\right)^{3/2}} q(\xi)\; d\xi \to \mp\tfrac{1}{2} q(x); \quad \text{as } y \to 0 \tag{1}$$

Indeed, integral kernel in (1) is $\frac{1}{2}$ of the half-space Poisson kernel, $P(x-\xi, y)$, approaches $\delta(x-\xi)$, as $y \to 0$. Hence, the jumps across $\Gamma$for u and its normal derivative,

$$[u]|_{\Gamma} = 0; \quad [\partial_y u]|_{\Gamma} = -q(x); \tag{2}$$

Next we take the double layer (dipole) potential

$$v(x,y) = \iint \frac{y}{\left(|x-\xi|^2 + y^2\right)^{3/2}} \mu(\xi)\; d\xi \to \pm\tfrac{1}{2}\mu(x); \quad \text{as } y \to 0$$

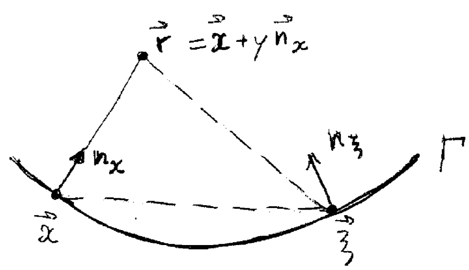
So $v(x,y)$ and its normal derivative (an odd function of y) have jumps

$$[v]|_{\Gamma} = \mu(x); \quad [\partial_y v]|_{\Gamma} = 0; \tag{3}$$

Hence an arbitrary potential with boundary sources on flat surface $\Gamma = \mathbb{R}^2$ can be uniquely expanded into the sum: $w = u + v$, of a single-layer and a double layer potentials. The former has charge density $q(x) = -[\partial_y w]|_{y=0}$, while the latter has dipole density $\mu(x) = [w]|_{y=0}$.

1.2 Arbitrary surface Γ

Next we evaluate the single and double-layers jumps on an arbitrary surface Γ. To that end we introduce normal coordinates: $\vec{r} = \vec{x} + y n_x$, on Γ, and write $K_0 = 1/\left|\vec{r} - \vec{\xi}\right| \approx 1/\sqrt{|x-\xi|^2 + y^2}$, as above.



For a single layer potential we get

$$u(r) = \iint_{\Gamma} \frac{q(\xi)}{|r-\xi|} d\xi \to \iint \frac{q(\xi)}{|x-\xi|} d\xi; \text{ as } y \to 0$$

- a continuous function across Γ, while its normal derivative

$$\begin{aligned} \partial_n u &= \iint n_x \cdot \nabla K_0 = -\iint \frac{n_x \cdot (x-\xi) + y}{\left(|x-\xi|^2 + y^2\right)^{3/2}} q(\xi) \to \\ & \mp \tfrac{1}{2} q(x) + \iint_{\Gamma} \frac{n_x \cdot (x-\xi)}{|x-\xi|^3} q(\xi) \end{aligned} \tag{4}$$

This formula represents normal derivatives of u on both sides of Γ similar to a "flat case" above, but with an additional integral operator $M(x,\xi) = \frac{n_x \cdot (x-\xi)}{|x-\xi|^3}$ applied to q.

By the same pattern the diploe density (double-layer)

$$v = \iint_\Gamma -\partial_{n_\xi}\left(\frac{1}{|r-\xi|}\right)\mu(\xi) = \iint \frac{(r-\xi)\cdot n_\xi}{|r-\xi|^3}\mu(\xi) \tag{5}$$
$$\to \pm\tfrac{1}{2}\mu(x) + \iint_\Gamma \frac{n_\xi\cdot(x-\xi)}{|x-\xi|^3}\mu(\xi)\,;\ \text{as } y\to 0$$

As above (planar case), the double layer potential near Γ is an odd function of normal variable y. Hence we get similar jump-terms (2-3) for both potentials across Γ, as in the planar case. But both functions (normal derivative of u, and limiting v) acquire additional integral operator terms,

$$\begin{aligned}\partial_n u &= \left(\mp\tfrac{1}{2}I + M\right) q\\ v &= \left(\pm\tfrac{1}{2}I + M^*\right)\mu\end{aligned}$$

Note that operators $M(x,\xi)$ and $M^*(x,\xi)$ are formal adjoints, their kernels being completely determined by the surface geometry.

2 Boundary value problem via surface sources

Solution of the Laplaces equation, $\nabla^2 u = 0$, with prescribed boundary value $u|_\Gamma = f$, (Dirichlet) could be sought in the double layer form

$$u(x) = \iint_\Gamma -\partial_{n_\xi}\left(\frac{1}{|r-\xi|}\right)\mu(\xi) \tag{6}$$

with yet undetermined density[1] μ. Passing to the boundary value of (6) we get the integral relation between dipole density μ and boundary value f

$$f = \left(\tfrac{1}{2}I + M^*\right)\mu \tag{7}$$

with kernel (5). One could solve integral equation (7) by formal inversion (series expansion)

$$\mu = \left(\tfrac{1}{2}I + M^*\right)^{-1} f = 2\left[I - 2M^* + 4M^{*2} - \ldots\right] f = 2(I+N) f \tag{8}$$

where N is another integral operator, with kernel

$$N(x,\xi) = -2M^*(x,\xi) + 4\int M^*(x,\eta)\,M^*(\eta,\xi)\,d\eta - \ldots$$

Series (8) converges (provided $-\frac{1}{2}$ is not an eigenvalue of M, M^*), and one could write an approximate solution, as

$$\begin{aligned}\mu(x) &\approx 2\left[f(x) - 2\iint_\Gamma \frac{n_\xi\cdot(x-\xi)}{|x-\xi|^3} f(\xi) + \ldots\right]\\ u(r) &\approx 2\left[\iint_\Gamma -\partial_{n_\xi}\left(\frac{1}{|r-\xi|}\right)\left\{f(\xi) - 2\iint_\Gamma \frac{n_\eta\cdot(\xi-\eta)}{|\xi-\eta|^3} f(\eta) + \ldots\right\}\right]\end{aligned}$$

[1] Indeed, if point source $K = \frac{1}{|x-\xi|}$ is replaced by the Green's function $K(x,\xi)$ in (6), we get the Poisson formula, $u = \int P(x,\xi)\,f(\xi)$, $P = -\partial_{n_\xi} K$.

The integral kernel in the r.h.s. gives an approximation of the Poisson kernel, expressed through $P_0\left(r,\xi\right)=-\partial_{n_\xi}\left(\frac{1}{|r-\xi|}\right)$ and operator kernel $M^*\left(x,\xi\right)$

$$\begin{aligned} P\left(r,\xi\right) &= -2\iint_\Gamma P_0\left(r,\,\eta\right)N\left(\eta,\xi\right)d\eta \\ &\approx -2\left[\left\{P_0\left(r\,,\xi\right)-2\iint_\Gamma P_0\left(r,\,\eta\right)M^*\left(\eta,\xi\right)d\eta+...\right\}\right] \end{aligned}$$

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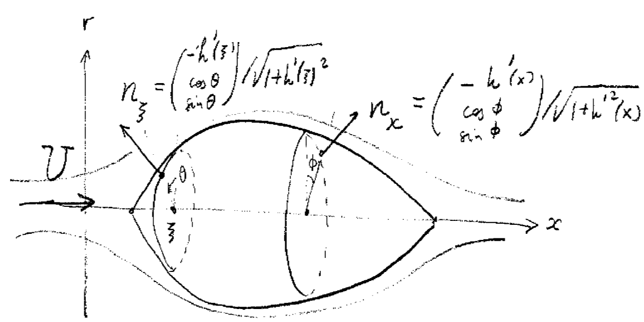
By the same pattern we solve Neumann problem: $\nabla^2 u=0$, $\partial_n u|_\Gamma=f$, via single layer potential $u=\iint_\Gamma\frac{q(\xi)}{|x-\xi|}$, where charge density q is obtained from integral equation, $f=\left(\frac{1}{2}I+M\right)q$, with kernel (4), via series expansion

$$q=2\left\{I-2M+\left(2M\right)^2-...\right\}f \tag{9}$$

In either case we get an approximate Poisson kernel $P\left(x,\xi\right)$ of the boundary value problem. We shall illustrate this method with the following example.

Example 1 *A nonlifting body of revolution Γ with axial profile $r=h\left(x\right)$ moves in a laminar flow with limiting velocity U. Its velocity potential Φ has zero normal derivative on Γ (tangential flow), we can write it as single-layer axisymmetric potential*



$$\Phi\left(x,r\right)=Ux-\iint_\Gamma\frac{q\left(\xi\right)\,dS}{4\pi\sqrt{\left(x-\xi\right)^2+\left|re^{i\phi}-h\left(\xi\right)e^{i\theta}\right|^2}} \tag{10}$$

with yet undetermined density q. Here we used polar variables in the cross-sectional plane to parameterize surface Γ: $\vec{\xi}=\left(\xi,h\left(\xi\right)e^{i\theta}\right)$, hence surface area element and normal vector,

$$dS=h\left(\xi\right)\sqrt{1+h'\left(\xi\right)^2}d\xi d\theta;\quad n_\xi=\left(h'\left(\xi\right),-e^{i\theta}\right)/\sqrt{1+h'^2}$$

The Neumann condition gives the following integral equation for q *in terms of* h

$$\partial_n \Phi|_\Gamma = -\frac{Uh'(x)}{\sqrt{1+h'(x)^2}} + \tfrac{1}{2}q(x) - \int M(x,\xi)\, q(\xi)\, d\xi = 0 \qquad (11)$$

Note that rotational symmetry allows us to integrate out the angular variable θ *in (10)., So* $M(x,\xi)$ *represents the reduced integral kernel on* Γ *in axial variables* x,ξ. *Precisely,*

$$M(x,\xi) = \int_0^{2\pi} \frac{n_{\vec{x}} \cdot \left(\vec{x}-\vec{\xi}\right)}{4\pi \left|\vec{x}-\vec{\xi}\right|^3} d\theta = \qquad (12)$$

$$h(\xi)\sqrt{\frac{1+h'^2(\xi)}{1+h'^2(x)}} \int_0^{2\pi} \frac{h(x) - h'(x)(x-\xi) - h(\xi)\cos\theta}{\left[(x-\xi)^2 + h(x)^2 + h(\xi)^2 - 2h(x)h(\xi)\cos\theta\right]^{3/2}}$$

Having computed kernel $M(x,\xi)$ *we can solve integral equation (11) via iterations (9) of function* $b(x) = \frac{Uh'(x)}{\sqrt{1+h'(x)^2}}$

$$q(x) = 2\left[b(x) + 2\int M(x,\xi)\, b(\xi)\, d\xi + ...\right]$$

Remark 2 *Integrand (12) has form* $\frac{a-b\cos\theta}{(c-d\cos\theta)^{3/2}}$ *with coefficients:*

$$\begin{aligned} a &= h(x) - h'(x)(x-\xi); \quad b = h(\xi); \\ c &= (x-\xi)^2 + h(x)^2 + h(\xi)^2; \quad d = 2h(x)h(\xi) \end{aligned}$$

Hence the integral could be expressed in terms of the complete elliptic integrals $K(m) = \int_0^{2\pi} \frac{d\theta}{\sqrt{1-m\cos^2\theta}}$, *and* $E(m) = \int_0^{2\pi} \sqrt{1-m\cos^2\theta}\, d\theta$, *of modulus* $m = \frac{2d}{d-c} = \frac{-2h(x)h(\xi)}{(x-\xi)^2+[h(x)-h(\xi)]^2}$, *as*

$$\int_0^{2\pi} ... = \frac{4\left[(ad-bc)E(m) + b(c+d)K(m)\right]}{\sqrt{c-d}\, d\,(c+d)}$$