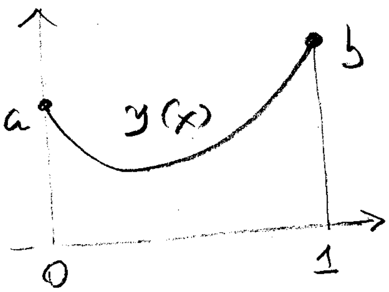
D. Gurarie

Variational problems:

Minization/optimization over f-n spaces

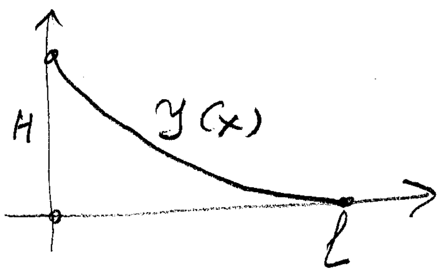
Examples (geometric/static)

1) 

Heavy chain:

$$\mathrm{Min}\left\{\underbrace{\int_0^1 y\,ds}_{P\text{ - pot. energy}} : \text{all } y(x) \text{ s.t. } \boxed{\begin{array}{l} y(0)=a \\ y(1)=b \\ \int_0^1 ds = L \end{array}}\right\}$$

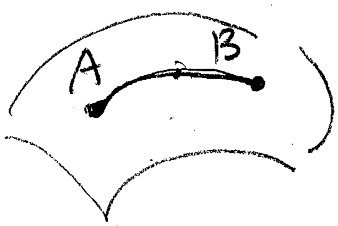
Length

2) 

$$\mathrm{Min}\left\{T=\int_0^l \frac{ds}{v} : \text{all } y(x) \;\middle|\; \begin{array}{l} y(0)=H \\ y(l)=0 \end{array}\right\}$$

time

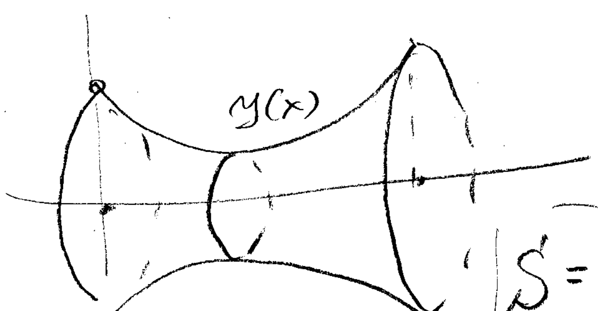
3) shortest path (geodesics) from A to B



$$\mathrm{Min}\left\{L=\int_A^B ds : \text{curves}(A,B)\right\}$$

4) Minimal surface:

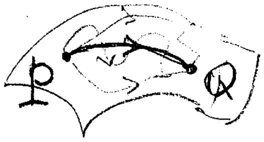
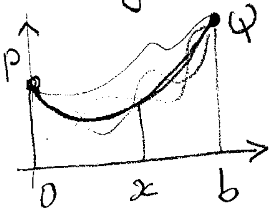
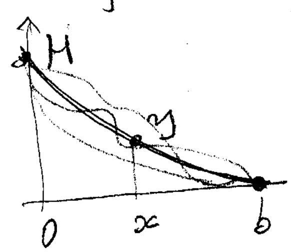
$$\mathrm{Min}\left\{\iint dS : \text{surfaces}\right\}$$



For S. revolution

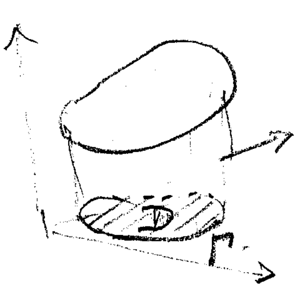
$$\boxed{S=2\pi\int y\,ds} : \quad \mathrm{Min}\{S : y(x)\}$$

Examples: geometric / static

Problem	Space	Lagrangian $\mathcal{L}$
Geodesics 	$\{X(t): 0 \le t \le 1\}$ $\underbrace{X(0)=P;\ X(1)=Q}_{B.C.}$	Length f-nal $= \int ds$ $\int_0^1 \sqrt{\sum_{ij} a_{ij}(x)\,\dot{x}_i \dot{x}_j}\,dt = \int \sqrt{\underbrace{A\dot{X}\cdot\dot{X}}_{\text{metric form}}}\,dt$ $\int \sqrt{(1+f_x^2)\dot{x}^2 + 2f_x f_y \dot{x}\dot{y} + (1+f_y^2)\dot{y}^2}$ for graph-surf. $z=f(x,y)$
Heavy chain 	$\{u(x): 0<x<b\}$ constraints: $u(0)=P;\ u(b)$ $S=\int_0^b ds = \int \sqrt{1+u'^2}\,dx = l$ fixed length	Potential energy: $P = \int u\,ds$ $\mathcal{L} = P - \lambda S = \int_0^b (u-\lambda)\sqrt{1+u'^2}\,dx$ ↑ Lagrange multip
Fastest slope 	$\{y(x): 0<x<b\}$ $y(0)=H;\ y(b)=0$	Time f-nal: $T = \int \frac{ds}{v} = \int_0^b \sqrt{\frac{1+y'^2}{2g(H-y)}}\,dx$ speed $v=\sqrt{2g(H-y)}$ from energy conserv: $\frac{mv^2}{2} + mgy = mgH$

Fermat: min time for light ray for refractive medium

$$T = \int ds / v(y,x)$$ local speed

Problem	Space	Lagrangian $\mathcal{L}$
Min surface (soap film) 	Param: $\{\mathbf{Z}(x,y): \}$ Graph: $\{u(x,y): \}$ $u\rvert_\Gamma = u_0(x,y)$ - fixed b-dary values	Area f-nal: $A = \iint_D \Vert \mathbf{Z}_x \times \mathbf{Z}_y \Vert\,dx\,dy$ $\iint \sqrt{1+\lvert\nabla u\rvert^2}\,dx\,dy$

Hamilton's "Minimal Action" principle

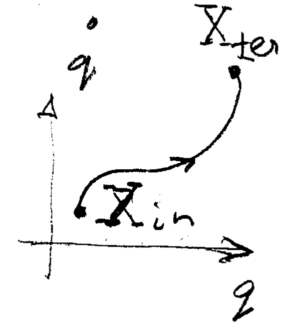
For mech. system with (generalized) coordin. $q = (q_i)$ and velocities $\dot{q} = (\dot{q}_i)$ the action functional is given by "Kinetic − Potential" energies:

$\underbrace{K(q,\dot{q})} = \frac{1}{2} \sum_{ij} a_{ij}(q)\, \dot{q}_i \dot{q}_j$ − quadratic form in $\dot{q}$

with "metric tensor" (generalized masses/inertia) $A(q) = [a_{ij}(q)]$

$\underbrace{P(q)}$ − f-n of coordinates

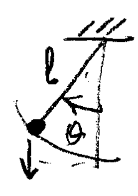
Action f-nal $\boxed{\mathcal{L} = \int_{t_1}^{t_2} (K - P)\, dt}$



The trajectories $\{(q(t), \dot{q}(t))\}$ in $(q,\dot{q})$ phase-space minimize $\mathcal{L}$

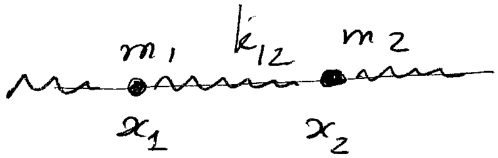
Examples:

1) Pendulum:



$K = \frac{l^2 \dot{\theta}^2}{2}$; $P = -gl\cos\theta$

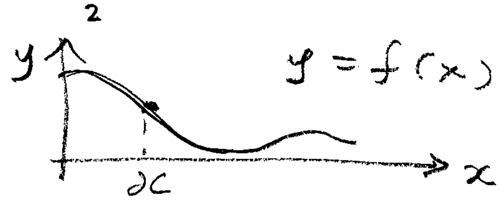
2) Coupled oscillators:



$U_i(x) = \frac{k_i x^2}{2}$

$K = \frac{1}{2} \sum m_i \dot{x}_i^2$; $P = \sum_i U_i(x_i - x_{i+1})$

3) Constrained motion on slope y = f(x):



$q = (x, y)$

$\dot{q} = (1, f_x)\dot{x}$ $\Rightarrow$ $\mathcal{L} = \underbrace{\frac{1}{2}(1 + f_x^2)\dot{x}^2}_{K} - \underbrace{g f(x)}_{P}$

M445

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Euler-Lagrange equation: variational derivative

To minimize/optimize functional $\mathcal{L}[u]$

we compute its variational derivative $\boxed{\frac{\delta \mathcal{L}}{\delta u} = 0}$

(1) Ordinary scalar $\{u(x)\}$, $\mathcal{L} = \int_a^b L(u, u' \ldots)\,dx$

$$\frac{\delta \mathcal{L}}{\delta u} = \boxed{\partial_u L - \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) = 0} \quad 2^{nd} \text{ ODE}$$

(2) Vector $\vec{u}(x) = (u, v, \ldots)$

$$\boxed{\partial_{\vec{u}} L - \frac{d}{dx}\left(\frac{\partial L}{\partial \vec{u}'}\right) = \vec{0}} \quad 2^{nd} \text{ ODS}$$

$$\begin{cases} \partial_u L - \frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) = 0 \\ \partial_v L - \frac{d}{dx}\left(\frac{\partial L}{\partial v'}\right) = 0 \end{cases} \quad \text{coupled system}$$

(3) Field $u(x,y)$

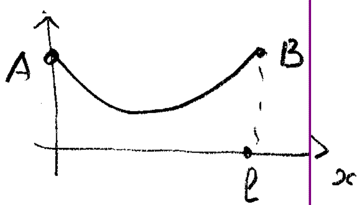
$$\boxed{\partial_u L - \nabla \cdot \frac{\partial L}{\partial (\nabla u)} = 0} \quad 2^{nd} \text{ pde in } (x,y)$$

$$\partial_u L - \partial_x\left(\frac{\partial L}{\partial u_x}\right) - \partial_y\left(\frac{\partial L}{\partial u_y}\right) = 0$$

Examples of E-L eq-ns : ODE / ODS

1) Heavy chain:

$\mathcal{L} = (y+\lambda)\sqrt{1+y'^2}$;



2 point BVP

+

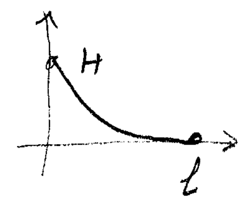
Alg. eq-n (constraint): $a = \int_0^\ell \sqrt{1+y'^2}\,dx > \ell$

2) Fast slope: $\sqrt{\dfrac{1+y'^2}{H-y}}$

3) Fermat: $\dfrac{\sqrt{1+y'^2}}{V(x,y)}$

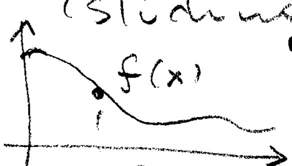
$$\boxed{\mathcal{L} = a(x,y)\sqrt{1+y'^2}}$$

$$\begin{cases} \left(\dfrac{a y'}{\sqrt{1+y'^2}}\right)' - a_y\sqrt{1+y'^2} = 0; \\ y(0) = H; \quad y(\ell) = 0; \end{cases}$$



BVP

3) Constrained motion (sliding slope)



$x(t)$

$\mathcal{L} = \dfrac{(1+f'^2)\dot{x}^2}{2} - g f(x)$

$\mathcal{L} = \dfrac{a\dot{x}^2}{2} - U(x)$ - action

$$\left.\begin{cases} \dfrac{d}{dt}\left[(1+f'^2)\dot{x}\right] = f'f''\dot{x}^2 - g f'(x) \\ x(0) = x_0; \quad \dot{x}(0) = V_0 \end{cases}\right] \text{IVP}$$

General form:

$$\dfrac{d}{dt}\left[a(x)\dot{x}\right] = \dfrac{a'(x)\dot{x}^2}{2} - U'(x)$$

4) Minimal length:

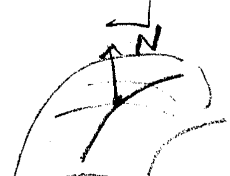
$\mathcal{L} = \sqrt{\sum a_{ij}(x)\dot{x}_i\dot{x}_j}$

Constrained motion w/o external force

$\mathcal{L} = K = \frac{1}{2}\sum_{ij} a_{ij}(x)\dot{x}_i\dot{x}_j$

$$\left.\begin{cases} \quad\cdots \\ \dfrac{d}{dt}\left(\sum_j a_{ij}\dot{x}_j\right) - \sum_{jk}\dfrac{\partial_i a_{jk}}{2}\dot{x}_j\dot{x}_k = 0 \\ \quad\cdots \end{cases}\right] \text{ODS}$$

solution = geodesics



has property : $\boxed{N_{curve} = N_{surface}}$

Multi-D problems

1) Min surface: $\mathcal{L} = \iint \sqrt{1+|\nabla u|^2}\, dx\,dy$; for graph $z = u(x,y)$

$$\frac{\delta \mathcal{L}}{\delta u} = \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0$$

$$\boxed{\partial_x \left(\frac{u_x}{\sqrt{1+|\nabla u|^2}} \right) + \partial_y \left(\frac{u_y}{\sqrt{1+|\nabla u|^2}} \right) = 0}$$

2) Vibrating string:

$$K = \frac{1}{2} \int_0^\ell \rho u_t^2 \; ; \quad P = \int_0^\ell T(ds - dx) = \int_0^\ell T\left(\sqrt{1+u'^2} - 1\right) dx$$

Action: $\mathcal{L}[u] = \int_{t_0}^{t_1} (K - P)\, dt = \int_{t_0}^{t_1} \int_0^\ell \left(\frac{\rho u_t^2}{2} - T\sqrt{1+u'^2} \right) dx\, dt$

$$\frac{\delta \mathcal{L}}{\delta u} = \partial_t \left(\frac{\partial L}{\partial u_t} \right) + \partial_x \left(\frac{\partial L}{\partial u_x} \right) - \partial_u L = 0$$

$$\boxed{\rho u_{tt} - \partial_x \left(\frac{T u_x}{\sqrt{1+u_x^2}} \right) = 0}$$

3) Membrane:

$$\mathcal{L} = \int_{t_0}^{t_1} \iint_D \left[\frac{\rho u_t^2}{2} - T\sqrt{1+|\nabla u|^2} \right]$$

$$\boxed{\rho u_{tt} - \nabla \cdot \left(\frac{T \nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0}$$

Hamiltonian formalism & conserved integrals

1) Mechanics: $L = \frac{m\dot{x}^2}{2} - U(x)$; $\left\{\begin{array}{l} \boxed{p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}} \\ h(x,p) = \frac{p^2}{2m} + U(x) \end{array}\right.$

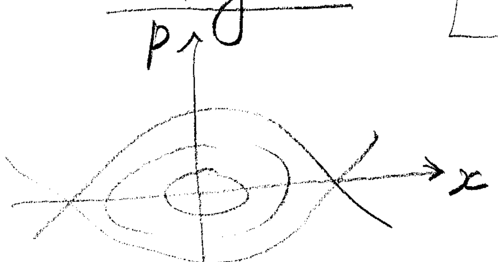
$\downarrow$ Hamiltonian

E-L / Newton: $\boxed{\frac{d}{dt}(m\dot{x}) + U'(x) = 0}$ (1)

$\Longleftrightarrow$ (Ham) $\left\{\begin{array}{l} \dot{x} = \frac{\partial h}{\partial p} = p/m \\ \dot{p} = -\frac{\partial h}{\partial x} = -\left[U'(x) + \frac{p^2}{2}\partial_x\left(\frac{1}{m}\right)\right] \end{array}\right.$ (2)

Either (1) or (2) have conserved integral $\boxed{h(x,p) = E - \text{const}}$

reduction



For (1) $\Rightarrow$ $\boxed{\frac{m\dot{x}^2}{2} + U(x) = E}$ $\Rightarrow$ $\boxed{\dot{x} = \pm\sqrt{\frac{2}{m}(E - U(x))}}$

(energy conserv.)

2) Lagrangian: $\boxed{\mathcal{L} = a(y)\sqrt{1+y'^2}}$

(chain; fastest slope; Fermat light ray)

has conjugate momentum: $p = \partial_{y'}\mathcal{L} = \dfrac{ay'}{\sqrt{1+y'^2}}$

and Hamiltonian $\boxed{h = \underbrace{y'p - \mathcal{L}} = -\dfrac{a}{\sqrt{1+y'^2}} = E}$

So 2^{nd} order E-L:

$$(3) \quad \boxed{\partial_x\left(\frac{ay'}{\sqrt{1+y'^2}}\right) - a'(y)\sqrt{1+y'^2} = 0}$$

is reduced to a 1^{st} order separable equation via conserved Hamiltonian $h(y,y')=E$

$$(4) \quad \boxed{y' = \pm\sqrt{(a/E)^2 - 1}}$$

Problems:

1) Derive conserved integral for more general Lagrangian $\mathcal{L} = a(y)\sqrt{1+y'^2} + b(y)$ with arbitrary coefficients a(y), b(y).

2) Apply conserved integral (4) to heavy chain: $a = (y-\lambda)$ and fastest slope: $a = 1/\sqrt{H-y}$. Solve the resulting ODEs and find the shape of heavy chain & fastest slope.