Shock propagation via Rankine-Hugoniot

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A quasilinear conservation law: $u_t + [f(u)]_x = 0$ with initial distribution $u(x,0) = h(x)$, develops shocks at the intersection of characteristics: $x = \xi + f'(h(\xi))\,t$, or more precisely at the caustics (folds) of the characteristic envelop (fig. 1). Here ξ - characteristic variable, $f'(u)$ - speed of propagation.

The shock is initiated at the first blow-up time, $t_{bl}(\xi) = -\frac{1}{(f'\circ h)'(\xi)}$, point ξ_0 that minimizes t_{bl} (fig.), and its speed of propagation obeys the Rankine-Hugoniot relation

$$\frac{dx}{dt} = \frac{f(h)_+ - f(h)_-}{h_+ - h_-} \tag{1}$$

Here h_+ and h_- represent the initial profile h at the forward (slow) and backward (fast) characteristics (ξ,η) overlapping at a given point (x,t) on the shock path[1] (fig. 2)So one has

$$x = \xi + t\,(f'\circ h)(\xi)\,; \quad x = \eta + t\,(f'\circ h)(\eta)$$

hence a pair of (h - dependent) characteristic coordinates in the vicinity of shock

$$\begin{aligned} x(\xi,\eta) &= \eta + \frac{(\xi-\eta)(f'\circ h)(\eta)}{(f'\circ h)(\eta) - (f'\circ h)(\xi)} \\ t(\xi,\eta) &= \frac{(\xi-\eta)}{(f'\circ h)(\eta) - (f'\circ h)(\xi)} \end{aligned} \tag{2}$$

Shock evolution could be described either in the natural (physical) variables, $x = x(t)$, or the characteristic ones: $\xi = \xi(\eta)$. The r.h.s. of Rankine-Hugoniot is expressed through ξ,η

$$\frac{dx}{dt} = \frac{x_\xi \frac{d\xi}{d\eta} + x_\eta}{t_\xi \frac{d\xi}{d\eta} + t_\eta} = V(\xi,\eta)$$

whence follows the evolution of shock in (ξ,η)

$$\frac{d\xi}{d\eta} = \frac{t_\eta V - x_\eta}{-x_\xi V + t_\xi} = F(\xi,\eta) \tag{3}$$

[1] For piecewise linear or constant profiles h (Riemann problem) one could solve (1) directly and produce a linear path of the shock.

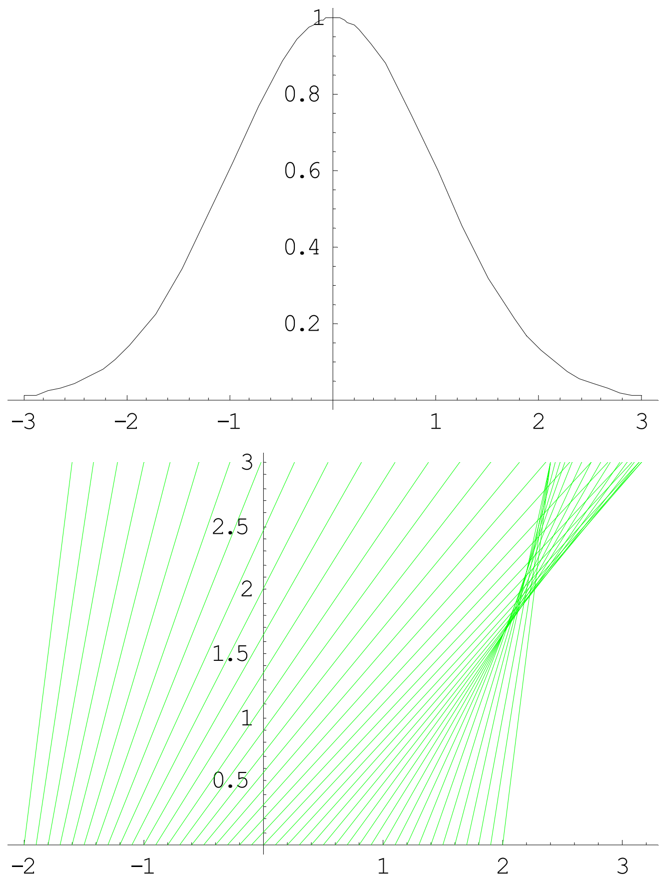

Figure 1: Characteristics of the inviscid Burgers equation: $u_t + u\, u_x = 0$, for Gaussian initial profile.

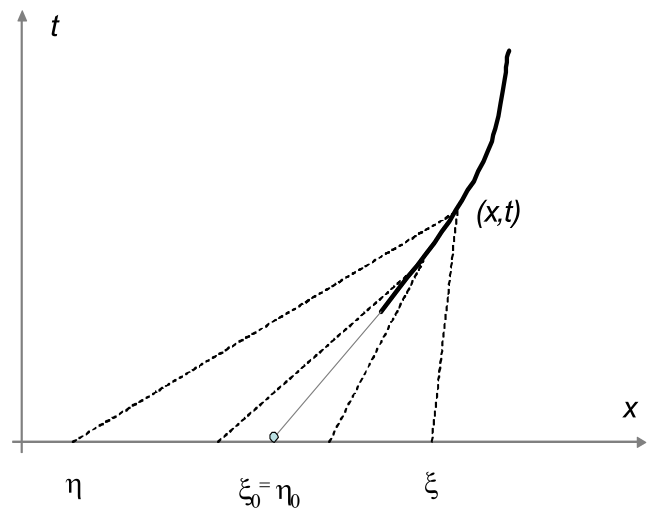

Figure 2: Characteristic coordinates ξ, η along a hypothetical shock path.

Function F in the r.h.s. has somewhat cumbersome yet explicit form (computed by Mathematica). Namely, it factors as $F = F_1 F_2$, where

$$F_1 = \frac{[(f \circ h)(\xi) - (f \circ h)(\eta) - (f' \circ h)(\xi)(h(\xi) - h(\eta))]}{[(f \circ h)(\xi) - (f \circ h)(\eta) - (f' \circ h)(\eta)(h(\xi) - h(\eta))]} \tag{4}$$

$$F_2 = \frac{\left[(f' \circ h)(\xi) - (f' \circ h)(\eta) - (f' \circ h)'(\xi)(h(\xi) - h(\eta))\right]}{\left[(f' \circ h)(\xi) - (f' \circ h)(\eta) - (f' \circ h)'(\eta)(h(\xi) - h(\eta))\right]}$$

Equation (3) is to be solved (numerically) for $\eta < \eta_0$, subject to the initial condition $\xi(\eta_0) = \eta_0$, as two coordinates coincide at the start. Clearly, $\eta = \xi$ makes (4), hence function F singular, geometrically the characteristic coordinate system becomes singular at $\xi_0 = \eta_0$.

So computationally one needs to take initially variable $\eta_0 - \varepsilon$, slightly below the shock value, while initial values $\xi(\eta_0 - \varepsilon) = \eta_0 + \varepsilon$. Two characteristics $\xi$and η merge at the start and we take them apart by ε, to initiate DE (3) at a nonsingular value.

We demonstrate our method by two examples.

Example 1 *Inviscid Burgers* $u_t + u\, u_x = 0$, *has* $f = \frac{u^2}{2}$, *hence*

$$F = \frac{h(\xi) - h(\eta) - h'(\eta)(\xi - \eta)}{h(\xi) - h(\eta) - h'(\xi)(\xi - \eta)} \approx 1 - \frac{h'''(\eta)}{3h''(\eta)}(\xi - \eta)$$

Its shock evolution for the Gaussian initial profile $h = e^{-x^2}$ *constructed by solving (3) is shown in plot 3.*

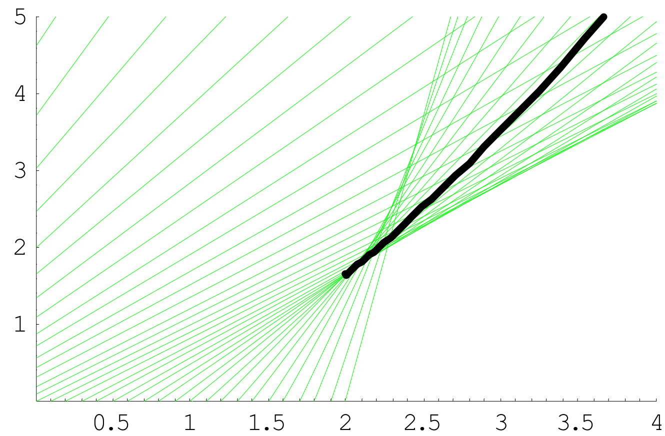

Figure 3: Shock propagation for inviscid Burgers and Gaussian initial profile.

Example 2 *Traffic flow* $\rho_t + [\rho\,(1-\rho)]_x = 0$*, has the same function* F*, but different shock initiation and evolution (fig. 4).*

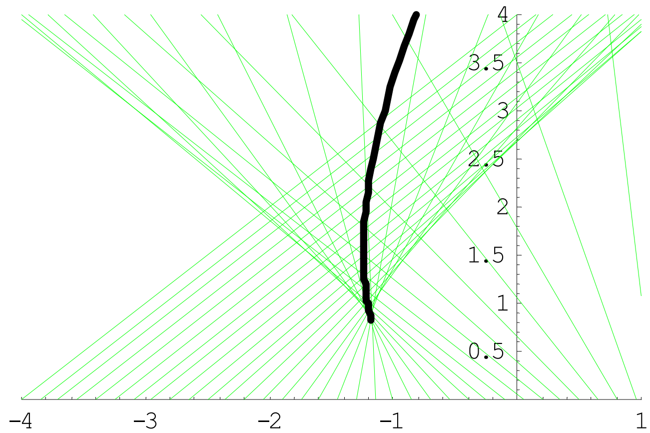

Figure 4: Shock propagation for the traffic flow and Gaussian initial profile.