

Derivation of the basic pde models of Mathematical Physics

Math. 445. Supplement

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The basic pde models appear either as balance equations (*conservation laws*) or as variational principles (minima/maxima of certain functionals). The latter includes large classes of geometric problems (minimal length, minimal area etc.), the problems in statics (minimal potential energy), and the foremost - the basic *dynamic laws* of motion (Newtonian mechanics in the Hamiltonian formulation). Here we shall introduce and review a few such examples.

Conservation (balance) laws

Traffic flow

One can approximately represent large (unidirectional) traffic flow along a stretch of highway by two continuous functions:

- $\rho = \rho(x, t)$ - cars density per unit length
- $c = c(x, t)$ - velocity density

Both are functions of space x and time t . We take a stretch of highway between two points $x_1 < x_2$ and consider the total number of cars over the range $[x_1, x_2]$,

$$N = \int_{x_1}^{x_2} \rho(x, t) dx$$

It changes in time as cars come in and get put of $[x_1, x_2]$ through the end-point. So the rate of change of $N(t)$ is determined by the flux density $c\rho$ at $\{x_1, x_2\}$

$$\begin{aligned} \frac{dN}{dt} &= \int_{x_1}^{x_2} \partial_t \rho(x, t) dx = c\rho \Big|_{x_1}^{x_2} \\ &= \underbrace{-c(x_2, t) \rho(x_2, t)}_{\text{outgoing}} + \underbrace{c(x_1, t) \rho(x_1, t)}_{\text{incoming}} \end{aligned} \quad (1)$$

We write the boundary “flux terms” as integral over $[x_1, x_2]$ and the (1) - the conservation (balance) of “cars” as the integral relation

$$\int_{x_1}^{x_2} \partial_t \rho(x, t) dx = - \int_{x_1}^{x_2} \partial_x (c\rho) dx \quad (2)$$

From the integral form one could easily pass to a differential form: since (2) holds for all $\{x_1, x_2\}$ the integrands

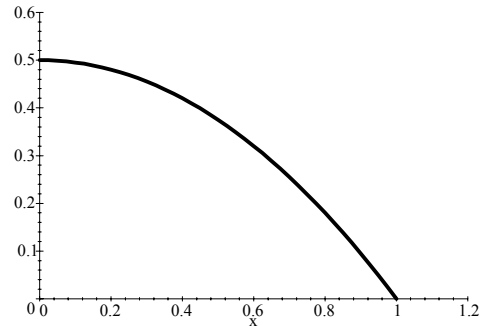


Figure 1:

are equal. Thus we get the conservation law in the form of first order pde for unknown function $\rho(x, t)$

$$\rho_t + (c\rho)_x = 0 \quad (3)$$

This form is typical of many other physical models, like fluid flows discussed below. Equation (3) however is not a well posed problem as it contains yet another undetermined quantity c (one equation with 2 unknown functions!). So one needs another equation relating ρ and c . Several choices are possible.

Constant speed c : Such (highly idealized) traffic flow is reduced to a simple linear *1D wave equation*


$$\rho_t + c\rho_x = 0 \quad (4)$$

Any solution of (4) on the real line \mathbb{R} is d'Alembert traveling wave $\rho = \rho_0(x - ct)$, so the initial car distribution density $\rho_0(x)$ will propagate to the right at a constant (legal) speed c without changing shape.

Variable speed: Of course, the real life cars would never go at the same speed. They will slow down before turns, highway exits etc., stop before the light (or speed up to cross it). The average car speed would thus be determined by many factors. Their cumulative effect however could be approximately related to the car density itself. The traffic tends to slow down as the car density increases and speed up at low densities (the traffic regulations require to maintain a safety distance of one car body per each 10 mi/hr of speed). That puts some natural constraints and on ρ, c . One of them $0 < \rho < 1$ - the car don't pile up. Also one can assume c to be a function of density $c = c(\rho)$, whose typical profile should look like the plot below (fig. 1) Now (3) turns into a nonlinear first order pde for ρ solved by the *method of characteristics*.

Heat-diffusion models

The propagation of heat or diffusing particles in a medium is also governed by certain conservation principles. We denote by $q(\mathbf{x}, t)$ -the density of heat (internal energy) per unit volume and by $\theta(\mathbf{x}, t)$ - the temperature distribution at space-time point \mathbf{x}, t . Here we allow \mathbf{x} to vary in space of any dimension (e.g. 1D rod, 2D plate or 3D solid). The basic principles here involve

- A suitable relation between q and θ . Typically one assumes $q = q(\theta)$ - function of temperature. A special case would be linear function $q = \alpha(\theta - \theta_0)$, where coefficient α measures the heat capacity (*specific heat*) of the medium. 
- The Fourier-Newton law of cooling.

The Fourier law states that the heat flows from the hot to cold parts at a rate proportional to the temperature-gradient $\nabla\theta = (\partial_1\theta, \partial_2\theta, \dots)$. So we introduce the heat-flux field $\Phi = \beta\nabla\theta = (\phi_1, \phi_2, \dots)$, whose coefficient (tensor) β measures the *heat conductivity* of the medium. Scalar β corresponds to the isotropic medium, whereas anisotropy results a matrix (tensor) array of coefficients $\beta = (\beta_{ij})$, so the i -th component of flux $\phi_i = \sum_{j=1}^3 \beta_{ij} \partial_j \theta$. More generally, the flux field could depend nonlinearly on θ or $\nabla\theta$.

Next we pick a finite (small) volume V of the body bounded by surface Σ and consider the total amount of heat inside V

$$Q = \iiint_V q(\mathbf{x}, t) d^3\mathbf{x}$$

and balance its rate of change by the heat-flux across Σ , according to the Fourier law

$$\frac{dQ}{dt} = \iiint_V \alpha\theta_t = - \oint_{\Sigma} \Phi \cdot N dS + F = - \oint_{\Sigma} \beta \partial_n \theta dS + F \quad (5)$$

where F represent external sources of heat. Here N denote the external normal on Σ , $\partial_n \theta = \nabla\theta \cdot N$ - normal derivative and dS the surface area element. The form of conservation law (5) is similar to (1) where Q plays the role of N and surface Σ - the role of two boundary points¹ $\{x_1; x_2\}$. As in (2) we transform surface integral in (5) to the volume integral by the divergence Theorem

$$- \oint_{\Sigma} \Phi \cdot N dS = \iiint_V \nabla \cdot \Phi$$

¹Indeed, in the 1D case V becomes an interval $[x_1; x_2]$, boundary Σ consists of two points $\{x_1; x_2\}$, flux $\Phi = \beta u_x$ and (5) becomes

$$\int_{x_1}^{x_2} \alpha u_t dx = - \beta u_x|_{x_1}^{x_2}$$

The resulting integral form of the conservation law

$$\iiint_V \alpha\theta_t = \iiint_V (\nabla \cdot \Phi + F)$$

(assuming external sources are continuously distributed over V) we pass to the differential *heat equation*

$$\alpha\theta_t - \nabla \cdot \beta\nabla\theta = F \quad (6)$$

This equation is second order in \mathbf{x} and first order in t , such beasts are called *parabolic*. It could be linear if α, β are independent of θ , but more complicated nonlinear models arise if any of 2 coefficients α, β depends on θ . In the simplest linear case with constant α, β (homogeneous, isotropic medium) (6) is reduced to

$$\theta_t - \frac{\beta}{\alpha} \Delta\theta = F$$

with Laplacian $\Delta = \sum \partial_i^2$. It can be solved exactly by the expansion methods (Fourier, Laplace etc.) or the Green's function techniques.

The diffusion model can be derived in a similar manner². Here θ denotes the density of the diffusing substance (tracer), and its flux is assumed to be proportional to its gradient $\Phi = \beta\nabla\theta$ (the *Fick law* of diffusion). In both cases (heat or diffusion) equation (6) could be augmented by an *advective* term $U \cdot \nabla\theta$, where U represents an external velocity field (like in a fluid flow), that transports the diffusing density or temperature along the flow. The combined advection-diffusion becomes

$$\alpha\theta_t - U \cdot \nabla\theta - \nabla \cdot \beta\nabla\theta = F \quad (7)$$

Initial-boundary conditions

The differential equation (6) or (7) is typically consider in space \mathbb{R}^n , or some region $D \subset \mathbb{R}^n$, bounded by surface Γ and must be supplemented by the *initial data* (initial temperature or density distribution),

$$\theta|_{t=0} = \theta_0(\mathbf{x})$$

and suitable *boundary constraints*, e.g.

$$\begin{aligned} \theta|_{\Gamma} &= \dots \text{ prescribed temp. (density) distribution on } \Gamma \\ \partial_n \theta|_{\Gamma} &= \dots \text{ prescribed temp. (density) flux across } \Gamma \end{aligned}$$

For instance, $\theta|_{\Gamma} = 0$ requires to maintain zero (or fixed) temperature on the boundary (a "cooler" or the "heat bath"), while $\partial_n \theta|_{\Gamma} = 0$ means zero flux across Γ , insulated body.

Fluid dynamics

The Eulerian description of a fluid/gas state is given by the basic fields of mass density $\rho(x, t)$, velocity $u(x, t)$ and (if needed) other quantities, like internal energy, temperature etc. The basic principles that determine the evolution of the fluid system are conservation of mass, momentum, energy.

²although more accurate procedure usually involves stochastic (random) modeling of the diffusing particles

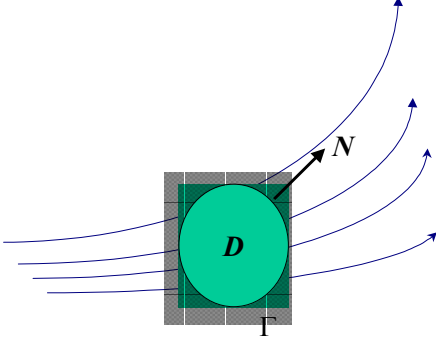


Figure 2: A flow passed a small volume D , bounded by surface Γ

Precisely, we balance the change of mass, momentum and energy densities: $\rho, \rho u_i, e = \rho \left(\frac{\vec{u}^2}{2} + \mathcal{E} \right)$ -sum of kinetic and internal (potential) energies, in a small volume D with their fluxes $\rho \vec{u}, \rho u_i \vec{u}, \rho e$ across boundary Γ (fig. ??)

$$\begin{aligned} \partial_t \iiint_D \rho + \oint_{\Gamma} \rho \vec{u} \cdot N dS &= 0 \\ \partial_t \iiint_D \rho u_i + \oint_{\Gamma} \rho u_i \vec{u} \cdot N dS &= - \oint_{\Gamma} p N_i dS + \iiint_D \rho F \\ \partial_t \iiint_D e + \oint_{\Gamma} p \vec{u} \cdot N dS &= \iiint_D \rho F \cdot \vec{u} \end{aligned}$$

where p is the (surface) pressure force, F -body forces (gravity, Coriolis, viscous dissipation etc.). Those yield via Gauss's theorem

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \vec{u}) &= 0 \text{ - continuity} \quad (8) \\ \partial_t (\rho u_i) + \nabla \cdot (\rho u_i \vec{u}) &= -\partial_i p + \rho F \\ \partial_t e + \nabla \cdot (e \vec{u}) &= -\nabla \cdot (p \vec{u}) + \rho F \cdot \vec{u} \text{ -energy} \end{aligned}$$

The second equation is transformed into its standard form

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{\nabla p}{\rho} + F \quad (9)$$

while the third one yields the internal energy evolution

$$D_t \mathcal{E} = -\frac{p}{\rho} \nabla \cdot \vec{u} \quad (10)$$

Here $D_t = \partial_t + \vec{u} \cdot \nabla$ is the complete derivative along the flow.

The entire system (8)-(9)-(10) must be supplemented by the (thermodynamic) equation of state $p = f(\rho, \mathcal{E})$. In the simplest case of isentropic gas $p = f(\rho) = C\rho^\nu$ - depends only on ρ , so the first two equations (8)-(9) form a closed system

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0 \\ \partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{f'(\rho) \nabla \rho}{\rho} + \dots \end{cases}$$

and the internal energy is driven by the density and volume change (10).

Shallow water equation

Another example is the shallow water equations. Here the role of density is played by the height function: $h(x, t)$, the pressure amounts to gravitational weight of the column: $p = \frac{gh^2}{2}$, and the flow \vec{u} is effectively horizontal (2D). So we get

$$\begin{aligned} h_t + \nabla \cdot (h \vec{u}) &= 0 \text{ - volume conservation} \\ \partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} &= -\frac{g \nabla h}{h} \text{ - momentum conservation} \end{aligned}$$

Remark 1 The general form of the conservation/balance law is often stated in terms of the density $q(x, t)$, (e.g. traffic-density, mass-density etc.) and its flux (or current), $J = J(x, t, q)$

$$q_t + \nabla \cdot J = 0$$

In the above examples: $q = \rho; J = c\rho$ (traffic); $q = \alpha(\theta)$ -heat/tracer density, $J = -\beta \nabla \theta$ (heat-diffusion problem). More generally, the heat-diffusion flux could combine the transport and diffusive terms: $J = U\theta - \beta \nabla \theta$ (7). The hydrodynamic system (8) in the absence of external/dissipation forces is also a conservation law, whose state variable $q = (\rho, m, e)$ with $m = \rho u$ - momentum field. In 1D case the current density $J = \left(m, \frac{m^2}{\rho} + p, \frac{m}{\rho}(e+p) \right)$, and we get

$$\partial_t \begin{pmatrix} \rho \\ m \\ e \end{pmatrix} + \partial_x \begin{pmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{m}{\rho}(e+p) \end{pmatrix} = \vec{0}$$

where $p = f\left(\rho, e - \frac{m^2}{2\rho}\right)$ - a given (thermodynamic) function of 3 variables. In the 2D case we have two components of the momentum: $m = \rho u, n = \rho v$, and the conservation law becomes

$$\partial_t \begin{pmatrix} \rho \\ m \\ n \\ e \end{pmatrix} + \partial_x \begin{pmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{mn}{\rho} \\ \frac{m}{\rho}(e+p) \end{pmatrix} + \partial_y \begin{pmatrix} n \\ \frac{mn}{\rho} \\ \frac{n^2}{\rho} + p \\ \frac{n}{\rho}(e+p) \end{pmatrix} = \vec{0}$$

Variational principles

Variational principles play important role in many problems of geometry, mechanics (finite-degree of freedom and continuum, classical and quantum) and physics in general. We shall give a few examples.

Geometric problems

Minimal length curves: One of classical geometric problems requires to find a path of *minimal length* on a given (graph) surface

$$\Sigma : z = f(x, y)$$

between two points $A = (x_0, y_0, z_0)$ and $B = (x_1, y_1, z_1)$. The path of minimal length are called *geodesics*.

To set up the problem we take all parameterized path between A and B : $\mathbf{r}(t) = (x(t), y(t), z(t)), 0 \leq t \leq 1$ in Σ and minimize the *length functional*

$$\mathcal{L}[\mathbf{r}] = \int ds = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2 + (f_x \dot{x} + f_y \dot{y})^2} dt \quad (11)$$

subject to boundary constraints: $\mathbf{r}(0) = A; \mathbf{r}(1) = B$. Problem (11) represent a typical variational problem, its integrand called the *Lagrangian density* $L(\mathbf{r}, \dot{\mathbf{r}}, t)$, so one asks to minimize

$$\mathcal{L}[\mathbf{r}] = \int_{t_0}^{t_1} L(\mathbf{r}, \dot{\mathbf{r}}, t) dt \quad (12)$$

over a suitable class of curves $\{\mathbf{r}(t)\}$ (scalar or vector functions).

Solution of (12) is given by the so called *Euler-Lagrange* equation³, written in terms of variational derivative

$$\frac{\delta L}{\delta \mathbf{r}} = \frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} \right) = 0 \quad (13)$$

- a second order ODE or ODS (ordinary differential system).

Specific example of the paraboloidal surface $z = \frac{1}{2}(x^2 + y^2)$ gives Lagrangian

$$L = \sqrt{(1+x^2)\dot{x}^2 + (1+y^2)\dot{y}^2 + 2xy\dot{x}\dot{y}}$$

and the EL-system

$$\frac{\delta L}{\delta \mathbf{r}} = \begin{cases} \frac{\delta L}{\delta x} = \frac{\dot{x}(x\dot{x}+y\dot{y})}{L} - \frac{d}{dt} \left(\frac{\dot{x}(1+x^2)+xy\dot{x}}{L} \right) = 0 \\ \frac{\delta L}{\delta y} = \frac{\dot{y}(x\dot{x}+y\dot{y})}{L} - \frac{d}{dt} \left(\frac{\dot{y}(1+y^2)+xy\dot{y}}{L} \right) = 0 \end{cases}$$

Minimal area surface

We consider a graph-surface $z = u(x, y)$ over a planar region D bounded by curve Γ , prescribe certain values to function u on the boundary $u|_{\Gamma} = f$ and ask for surface (function) u that would minimize the surface area, functional

$$\mathcal{A}[u] = \iint_D dS = \iint_D \sqrt{1 + (\nabla u)^2} dx dy \quad (14)$$

A physical model would be a soap film attached to a carcass (boundary condition).

This time the ‘undetermined parameter’ is a two-variable function $u(x, y)$, the functional being given by the Lagrangian

$$L(u, \nabla u, \dots) = L(u, u_x, u_y, \dots)$$

³Derivative $\frac{\delta}{\delta}$ plays the role of gradient ∇ in the multivariable calculus and condition (13) is the variational analogue of $\nabla f(\mathbf{x}) = \mathbf{0}$ for critical (max/min/...) points on n-variable function $f(\mathbf{x})$.

The EL-equation takes the form of a 2-nd order *pde*

$$\frac{\delta L}{\delta u} = \frac{\partial L}{\partial u} - \nabla \cdot \frac{\partial L}{\partial (\nabla u)} = \frac{\partial L}{\partial u} - \partial_x \left(\frac{\partial L}{\partial u_x} \right) - \partial_y \left(\frac{\partial L}{\partial u_y} \right) = 0$$

-typically highly nonlinear⁴. In particular, the minimal area equation for (14) becomes

$$\begin{aligned} \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + (\nabla u)^2}} \right) &= \partial_x \left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}} \right) \\ + \partial_y \left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}} \right) &= 0 \end{aligned} \quad (15)$$

For small ∇u (gently slopping surfaces) the area functional is approximated by the quadratic expression

$$\sqrt{1 + (\nabla u)^2} \approx 1 + \frac{1}{2} (\nabla u)^2$$

hence E-L (15) is simplified to a linear Laplace’s equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

Hamilton’s principle of classical mechanics: oscillations and waves

So far we produced differential equation models based on simple balance (conservation) laws. A large number of physical systems however, obeys the dynamic Newton’s law of motion that involve the balance of forces and higher derivatives (acceleration). The dynamic equations for such systems result from two forms of energy: the kinetic one K and potential P . We assume that the state of system is described by a set of ‘generalized coordinates’ $\{u_i\}$ and the corresponding velocities $\{\dot{u}_i\}$. Typically the kinetic energy depends on velocities $K = K(\dot{u}, \dots)$ (it could also involve $\{u_i\}$), while the potential one depends on coordinates only $P = P(u)$. The generalized momenta are derivatives

$$p_i = \frac{\partial K}{\partial \dot{u}_i}$$

while generalized (potential) forces $f_i = -\frac{\partial P}{\partial u_i}$, and the Newton’s law reads

$$\frac{dp_i}{dt} = f_i \text{ or } \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{u}_i} \right) - \frac{\partial P}{\partial u_i} = \dots \quad (16)$$

The r.h.s. is either 0 (pure potential case) or contains other non-potential forces $\{F_i\}$, like external driving, friction, etc. For systems of finite number n of degree of freedom (16) becomes a second order ordinary differential system in n variables⁵. We shall first illustrate (16) with a simple mechanical model of oscillator.

⁴The only case when E-L gives a linear equation/system is when the Lagrangian is quadratic in u and its derivatives.

⁵It represents a special case of so called *Euler-Lagrange* equation for the action functional (Lagrangian) $L = K - P$, and could be derived from the variational Hamilton’s principle of minimal action.

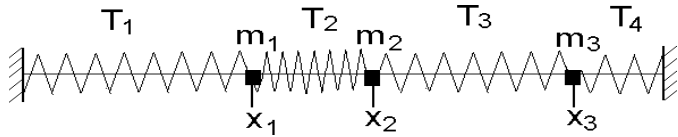


Figure 3:

Oscillator

Oscillator is a mass-spring system that obeys the Hook's law (fig. 3)

The i -th coordinate here represents the displacement of the i -th mass. The Newton's equations of motion are

$$\begin{cases} \dots \\ m_i \ddot{u}_i = V'_{i-1,i} + V'_{i,i+1} + \dots \\ \dots \end{cases} \quad (17)$$

where $V_{i,i+1} = V(u_{i+1} - u_i)$ denotes the interaction potential between the i -th and $i + 1$ nodes. In the simplest (Hook's) case potentials are quadratic functions of relative displacements $V_{i,i+1} = \frac{k}{2}(u_{i+1} - u_i)^2$ with spring constant k . Hence forces (gradients of potentials) depend linearly on $u_{i+1} - u_i$. In addition to potential forces (derivatives of $V_{i,i+1}$) the r.h.s. could include other (external) forces. The kinetic and potential energies here are

$$\begin{aligned} K &= \frac{1}{2} \sum m_i \dot{u}_i^2 \\ P(u_1, \dots, u_n) &= \sum_i V_i(u_{i+1} - u_i) \end{aligned} \quad (18)$$

In the linear case $P = \frac{1}{2} \sum_i \kappa_i (u_{i+1} - u_i)^2$, with spring constants $\{\kappa_i\}$, so both K and P are quadratic in $\{u; \dot{u}\}$. One readily verifies the "Lagrangian" form (16) of equations (17).

Vibrating string.

The stretched string is characterized by its tension T - force per cross-sectional area. As the latter is negligibly small, we can think of T as force directed along the string at each point x . We call the length of the string ℓ and its density (at point x) $\rho(x)$. Any vertical displacement $u(x)$ would increase the tension. So the latter could be viewed as a potential force with potential energy density $P = T ds$ or $T(ds - dx)$, i.e. proportional to the change in the arc-length $ds = \sqrt{u_x^2 + 1} dx$ due to the displacement. Introducing state variables $\{u(x); u_t(x)\}$ (positions and velocities) we write the kinetic and potential energies of the string as

$$K = \frac{1}{2} \int_0^\ell \rho u_t^2 dx \quad (19)$$

$$P = \int_0^\ell T (\sqrt{u_x^2 + 1} - 1) dx$$

There is an analogy between the continuous system (19) and the oscillator (18). Indeed, the former could be approximated by certain discrete oscillator models where horizontal displacements are replaced by the vertical ones along vertical rods.

We shall give two derivations of the equation of motion, based on the Newtonian balance of forces and the Euler-Lagrange formalism.

Newtonian forces.

We take a small portion of the string $[x_0; x_1]$ inside $[0, \ell]$ and write the tension force vector as $T \times$ "unit tangent" along the string (fig. 4)

$$\begin{aligned} \vec{T} &= T \left(\frac{\Delta x}{\sqrt{(\Delta u)^2 + (\Delta x)^2}}, \frac{\Delta u}{\sqrt{(\Delta u)^2 + (\Delta x)^2}} \right) \\ &= \left(\frac{T}{\sqrt{1 + u_x^2}}, \frac{T u_x}{\sqrt{1 + u_x^2}} \right) \end{aligned}$$

Only the vertical component will enter the dynamics by the Newton's $ma = F$ law which now becomes

$$\int_{x_0}^{x_1} \rho u_{tt} = \frac{T u_x}{\sqrt{1 + u_x^2}} \Big|_{x_0}^{x_1}$$

The boundary forces could be brought to the integral over $[x_0; x_1]$ via the fundamental theorem of calculus

$$\frac{T u_x}{\sqrt{1 + u_x^2}} \Big|_{x_0}^{x_1} = \int_{x_0}^{x_1} \frac{\partial}{\partial x} \left(\frac{T u_x}{\sqrt{1 + u_x^2}} \right)$$

As the integral relation holds for all subintervals $[x_0; x_1]$ we end up with a partial differential equation for $u(x, t)$

$$\rho u_{tt} - \frac{\partial}{\partial x} \left(\frac{T u_x}{\sqrt{1 + u_x^2}} \right) = 0 \text{ or external force } F \quad (20)$$

Euler-Lagrange formulation

Here we shall use the kinetic and potential energies (19) and follow the finite-D oscillator procedure (16). The position-velocity (state) variables $\{u; u_t\}$ are labeled by the continuous parameter $x \in [0, \ell]$, and the role of standard partial derivatives $\frac{\partial K}{\partial \dot{u}_i}$ (generalized momenta) and $\frac{\partial P}{\partial u_i}$ (potential forces) will be played by *variational derivatives*

$$\begin{aligned} \frac{\partial K}{\partial \dot{u}_i} &\rightarrow \frac{\delta K}{\delta u} = \frac{\partial K}{\partial u_t} = \rho u_t \\ \frac{\partial P}{\partial u_i} &\rightarrow \frac{\delta P}{\delta u} = \partial_u P - \partial_x \left(\frac{\partial}{\partial u_x} P \right) \\ &= \frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left(\frac{T u_x}{\sqrt{1 + u_x^2}} \right) \end{aligned} \quad (21)$$

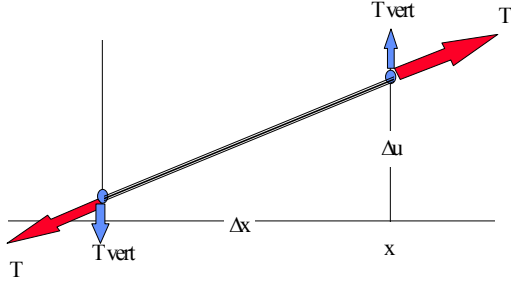


Figure 4:

The result is the same second order nonlinear pde (20) for $u(x; t)$.

Let us remark that for the “gently sloping” deformations ($|u_x| \ll 1$) equation (20) can be linearized by dropping $\sqrt{u_x^2 + 1}$ in the denominator, equivalently approximating the arc-length differential in the potential energy form (19) by the quadratic term

$$\sqrt{u_x^2 + 1} - 1 \approx \frac{1}{2}u_x^2$$

Clearly, quadratic energy-forms make the Newton’s equations of motion linear. Hence (20) becomes a linear equation

$$\rho u_{tt} - (T u_x)_x = 0 \quad (22)$$

Other external forces could be added to (20) or (22), e.g. friction $F = \alpha u_t$, gravity $F = gu$, driving force $F(x, t)$ etc.

Vibrating membrane.

The derivation is similar to the 1D string case. The space variable $\mathbf{x} = (x, y)$ now varies over the plane region D (the shape of the membrane). If $u(\mathbf{x}; t)$ denotes as above the vertical displacement, the kinetic and potential energies take the form

$$\begin{aligned} K &= \int_D \frac{1}{2} \rho u_t^2 d^2 \mathbf{x} \\ P &= \int_D \left\{ T \left(\sqrt{\nabla u^2 + 1} - 1 \right) + F(u) \right\} d^2 \mathbf{x} \end{aligned} \quad (23)$$

So the change in the surface area due to deformation

$$T \left(\sqrt{\nabla u^2 + 1} - 1 \right) d^2 \mathbf{x} = dS - d^2 \mathbf{x}$$

is the main contributing factor to potential energy. Here T is the membrane tension, assumed for simplicity to be scalar (isotropic case).

In the Newtonian derivation we take a small patch of the membrane surface U bounded by curve Γ , and consider the tension force distributed along the boundary (fig.5).

The effect of vertical tension force at a point Γ in the normal direction N is found similar to the string case

$$F = T \frac{\partial_n u}{\sqrt{1 + (\nabla u)^2}} = N \cdot T \frac{\nabla u}{\sqrt{1 + (\nabla u)^2}}$$

So the resulting balance of forces over patch U gives

$$\iint_U \rho u_{tt} = \oint_{\Gamma} N \cdot T \frac{\nabla u}{\sqrt{1 + (\nabla u)^2}} ds \quad (24)$$

Applying the divergence theorem to the r.h.s.

$$\oint_{\Gamma} N \cdot T \nabla u / \sqrt{\nabla u^2 + 1} ds = \iint_U \nabla \cdot \left(\frac{T \nabla u}{\sqrt{\nabla u^2 + 1}} \right)$$

of (24) both sides are brought to the area integral over U , hence results the pde for u

$$\rho u_{tt} - \nabla \cdot \left(\frac{T}{\sqrt{u_x^2 + u_y^2 + 1}} \nabla u \right) = \dots \quad (25)$$

The Lagrangian derivation repeats the above (string) argument verbatim. We write momentum variables

$$\frac{\partial K}{\partial u_t} = \rho u_t$$

and the potential forces for P of (23)

$$\begin{aligned} \partial_u P - \nabla \cdot \left(\frac{\partial}{\partial (\nabla u)} P \right) &= F_u - \\ - \partial_x \left(T \frac{u_x}{\sqrt{u_x^2 + u_y^2 + 1}} \right) - \partial_y \left(T \frac{u_y}{\sqrt{u_x^2 + u_y^2 + 1}} \right) \end{aligned} \quad (26)$$

The result is the same equation (25) - nonlinear pde for u , or its linearized version

$$\rho u_{tt} - \nabla \cdot (T \nabla u) = F_u$$

In general ρ and T are functions of \mathbf{x} and T is either scalar (isotropic medium) or matrix (tensor) to account for possible anisotropy. In the simplest homogeneous isotropic case ($\rho = \text{Const}$; $T = \text{Const}$) we get

$$u_{tt} - c^2 \Delta u = \dots$$

in terms of the Laplacian Δ . Here $c = \sqrt{\frac{T}{\rho}}$ measures the speed of wave propagation.

Initial-boundary conditions

In all examples above the differential equation holds in a space-time region $\{\mathbf{x} \in D, t > 0\}$ and must be augmented with suitable initial-boundary conditions. Since the equations are second order in space and time (and classified as hyperbolic) the initial value problem requires the complete initial state, that is initial position $u|_{t=0} = u(x, 0) = u_0(x)$ and initial velocities $u_t|_{t=0} = u_t(x, 0) = u_1(x)$.

The boundary conditions usually come in 3 types. We shall describe them specifically in each case

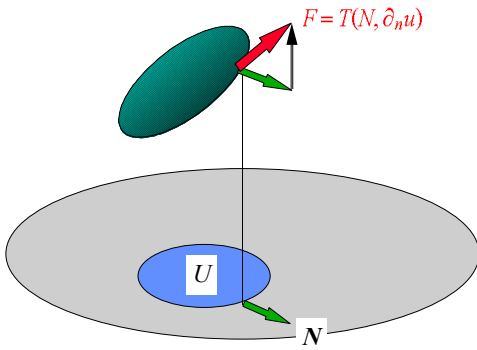


Figure 5:

String Here the range of x is $[0, \ell]$ and 3 types of boundary conditions become

- Dirichlet: $u|_{x=0} = u(0, t) = \dots; u|_{x=\ell} = u(\ell, t) = \dots$ So the ends of the string are fixed
- Neumann: $u_x|_{x=0} = \dots; u_x|_{x=\ell} = \dots$ Here we fix the slopes, rather than positions of the ends
- Mixed (Robin) combines the values and the derivatives at the ends:

$$(a_0 u + b_0 u_x)|_{x=0} = \dots; (a_1 u + b_1 u_x)|_{x=\ell} = \dots$$

In other words we maintain certain ratio of values to slopes at the ends.

Membrane Here the range on \mathbf{x} is plane region D bounded by curve Γ and three types of boundary conditions take the form

- Dirichlet: $u|_{\Gamma} = \dots$ (prescribed values on the boundary)
- Neumann: $\partial_n u|_{x=0} = \dots$ (prescribed normal derivative on the boundary)
- Mixed (Robin): $(au + b\partial_n u)|_{x=0} = \dots$ (prescribed combination of values and normal derivatives).

Elastic rod

We shall consider an elastic rod in 3D, described by a vector function $\mathbf{u}(x)$ measuring its displacement from the equilibrium, so the state (phase) space consists of the position-velocity configurations $\{\mathbf{u}, \mathbf{u}_t\}$ as above for string. The potential energy of elastic rod however is more complicated. It involves besides stretching (change in the arc-length) the *bending* and *twisting* moments of the rod that take into account its thickness

$$P = a\kappa^2 + b\tau^2$$

The first term is proportional to the curvature squared (bending) and torsion squared (twisting, for 3-D motions)

$$\kappa = \frac{|\mathbf{u}' \times \mathbf{u}''|}{|\mathbf{u}'|^3} \quad (27)$$

$$\tau = \frac{\mathbf{u}_s \times \mathbf{u}_{ss} \cdot \mathbf{u}_{sss}}{\kappa^2} = \frac{[\mathbf{u}', \mathbf{u}'', \mathbf{u}''']}{|\mathbf{u}' \times \mathbf{u}''|^2}$$

Here $'$ denotes the derivative in x , while subscript \mathbf{u}_s indicates differentiation with respect to the arc-length $ds = |\mathbf{u}'(x)| dx$. So the complete Lagrangian (kinetic + potential) becomes

$$L[\mathbf{u}] = \frac{1}{2} \iint \{ \rho \mathbf{u}_t^2 - a\kappa^2(\mathbf{u}) - b\tau^2(\mathbf{u}) \} dx dt$$

and the Euler-Lagrange equations⁶ are

$$\rho \mathbf{u}_{tt} - \partial_x \left(\frac{\partial P}{\partial \mathbf{u}_x} \right) + \partial_x^2 \left(\frac{\partial P}{\partial \mathbf{u}_{xx}} \right) - \partial_x^3 \left(\frac{\partial P}{\partial \mathbf{u}_{xxx}} \right) = F \quad (28)$$

where F represents the external/boundary forces acting upon the rod. It is convenient to take $x = s$ -the arc-length, then pde (28) is considered on the fixed range $[0, L]$ and suitable boundary conditions are imposed at two endpoints $x = 0; L$. The resulting equations look fairly complicated: nonlinear 4-th or 6-th order pde system.

In the planar case (28) is somewhat simplified, as twist disappears, while curvature (27) of the graph $u = u(x)$, assumes the form

$$\kappa^2 = \frac{u''^2}{(1 + u'^2)^3}$$

That turns (28) into a 4-th order scalar pde

$$\rho u_{tt} + 3a \left[\frac{u_x u_{xx}^2}{(1 + u_x^2)^{5/2}} \right]_x - 2a \left[\frac{u_{xx}}{(1 + u_x^2)^{3/2}} \right]_{xx} = 0 \text{ or } f \quad (29)$$

whose linearized form of (29)

$$\rho u_{tt} - 2a u_{xxxx} = \dots$$

Equations (28)-(29) could be further simplified (reduced to a 2-nd order equation) by a clever change of variable, namely replacing function u by angle θ between unit tangent and the x -axis, and using arclength ds in place of dx (fig. 6). As curvature $\kappa = \frac{d\theta}{ds}$ we get quadratic potential energy $P = a \left(\frac{d\theta}{ds} \right)^2$. The length constraint is also easy to accommodate here, as coordinate functions are given by integrals of the unit tangent $T = (\cos \theta, \sin \theta)$

$$x = \int_0^s \cos \theta ds; \quad u = \int_0^s \sin \theta ds$$

⁶In general, higher order Lagrangians $L(u, u_x, u_{xx}, \dots)$ have their first variation (E-L) equal to $\frac{\delta L}{\delta u(x)} = \partial_u L - \frac{d}{dx} \left(\frac{\partial L}{\partial u_x} \right) + \frac{d^2}{dx^2} \left(\frac{\partial L}{\partial u_{xx}} \right) - \dots$

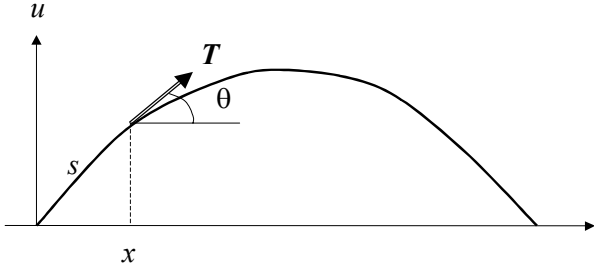


Figure 6:

Hence the constrained Lagrangian $\mathcal{L}[\theta] = \int_0^L a\theta'^2 - \lambda \sin \theta$, with multiplier λ . Hence Euler-Lagrange equation turns into a second order nonlinear ODE

$$\theta_{ss} + \lambda \sin \theta = 0, \text{ on } 0 < s < L \quad (30)$$

equation (30) is supplemented with Neumann boundary conditions: $\theta'|_{s=0,L} = 0$.

It has conserved integral: $e = \frac{1}{2}\theta'^2 - \lambda \cos \theta$, with constant of integration $e = -\lambda \cos \theta_0$ (due to boundary conditions). Conserved integral yields an expression of θ in terms of the elliptic integral :

$$s = \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{2(e + \lambda \cos \theta)}} = \sqrt{\frac{2}{\lambda + e}} F\left(\frac{\theta}{2} \middle| m\right)$$

of modulus $m = \frac{2\lambda}{\lambda + e} = \frac{2}{1 - \cos \theta_0}$. Hence we get solution θ expressed through the Jacobi amplitude

$$\theta = 2am \left(\sqrt{\frac{\lambda + e}{2}} (s - s_0) \middle| m \right)$$

We show solutions θ for several initial values: $\cos \theta_0 = .1 \div .9$

