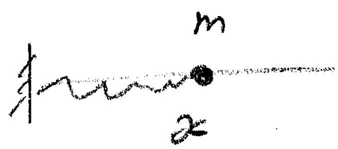
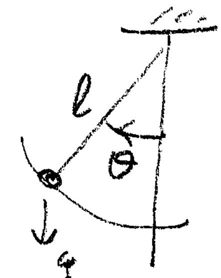
445 # Mechanical systems

D. Gurarie

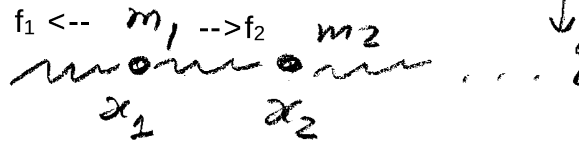
I. Finite - D

a) oscillators:

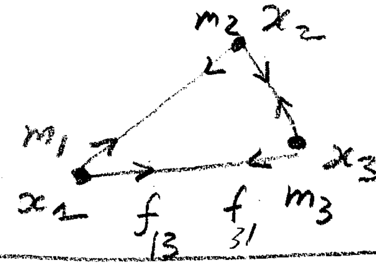




b) coupled:



c) gravitational n-body:



---

① 2^{nd} order ODE/ODS

a) $m\ddot{x} = f(x, \dot{x} \dots)$; $l\ddot{\theta} = -g \sin\theta$

b) $\left\{ m_i \ddot{x}_i = f_i \right.$; $f_i = \begin{cases} -f_i + f_{i+1} ; & (b) \\ \sum_{j \neq i} f_{ij} & (c) \end{cases}$

$$\underbrace{\qquad\qquad\qquad}_{f_{ij} = f_{ij}(x_i - x_j)}$$

Vector form: $\boxed{M\ddot{X} = F(X, \dots)}$

mass/inertia matrix/tensor $M = \begin{bmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{bmatrix}$

② Potential forces: $\boxed{f_i = -\partial_i U ; \quad F = -\nabla U \text{ - pot. energy}}$

a) $f = -kx$; $U = \frac{kx^2}{2}$; $f = -g \sin\theta \Rightarrow U = -g\cos\theta$

b) $f_i = -k_i(x_i - x_{i-1}) \Rightarrow U = \frac{1}{2}\sum_i k_i (x_i - x_{i-1})^2$

c) $f_{ij} = \frac{G m_i m_j}{r_{ij}^3}(x_i - x_j)$; $r_{ij} = |x_i - x_j|$

$$\Rightarrow \boxed{U = -\sum_{i \neq j} \frac{G m_i m_j}{r_{ij}}}$$ grav. potential

---

(3) Energy conservation:

Kinetic: $K = \frac{1}{2}\sum m_i v_i^2$; $v_i = \dot{x}_i$

Potential: $P = U(x_1, \dots x_n)$

Total: $E = K + P$ - const $\boxed{\frac{d}{dt}E = 0}$

---

(4) Hamiltonial form

Momenta: $p_i = m_i \dot{x}_i$; Ham. f-n: $h(x,p) = \frac{1}{2}\sum \frac{p_i^2}{m_i} + U$

$$(\text{Ham})\begin{cases} \dot{x}_i = \partial_{p_i} h \\ \dot{p}_i = -\partial_{x_i} h \end{cases} \iff \begin{cases} \dots \\ m_i \ddot{x}_i = f_i \\ \dots \end{cases} \text{(Newton)}$$

---

(5) For 1D systems a), b) etc. energy cons. reduce 2nd order Newton to 1st order

$$\boxed{m\ddot{x} = -U'(x)} \Longrightarrow \frac{m\dot{x}^2}{2} + U(x) = E - \text{const} \Rightarrow$$

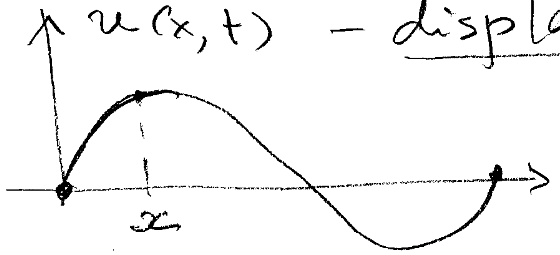
$$\boxed{\dot{x} = \pm\sqrt{\frac{2}{m}(E - U(x))}} \leftarrow \text{1st order separable ODE}$$

$$\int_{x_0}^{x} \frac{dx}{\sqrt{2/m\,(E-U)}} = t - t_0$$

II. Continuous (∞-D) systems

1. Vibrating string

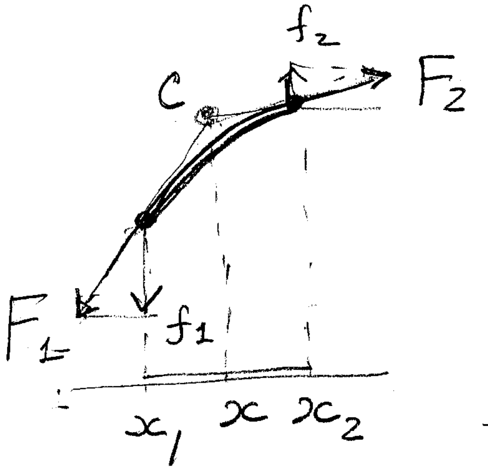
$u(x,t)$ – displacement; $\rho(x)$ – mass density



string tension

Restoring force $= \tau\, ds = \tau\sqrt{1+u_x^2}\, dx$

arc length



Forces at end-points of Δx

$$\mathbb{F}_1 = -\tau \underbrace{\left(\frac{ds}{dx}\right)}\,\mathbb{T}\Big|_{x_1}\ ; \quad \mathbb{F}_2 = \tau\frac{ds}{dx}\mathbb{T}\Big|_{x_2}$$

unit tangent

$$\mathbb{F}_1 = -\tau(1, u_x)\big|_{x_1} \qquad \mathbb{T} = \frac{(1, u_x)}{\sqrt{1+u_x^2}}$$

$$\mathbb{F}_2 = \tau(1, u_x)\big|_x$$

Vertical forces: $\quad f_1 = -\tau u_x\big|_{x_1}$

$$f_2 = \tau u_x\big|_{x_2}$$

Acceleration of approximate center-mass $\approx$ acceleration of arc ??

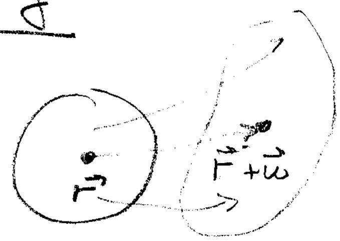
$$\int_{x_1}^{x_2} \rho u_{tt}\, dx \overset{?}{\approx} f_2 - f_1 = \tau u_x\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \partial_x(\tau u_x)\, dx$$

$$\Longrightarrow \quad \boxed{\rho\, u_{tt} \approx (\tau u_x)_x} \quad \text{or} \quad \boxed{u_{tt} = c^2 u_{xx}}$$

with speed $c = \sqrt{\tau/\rho}$

Q: is $\approx$ valid?

D. GURARIE   Ellasticity

* Deformation: 

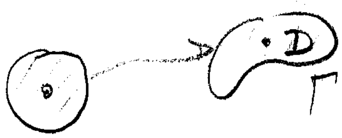
$\vec r \to \vec r + \vec w(\vec r, t)$

Infinites. deformation:

$$\vec w(\vec r+\Delta \vec r) \approx \underbrace{\vec u(\vec r)}_{\text{shift}} + \underbrace{\left(\frac{\partial u_i}{\partial r_j}\right)}_{\text{lin. deform. tensor}} \Delta \vec r$$

* Strain tensor: $\boxed{S_{ij} = \frac{\partial_i u_j + \partial_j u_i}{2}}$ - symmetrized deform. tensor

* $\boxed{\sigma_{ij} = \phi_{ij}(S)}$ - stress tensor (surface forces due to deformation)



$$\boxed{\partial_t^2 \left(\iint_D \rho \vec u\right) dV = \oint_\Gamma \sigma \cdot N\, dS}$$ Newton's

$$\iint_D \rho\, \partial_t^2 w_i\, dV = \oint_\Gamma \sum_j \sigma_{ij} N_j\, dS = -\iint_D \sum_j \partial_j(\sigma_{ij})\, dV$$

$$\Rightarrow \boxed{\rho\, \partial_t^2 w_i = -\sum_j \partial_j \sigma_{ij}(\partial_r \vec w)}$$

* Linear ellasticity (Hook): $\sigma_{ij} = \sum \overset{\text{ellastic moduli}}{\phi_{ijkl}} S_{kl}$

* Rotation symmetry (Cauchy): $\boxed{\sigma_{ij} = \lambda\, \mathrm{tr}\, S\, \delta_{ij} + 2\mu S_{ij}}$

λ, μ - Lame coefficients.

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdot \\ \sigma_{21} & \cdots & \\ \cdots & & \end{bmatrix} = \lambda (\nabla \cdot \vec w) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mu \begin{bmatrix} 2\partial_1 u_1 & (\partial_1 u_2 + \partial_2 u_1) & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

PDE system:

$$\rho \partial_t^2 u_i = \lambda \partial_i \nabla\cdot\vec{u} + \mu\left(\partial_i \nabla\cdot\vec{u} + \nabla^2 u_i\right)$$

Vector form:

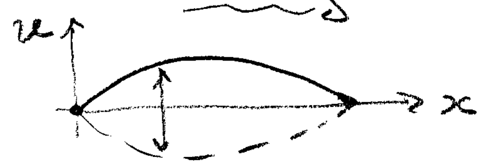
$$\boxed{\rho \vec{u}_{tt} = (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) + \mu\nabla^2\vec{u}} \qquad (*)$$

Coordinate form (2D):

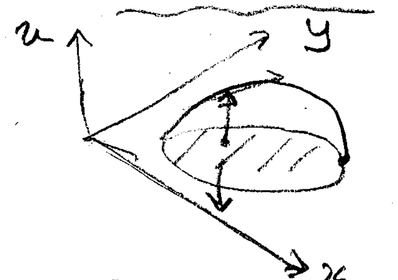
$$\rho \partial_t^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} (\lambda+2\mu)\partial_x^2 \cancel{+\mu\partial_x^2} & (\lambda+\mu)\partial_{xy}^2 \\ (\lambda+\mu)\partial_{xy}^2 & \cancel{\mu\partial_x^2} + (\lambda+2\mu)\partial_y^2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Example 1:

string



membrane



Transverse displacement: $\vec{u} = (0, u(x,t))$ $\qquad$ $\vec{u} = (0, 0, u(x,y,t))$

$$\boxed{\nabla\cdot\vec{u} = 0}$$

$$\boxed{\rho u_{tt} = \mu u_{xx}} \qquad \boxed{\rho u_{tt} = \mu \nabla^2 u}$$

Wave eq-ns with speed $c = \sqrt{\mu/\rho}$

Helmholtz decomposition

$$\vec{u} = \underbrace{\nabla\phi}_{\text{irrotational}} + \underbrace{\nabla\times\vec{\psi}}_{\text{solenoidal}}$$

gives after substitution in (*) two scalar wave eq-ns with different speed

$$\nabla\underbrace{\left(\rho\phi_{tt} - (\lambda+2\mu)\nabla^2\phi\right)}_{=0} + \nabla\times\underbrace{\left(\rho\vec{\psi}_{tt} - \mu\nabla^2\vec{\psi}\right)}_{=0} = 0$$

Compression waves

Potential field $\vec{u} = \nabla\phi$ propagates at speed $c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$

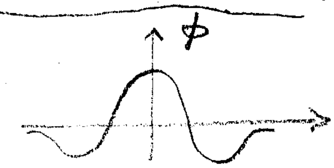
Solenoidal —"— $\vec{u} = \nabla\times\vec{\psi}$ —"— $c_s = \sqrt{\mu/\rho}$

Shear waves

Example 2:

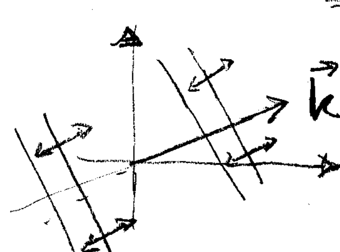
i) Compression (longitudinal) plane-wave:

$\vec{u} = \nabla[\phi(\vec{k}\cdot\vec{x})] = \vec{k}\,\phi'(\vec{k}\cdot\vec{x})$



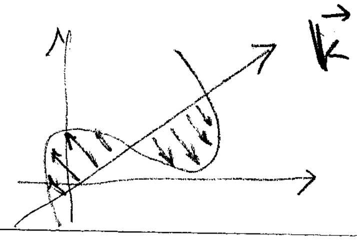
$\vec{k}$ – wave vector

speed $= c_p$



ii) Solenoidal (shear) wave: $\vec{u} = \nabla\times[\underbrace{\vec{a}\times\vec{k}\,\psi(\vec{k}\cdot\vec{x})}_{\vec{\psi}}]$

$\vec{u} = k^2\vec{a}\,\psi'(\vec{k}\cdot\vec{x})$



speed $= c_s < c_p$

(string & membrane are shear waves)

Maxwell eq-ns of electrodynamics

(I) Vacuum: $\mathbb{E}$ - electric f.; $\mathbb{B}$ - magnetic induction

I $\quad \nabla \cdot B = 0$ - no magn. sources

II $\quad \nabla \times E + \frac{1}{c}\partial_t B = 0$ - Faraday law of EMF

III $\quad \nabla \cdot E = 4\pi q$ - Coulomb law: "E-source" = "charge"

IV $\quad \nabla \times B - \frac{1}{c}\partial_t E = \frac{4\pi}{c} j$ - Ampere law: B produced by current

Taking div of eq-n IV we get
"charge conservation" law: Dt(q)=-div(J)

c = light speed

Potentials: $(I) \Rightarrow B = \nabla \times A$ $\quad (II) \Rightarrow E = -\frac{1}{c}A_t - \nabla\phi$

vector $\qquad$ scalar

Gauge transform:
$$A \to A + \nabla\chi$$
$$\phi \to \phi - \frac{1}{c}\chi_t$$

Adjust χ? $\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\chi = \dots$

Lorentz gauge: $\boxed{\nabla \cdot A + \frac{1}{c}\phi_t = 0}$

$$\Rightarrow \begin{matrix} III & \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\phi = 4\pi q \\ IV & \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)A = \frac{4\pi}{c} j \end{matrix} \Longrightarrow \text{Wave eq-n with speed } c$$

(II) Medium: $(E, B) \to (D, H)$ new fields

Constitutive relations: $\boxed{D = \epsilon E; \quad H = \frac{1}{\mu} B}$ w/o polarization

ϵ dielectric $\quad \mu$ magnetic permeabil.

$$\Rightarrow \left\{ \begin{array}{l} \left.\begin{array}{l} \nabla \cdot B = 0 \\ \nabla \times E + \frac{1}{c}\partial_t B = 0 \end{array}\right] \to \text{potentials } (A, \phi) \\ \left.\begin{array}{l} \nabla \cdot D = 4\pi q \\ \nabla \times H - \frac{1}{c}\partial_t E = \frac{4\pi}{c} j \end{array}\right\} \Rightarrow \boxed{\left(\epsilon\mu\,\partial_t^2 - \nabla^2\right)\begin{pmatrix}\phi \\ A\end{pmatrix} = \begin{pmatrix} q \\ j \end{pmatrix}} \end{array} \right.$$

Wave eq-n with speed $c' = \frac{1}{\sqrt{\mu\epsilon}} \le c$