Green's f-ns of elliptic $(\mathcal{L}, \mathcal{B})$ and associated problems

Elliptic: $\mathcal{L} = -\partial_x p \partial_x + q$ or $-\nabla \cdot p \nabla + q$ (D) Γ

BVP $\quad \mathcal{B} = a + b\partial_x \big|_{0,\ell}$ or $a + b\partial_n \big|_{\Gamma}$

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(1) Elliptic BVP: $\begin{cases} \mathcal{L}[K] = \delta(x, \xi) \\ \mathcal{B}[K]\big|_{\Gamma} = 0 \end{cases}$

Formal: $\boxed{K = \mathcal{L}^{-1}}$; Expansion $\boxed{K(x,\xi) = \sum_k \frac{1}{\lambda_k} \frac{\psi_k(x)\overline{\psi_k(\xi)}}{\|\psi_k\|^2}}$

(2) Heat: $\begin{cases} \partial_t G + \mathcal{L}[G] = 0; \ t > 0 \\ G\big|_{t=0} = \delta(x-\xi); \ \mathcal{B}[G] = 0 \end{cases}$

$\boxed{G = e^{-t\mathcal{L}}}$; $\boxed{G(x,\xi,t) = \sum_k e^{-\lambda_k t} \frac{\psi_k(x)\overline{\psi_k(\xi)}}{\|\psi_k\|^2}}$

(3) Wave: $\begin{cases} \partial_t^2 S + \mathcal{L}[S] = 0; \ t > 0 \\ S\big|_{t=0} = 0; \ S_t\big|_{t=0} = \delta(x,\xi); \ \mathcal{B}[S] = 0 \end{cases}$ (sin-propag)

$\boxed{S = \frac{\sin t\sqrt{\mathcal{L}}}{\sqrt{\mathcal{L}}}}$ $\boxed{S(x,\xi,t) = \sum_k \frac{\sin\sqrt{\lambda_k}\,t}{\sqrt{\lambda_k}} \frac{\psi_k(x)\overline{\psi_k(\xi)}}{\|\psi_k\|^2}}$

From Green's f-n kernel on can comput Poisson kernel to represent Boundary sources

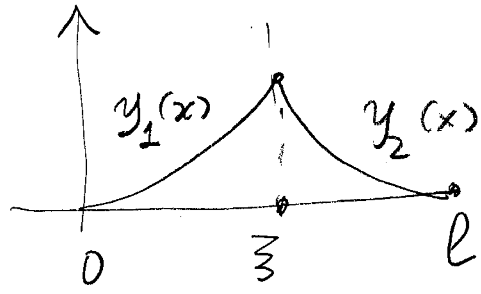
Green's f-n for 1D 2^{nd} order BVP

Operator $\mathcal{L} = -p\,d_x^2 + \ldots$ on $[0, \ell]$ with

B.C. $B\big|_{0,\ell} = \ldots$

<u>Green's f-n</u> $K(x,\xi)$ – integral kernel

Solves $\begin{cases} \mathcal{L}[K] = \delta(x-\xi) \\ B[K]\big|_{0,\ell} = 0 \end{cases}$



Take 2 solutions $\mathcal{L}[y_i] = 0$ on $[0,\xi]$ and $[\xi,\ell]$, each one satisfying its B.C. (at 0 & ℓ) and <u>stich them</u> together at $x=\xi$, so that $y_1 = y_2\big|_\xi$ – contin. & derivatives have <u>jump discont.</u>

$$\left.\begin{cases} y_1 - y_2\big|_\xi = 0 \\ y_1' - y_2'\big|_\xi = \dfrac{1}{p(\xi)} \end{cases}\right] \Rightarrow \boxed{K(x,\xi) = \frac{1}{pW}\begin{cases} y_1(x)\,y_2(\xi); & x<\xi \\ y_1(\xi)\,y_2(x); & x>\xi \end{cases}}$$

where $W(\xi) = \text{Vronskin} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$; p – leading coeff of $\mathcal{L}$

Examples:

$\mathcal{L}$	B	y_1 y_2	W	K
1) $-\partial_x^2$	(D)	x ; $l-x$	l	$\frac{1}{l}\begin{cases} x(l-\xi) \\ \xi(l-x)\end{cases}$
2) $-\partial_x^2-a^2$	(D)	$\sin ax$; $\sin a(l-x)$	$a\sin al \neq 0$ $al \neq \pi k$	$\frac{1}{a\sin al}\begin{cases}\sin ax \sin a(l-\xi) \\ \sin a\xi \sin a(l-x)\end{cases}$
3) $-\partial_x^2+a^2$	(D)	$\operatorname{sh} ax$; $\operatorname{sh} a(l-x)$	$a \operatorname{sh}(al)$	$\frac{1}{a\operatorname{sh}(al)}\begin{cases}\operatorname{sh} a\xi \operatorname{sh} \ldots \\ \operatorname{sh} \quad \operatorname{sh} \ldots\end{cases}$
4) $\partial_r^2+\frac{1}{r}+\frac{m^2}{r^2}$		r^m ; $\left(\frac{r}{l}\right)^m-\left(\frac{l}{r}\right)^m$	...	...
		For $m=0$ $\{1;\ \ln r\}$	...	...

Expansion: $$\boxed{K(x,\xi) = \sum \frac{\psi_k(x)\psi_k(\xi)}{\lambda_k \|\psi_k\|^2}}$$

Ex: Case 1) $$K = \sum_1^\infty \frac{\sin\frac{\pi k x}{l}\sin\frac{\pi k \xi}{l}}{\left(\frac{\pi k}{l}\right)^2 l/2}$$

Solution of source BVP: $\begin{cases}\mathcal{L}[u] = F \\ B\, u|_{..} = A, B\end{cases} \Rightarrow$

$$\boxed{u(x) = \int_0^l K(x,\xi)F(\xi)\,d\xi + A P_0(x) + B P_1(x)}$$

$$P_0 = -\partial_\xi K(x,0) \; ; \; P_1 = \partial_\xi K(x,l)$$

Problems:

1) Check $K(x,\xi)$ of (1) by $\mathcal{F}$ expansion in x (ξ - fixed)

2) Solve stationary problem: $\begin{cases} -u_{xx} = 1 & \text{on } [0,l] \\ u\big|_{\partial l} = 0 \end{cases}$ by 3 methods: Fourier, Green's, direct & compare solutions

3) Compute Green's f-n in (4) for $m=0$ and $m \neq 0$

6° Green's f-n of heat-diff. on $[0,\ell]$

$$G(x,\xi,t) = \frac{2}{\ell}\sum e^{-\lambda_k t}\,\psi_k(x)\,\overline{\psi_k}(\xi) = \dots$$

Poisson kernels at $x=0,\ell$

$$P(x,0,t) = \frac{2}{\pi}\sum \frac{\lambda_k}{k}\,e^{-\lambda_k t}\,\psi_k(x) = \frac{2\pi}{\ell}\sum \frac{k}{\ell}\sin\frac{\pi k x}{\ell}\,e^{-\lambda_k t}$$

$$P(x,\ell,t) = \frac{2}{\pi}\sum \frac{(-1)^{k-1}}{k}\lambda_k\,e^{-\lambda_k t}\,\psi_k(x) = \frac{2\pi}{\ell}\sum (-1)^{k-1}\frac{k}{\ell}\sin\frac{\pi k x}{\ell}\,e^{-\lambda_k t}$$

Note: $P(x,0,t) = \partial_\xi G(x,\xi,t)\big|_{\xi=0}$

$P(x,\ell,t) = -\partial_\xi G(\dots)\big|_{\xi=\ell}$

$$\Rightarrow \boxed{u = \int_0^t P(x,0,t-\tau)\,A(\tau)\,d\tau + \int_0^t P(x,\ell,t-\tau)\,B(\tau)}$$

Poisson rep-n

Same works for wave-eq-n:

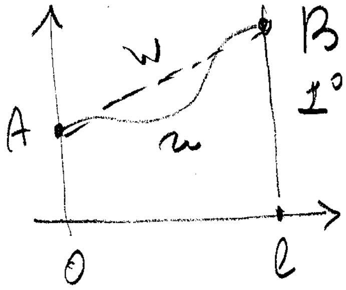
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u|_0 = A;\quad u|_\ell = B \end{cases} \Rightarrow \text{Green's: } N(x,\xi,t) = \sum \frac{\sin\sqrt{\lambda_k}\,t}{\sqrt{\lambda_k}}\,\psi_k(x)\,\overline{\psi_k}(\xi)$$

$P(x,0,t) = \partial_\xi N\big|_{\xi=0}$

$P(x,\ell,t) = -\partial_\xi N\big|_{\xi=\ell}$

III. B-dary sources:

$$\begin{cases} u_t - D u_{xx} = 0 & \text{on } [0,l] \\ u\big|_{x=0} = A(t); \quad u\big|_{x=l} = B(t) \end{cases}$$



1° Subtract auxiliary f-n $w(x,t)$ that satisfies B.C. $\boxed{w = \dfrac{A(l-x)+Bx}{l}}$ to convert b-dary sources into contin. source: $v = u - w$ solves

$$(2)\ \begin{cases} v_t - D\, v_{xx} = -\dfrac{A'(t)(l-x) + B'(t)x}{l} = F \\ v\big|_{x=0,l} = 0 \end{cases}$$

2° Use F-expansion for source F

$$\boxed{\frac{l-x}{l} = \frac{2}{\pi}\sum_1^\infty \frac{1}{k}\sin\frac{\pi k x}{l}}; \quad \boxed{\frac{x}{l} = \frac{2}{\pi}\sum_1^\infty \frac{(-1)^{k-1}}{k}\sin\frac{\pi k x}{l}}$$

3° Get:

$$v = -\frac{2}{\pi}\sum_k \Big\{ \underbrace{\big[A(0)e^{-\lambda_k t} + (e^{-\lambda_k t} * A'(t))\big]}_{A(t)-\lambda_k(e^{-\lambda_k t} * A)}\frac{1}{k} + \underbrace{\big[B(0)e^{-\lambda_k t} + (e^{-\lambda_k t} * B'(t))\big]}_{B(t)-\lambda_k(e^{-\lambda_k t} * B)}\frac{(-)^{k-1}}{k} \Big\} \sin\frac{\pi k x}{l}$$

Check:

Use convolut. identity: $\boxed{e^{-\lambda t} * f'(t) = f(t) - e^{-\lambda t} f(0) - \lambda e^{-\lambda t} * f}$

4° $$v(x,t) = -A(t)\Big(\underbrace{\textstyle\sum \dots}_{\frac{l-x}{l}}\Big) - B(t)\Big(\underbrace{\dots}_{x/l}\Big) + \int_0^t P_0(x,t-\tau)A(\tau) + \int_0^t P_1(x,t-\tau)B(\tau)$$

Poisson kernels at 0, l of Green's f-

$$P_0(x,t) = \frac{2}{\pi}\sum \frac{\lambda_k}{k}\psi_k(x) e^{-\lambda_k t}$$
$$P_1 = \frac{2}{\pi}\sum \frac{\lambda_k}{k}(-)^{k-1}\psi_k(x) e^{-\lambda_k t}$$
$$= \partial_\xi G\big|_{\xi=0,l}$$

$$\boxed{G(x,\xi,t) = \frac{2}{\pi}\sum e^{-\lambda_k t}\frac{\psi_k(x)\psi_k(\xi)}{\|\psi\|^2}}$$