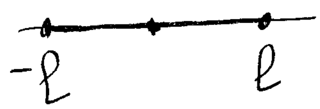
D. GURARIE

445

Fourier Series

Trig. systems (harmonics) on 

$$\left.\begin{array}{ll}\left\{\cos\frac{\pi k x}{\ell}\right\}_{k=0}^{\infty} & \text{- even}\\ \left\{\sin\frac{\pi k x}{\ell}\right\}_{k=1}^{\infty} & \text{- odd}\\ \left\{e^{i\frac{\pi k x}{\ell}}\right\}_{k=-\infty}^{\infty} & \text{- complex}\end{array}\right]\ 2\ell\text{-periodic}$$

Inner product: $\langle f|g\rangle = \int_{-\ell}^{\ell} f(x)\,\overline{g(x)}\,dx$

Orthogonality:

1) on $[-\pi,\pi]$: $\langle e^{ikx}|e^{imx}\rangle = 2\pi \boxed{\delta_{k-m}}$ ← Kronecker δ

$\Rightarrow \|e^{ikx}\|^2 = 2\pi$

2) $\langle \cos|\cos\rangle = \begin{cases}\frac{\ell}{2}\delta_{km}; & k,m\neq 0\\ \ell; & k=m=0\end{cases}$ on $[0,\ell]$

$\langle \sin|\sin\rangle = \frac{\ell}{2}\delta_{km}$; on $[0,\ell]$

$\|\cos\|^2 = \begin{cases}\ell; & k=0\\ \ell/2; & k>0\end{cases}$ $\quad \|\sin\|^2 = \ell/2$

F-modes form a complete, orthogonal system in space $L^2 = \{f(x): \|f\|^2 = \int |f|^2 dx < \infty\}$

$\mathcal{F}$-Expansion. Planchetel f-la

Any f-n $f(x)$ on $[-l, l]$ or $[0, l]$
can be expanded in $\mathcal{F}$-series:

$$f(x) = \sum_{-\infty}^{\infty} \hat{f}_k e^{ikx}; \quad \hat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx} = \frac{\langle f | e^{ikx} \rangle}{\| .. \|^2}$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos\frac{\pi k x}{l} + b_k \sin\frac{\pi k x}{l} \right) \quad \text{on } [-l, l]$$

$$a_k = \frac{\langle f | \cos .\rangle}{\| \cos \|^2}; \quad b_k = \frac{\langle f | \sin ..\rangle}{\| \sin \|^2}$$

Complex & real $\mathcal{F}$-coeff. are related via

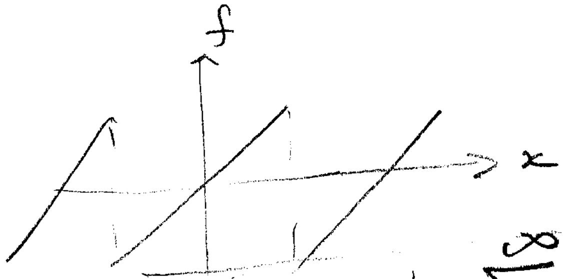
$$\hat{f}_k = \frac{a_k + i b_k}{2}$$

so $\boxed{\hat{f}_k e^{ikx} + \hat{f}_{-k} e^{-ikx} = a_k \cos kx + b_k \sin kx}$

Planchetel: For any complete, orthogonal system $\{\varphi_k\}$: $f(x) = \sum \hat{f}_k \varphi_k(x)$

$$\boxed{\| f \|^2 = \sum |\hat{f}_k|^2 \, \| \varphi_k \|^2}$$

Examples:



1) $f = x$ on $[-\pi, \pi]$

$$f = \sum_{k \neq 0} \frac{(-1)^k}{ik} e^{ikx} \Rightarrow \|f\|^2 = \frac{2}{3}\pi^3 = 4\pi \sum_{1}^{\infty} \frac{1}{k^2} = \sum |\hat{f}_k|^2$$

$$\Rightarrow \text{Euler } \boxed{\sum_{1}^{\infty} \frac{1}{k^2} = \pi^2/6}$$

2) From expansion of x^2 on $[-\pi, \pi]$ $\Rightarrow$

$$\boxed{\sum_{1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{\dots}}$$

Values of Riemann zeta f-n: $\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s}$

Many appications

* Math (analysis: harmonic, complex; number theory). Probabity, statistics
* Time series & imaging
* spectral methods (computing)
* Differential eq-ns

Completness & convergence

Basic questions:

(I) Do partial sums $S_n(f) = \sum_{-n}^{n} \hat{f}_k e^{ikx}$ converge to $f(x)$, as $n \to \infty$?

(II) Do other "regularization/resummation" procedures give "convergent truncations"?

Answer:

(I) $S_n(f) = K_n * f$ – convolution with Dirichlet kernel: $K_n(x) = \frac{1}{2\pi} \sum_{-n}^{n} e^{ikx} = \frac{\sin(n+\frac{1}{2})x}{\sin x/2}$

(II) Abel means: $\{a_n\} \Rightarrow \{b_n = \frac{1}{n} \sum_{1}^{n} a_k\}$ – Better convergence

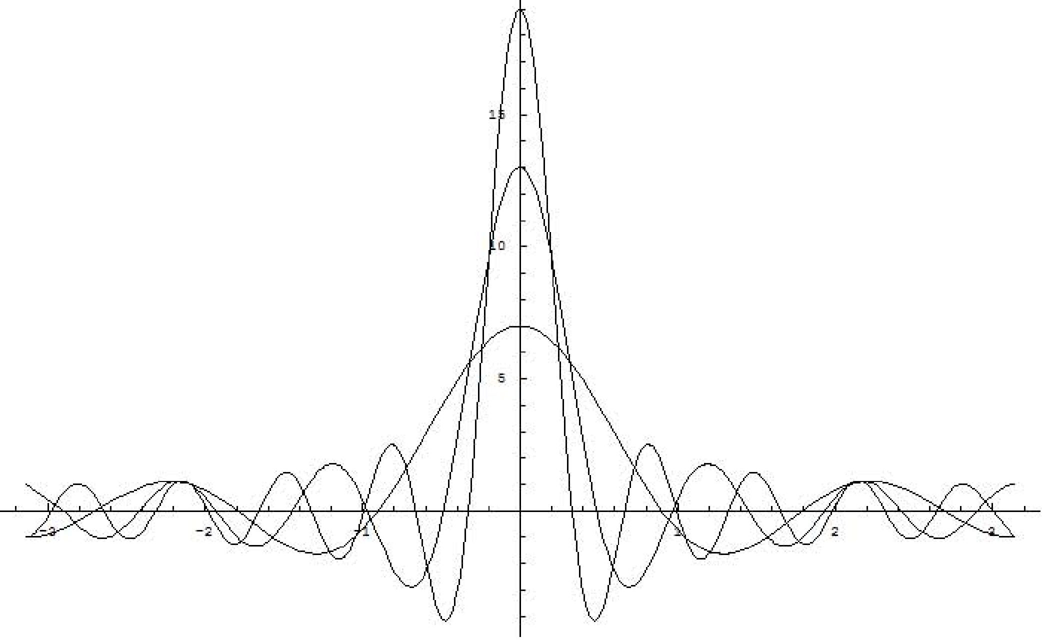
$$F_n(x) = \frac{1}{n} \sum_{m=1}^{n} K_m = \frac{1}{2\pi n} \left[\frac{\sin(\frac{n}{2} x)}{\sin \frac{x}{2}} \right]^2$$

Thm: Trig. polynomials $F_n * f \underset{n \to \infty}{\longrightarrow} f(x)$

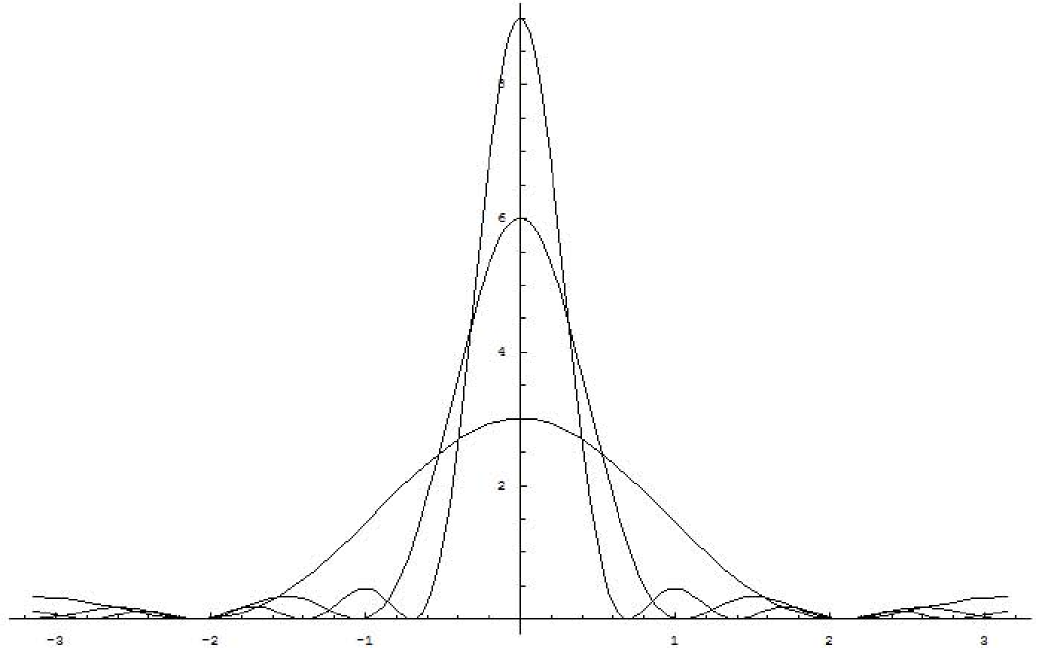
(i) uniformly for any contin. $f(x)$

(ii) in L^2-norm: $\|f - F_n * f\|^2 \le \|f - S_n(f)\|^2 \underset{n \to \infty}{\longrightarrow} 0$

Discontinuities of $f(x)$ create Gibbs phenomena overshoot in partial sums.



Dirichlet kernel: n=3,6,9



Feier kernel: n=3,6,9