Flow passed obstacle

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1 Inviscid ideal fluid

A uniform velocity U of inviscid fluid passed an obstacle D creates a potential flow $\vec{u} = \nabla\phi$. Function ϕ is harmonic in the exterior region D (incompressibility), and has zero normal derivative on boundary Γ (tangent boundary flow)

$$\begin{aligned} \Delta\phi &= 0 \\ \partial_n \phi|_\Gamma &= 0 \end{aligned} \tag{1}$$

Clearly, potential ϕ is made of two terms: $\phi = U \cdot \vec{r} + \phi'$, where the harmonic correction ϕ' should balance the boundary value of the base uniform flow

$$\partial_n \phi' + U \cdot N\big|_\Gamma = 0$$

One could find such ϕ' by solving the exterior Neumann problem for the Laplacian. Another approach is to take complex potential (for incompressible-irrotational flow) $f = \phi + i\psi$. We consider a few examples.

1.1 2D-flow passed cylinder

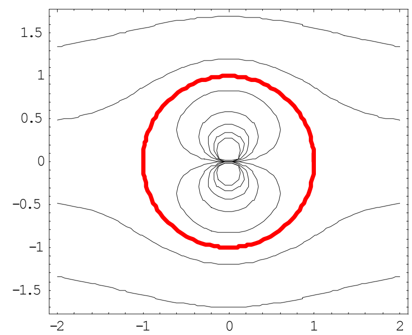
Here region D is disk $\{|z| \le a\}$, we have complex potential $f(z) = \phi + i\psi$ - a holomorphic function of $z = x + iy$, subject to $\operatorname{Im} f|_\Gamma = 0$, and $f \approx Uz$ at ∞. The obvious choice is $f = U\left(z + \frac{a^2}{z}\right)$, i.e.

$$\phi = Ux\left(1 + \frac{a^2}{r^2}\right);\ \psi = Uy\left(1 - \frac{a^2}{r^2}\right)$$

where $r = |z|$. The corresponding velocity

$$\vec{u}(\vec{r}) = U + \frac{a^2}{r^2}\left(U - \frac{U \cdot \vec{r}}{r^2}\vec{r}\right)$$

on the boundary $|\vec{r}| = a$ is twice the tangent component of U at $\vec{r}$. When formally extended inside the disk, the stream field of a cylinder takes on the same form as the vortex dipole, so the cylinder flow in an ideal fluid is equal to the vortex dipole flow.

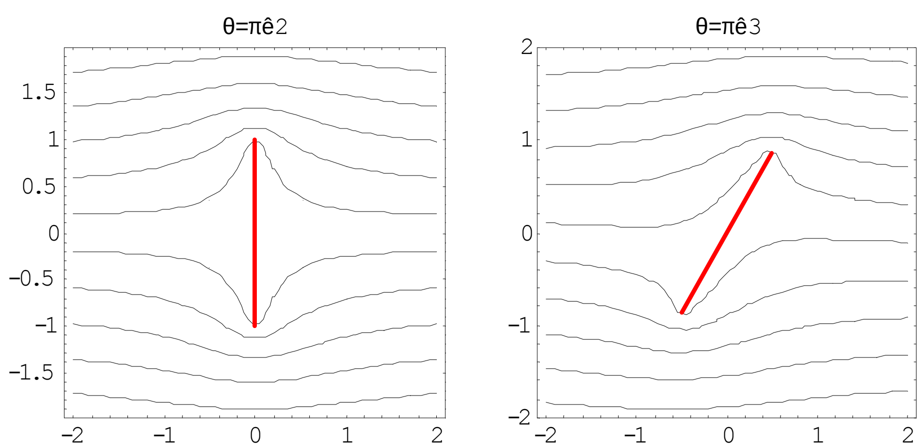


1.2 General 2D profile

As above we need a holomorphic correction $f(w)$ for exterior of B, with prescribed imaginary part $\operatorname{Im} f = -U \operatorname{Im} w$ on its boundary Γ. We have done it on the unit disk. The simplest approach is to take a conformal map $w = g(z)$ from the exterior of B onto exterior of D, so that $g(z) \approx z$ at ∞, and get the body potential from the disk potential (above), $f_B(z) = f_D(g(z))$. Thus for a plate at an angle of attack $a = e^{i\theta}$ function $w = g(z) = z \pm \sqrt{z^2 - a^2}$. Such g, e.g. for $a = 1$ maps the upper and lower branches of the interval $-1 < z < 1$ onto the upper/lower semicircle $w = e^{\pm i\theta}$. Hence the plate potential

$$f = \frac{U}{2}\left(z \pm \sqrt{z^2 - a^2} + \frac{1}{z \pm \sqrt{z^2 - a^2}}\right)$$

plotted below



1.3 3D ball

Here exterior harmonic potential

$$\phi = \vec{U} \cdot \vec{r} \left(1 + \frac{a^3}{2r^3}\right) \tag{2}$$

- a consequence of two formulae

$$\begin{aligned} \Delta \left[f U \cdot \vec{r}\right] &= \left(f'' + \frac{4}{r} f'\right) U \cdot \vec{r} \\ \partial_r \left[f U \cdot \vec{r}\right] &= \left(f' + \frac{1}{r} f\right) U \cdot \vec{r} \end{aligned}$$

for any radial function $f = f(r)$. The general solution of the former (Laplace's) equation is $f = C_1 + C_2 r^{-3}$, and the latter (boundary condition at $r = a$) gives coefficients $C_1 = 1, C_2 = \frac{1}{2} a^3$. The corresponding velocity

$$\vec{u} = U + \frac{a^3}{2r^3} \left[U - 3 \frac{U \cdot \vec{r}}{r^2} \vec{r}\right]$$

On the boundary velocity $\vec{u}(\vec{r}) = 3/2 \times$ tangent (to the ball) component of U at point $\vec{r}$. Field $\vec{u}$ has also a stream-field, given by the vector-potential

$$\psi = \tfrac{1}{2} \left(1 + \frac{a^3}{r^3}\right) U \times \vec{r}$$

which follows from the general formula $\nabla \times (f \, U \times \vec{r}) = (r f' - 2f) \, U - \left(\frac{f' U \cdot \vec{r}}{r}\right) \vec{r}$.

1.4 Moving ball

We take a ball of radius a moving throuth the fluid with velocity $\boldsymbol{u} = (u, 0)$, and creating a potential flow with velocity field $\boldsymbol{v}$

$$\begin{aligned} \phi &= -\frac{a^3}{2r^2} \boldsymbol{u} \cdot \boldsymbol{n} \\ \boldsymbol{v} &= \nabla \phi = \frac{a^3}{2r^3} \left(3 \boldsymbol{u} \cdot \boldsymbol{n} \, \boldsymbol{n} - \boldsymbol{u}\right) \end{aligned}$$

Here $\boldsymbol{n} = \vec{r}/r$ denotes a unit orth in the direction $\vec{r}$. The pressure on a moving ball

$$p = p_0 - \rho \phi_t - \tfrac{\rho}{2} v^2 \tag{3}$$

We compute both terms ϕ_t and v^2. The potential is advected by the ball-velocity $\boldsymbol{u}$, so

$$\phi_t = \frac{\partial \phi}{\partial u} \cdot \dot{\mathbf{u}} - \boldsymbol{u} \cdot \nabla \phi = \frac{a^3}{2r^2} \left(-\dot{\mathbf{u}} \cdot \boldsymbol{n} + \frac{u^2 - 3 \left(\mathbf{u} \cdot \mathbf{n}\right)^2}{r}\right)$$

while

$$\tfrac{v^2}{2} = \frac{a^6}{8r^6}\left(3\left(\boldsymbol{u}\cdot\boldsymbol{n}\right)^2 + u^2\right)$$

Combining two terms and evaluating (3) on the surface of the sphere $r = a$ we get

$$p = p_0 + \tfrac{\rho u^2}{2}\left(9\cos^2\theta - 5\right) + \tfrac{\rho a}{2}\boldsymbol{n}\cdot\dot{\mathbf{u}}$$

where θ is the angle between $\boldsymbol{u}$ and $\vec{r}$. Two terms are thus contribute to the pressure, one proportional to u^2, another one proportional to acceleration $\dot{u}$. However, integrated over the entire surface the net drag and lift in the ideal fluid are zero.

Remark 1 *Exterior harmonic functions are convienienly expanded in spherical harmonics*

$$u = \sum_{l=0}^{\infty} c_l \frac{Y_l}{r^{l+1}} \tag{4}$$

For instance, the exterior Dirichlet problem $u|_{\Gamma} = f$ *on surface* Γ *parametrized by radius* $r = \rho(\phi,\theta)$ *could be transplanted on the unit sphere by writing (4) with unknown coefficients* $\{c_l\}$*, then evaluating* u *on* Γ*, and expanding in* $\{Y_l\}$

$$f = \sum_{l=0}^{\infty} c_l \frac{Y_l}{\rho^{l+1}} = \sum_{m=0}^{\infty} \hat{f}_m Y_m$$

Using orthogonality of $\{Y_l\}$ *we get a linear system for unknown coefficients* $\{c_l\}$ *in terms of the known "Fourier" coefficients* $\left\{\hat{f}_m\right\}$ *and the* L^2*-inner products*

$$\sum_{l=0}^{\infty}\left\langle \frac{Y_l}{\rho^{l+1}} \middle| Y_m \right\rangle c_l = \hat{f}_m$$

2 Viscous drag on the ball (Stokes)

In case of viscous fluid the stationary N-S equation (in the body moving frame) takes the form

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\frac{\nabla p}{\rho} + \nu\Delta\boldsymbol{v}$$

$\boldsymbol{v} \approx \boldsymbol{u}$ -body velocity at ∞.

For small Reynolds, $\mathrm{Re} = O\left(\frac{v\cdot\nabla v}{\nu\Delta v}\right) \ll 1$, the nonlinear term could dropped, so $-\frac{\nabla p}{\rho} + \nu\Delta\boldsymbol{v} = 0$. Taking curl we get a harmonic vorticity-field $\zeta = \nabla\times\boldsymbol{v}$, $\Delta\zeta = 0$.

As above $\boldsymbol{v}$ has stream-field of the form $\psi = g(r)\, \boldsymbol{u} \times \vec{r}$, due rotational symmetry, $\boldsymbol{u}$-axi-symmetry and linear dependence of $\boldsymbol{v}$ (resp. ψ) on $\boldsymbol{u}$. Indeed, ψ itself could be chosen solenoidal, $\psi = \nabla \times (f\boldsymbol{u})$ with radial f, so that $g = \frac{f'}{r}$.

Since $\zeta = \Delta\psi$, we get a biharmonic stream $\Delta^2\psi = 0$, hence a biharmonic scalar (radial) component $\Delta^2 f = 0$.

We solve biharmonic equation in two steps: $h = \Delta f$, $\Delta h = 0$. The general solution of the latter is $h = a + \frac{2b}{r}$. Coefficient $a = 0$, as the flow should vanish at ∞, then we get $\Delta f = f'' + \frac{2}{r} f' = \frac{2b}{r}$, which yields

$$\begin{aligned} f &= br + \frac{c}{r} \\ \boldsymbol{v} &= \boldsymbol{u} + \nabla \times \nabla \times (f\boldsymbol{u}) = \boldsymbol{u} - b\frac{\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{n})\ \boldsymbol{n}}{r} + c\frac{3\,(\boldsymbol{u}\cdot\boldsymbol{n})\ \boldsymbol{n} - \boldsymbol{u}}{r^3} \end{aligned}$$

Coefficients b, c are determined from the boundary condition $\boldsymbol{v}|_{r=a} = 0$ (no boundary slip). Hence $b = \frac{3a}{4}$, $c = \frac{a^3}{4}$, and we get function f and velocity $\boldsymbol{v}$

$$\begin{aligned} f &= \frac{3a}{4} r + \frac{a}{4r} \qquad (5) \\ \boldsymbol{v} &= \boldsymbol{u} - \frac{3a}{4}\frac{\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{n})\ \boldsymbol{n}}{r} + \frac{a^3}{4}\frac{3\,(\boldsymbol{u}\cdot\boldsymbol{n})\ \boldsymbol{n} - \boldsymbol{u}}{r^3} \end{aligned}$$

The flow is laminar, but not potential, as its vorticity

$$\zeta = \Delta f\ \boldsymbol{u} \times \vec{r} = \frac{3a}{2}\frac{\boldsymbol{u} \times \vec{r}}{r} \neq 0.$$

We can write $\boldsymbol{v}$ in polar form (v_r, v_θ) (using $\boldsymbol{u}$ as principal direction) as

$$\begin{aligned} v_r &= u\cos\theta \left[1 - \tfrac{3a}{2r} + \tfrac{a^3}{2r^3}\right] \qquad (6) \\ v_\theta &= -u\sin\theta \left[1 - \tfrac{3a}{4r} + \tfrac{a^3}{4r^3}\right] \end{aligned}$$

The pressure is computed from the linearized momentum equation $-\nabla p + \eta\Delta\boldsymbol{v} = 0$, ($\eta = \rho\nu$ -dynamic viscosity) given that

$$\boldsymbol{v} = \nabla \times \nabla \times (f\,\boldsymbol{u}) = (\boldsymbol{u}\cdot\partial)\nabla f - \Delta f\,\boldsymbol{u}$$

hence $\Delta\boldsymbol{v} = \nabla(\boldsymbol{u}\cdot\partial f)$. So we get

$$p = p_0 + \eta\,(\boldsymbol{u}\cdot\partial)\,\Delta f = p_0 - \tfrac{3a\eta}{2}\frac{\boldsymbol{u}\cdot\boldsymbol{n}}{r^2}$$

The drag-buoyancy force is obtained by integrating surface forces (pressure plus viscous stresses) over the surface

$$F_i = \oint \left\{-pn_i + \sum \sigma_{ij} n_j\right\} dS$$

Using polar form of the stress-tensor

$$
\begin{aligned}
\sigma_{rr} &= 2\eta \frac{\partial v_r}{\partial r} \\
\sigma_{r\theta} &= \eta \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)
\end{aligned}
$$

The substitution of the (6) gives

$$
\begin{aligned}
\sigma_{rr} &= 3\eta u \left(\frac{a}{r^2} - \frac{a^3}{r^4} \right) \cos\theta = 0, \text{ for } r = a \\
\sigma_{r\theta} &= ... = -\tfrac{3\eta u}{2a} \sin\theta, \text{ for } r = a
\end{aligned}
$$

The off-shot is zero buoyancy and drag -the Stokes formula

$$
F_d = \tfrac{3\eta u}{2a} \oint dS = 6\pi a \eta u
$$