Conformal maps, Green's function and Poisson kernel.

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1 The reflection method for half-space problems

To get Green's functions (for Laplaces; heat; wave) on the half-space $\mathbb{R}_+^{n+1} = \{z = (x, y) : y > 0\}$ we reflect the free-space source $K_0(z - w)$ about the boundary plane

$$w = (\xi, \eta) \to w^* = (\xi, \eta)$$

and take a suitable combination of $K(z, w)$ and $K(z, w^*)$, i.e. use odd/even extensions of K_0 in the vertical variable y,

$$\begin{array}{ll} K(z, w) = K_0(z - w) - K_0(z - w^*) & \text{Dirichlet} \\ K(z, w) = K_0(z - w) + K_0(z - w^*) & \text{Neumann} \end{array}$$

In a similar way one handles the heat and wave propagators, e.g.

$$G(z, w; t) = G_0(z - w; t) - G_0(z - w^*; t)$$

in terms of the free-space Gaussian G_0.

The corresponding Poisson kernels are computed from $K; G; ...$ via standard relations, e. g.

$$\begin{array}{ll} P(z, w) = \partial_n K(z, w)|_{w \in \Gamma} = \partial_\eta K|_{\eta=0} & \text{Dirichlet} \\ K(z, w) = K(z, w)|_\Gamma = K|_{\eta=0} & \text{Neumann} \end{array}$$

In particular for spherically-symmetric problem, like Laplacian/Helmholtz potentials $K_0 = K_0(r)$; $r = |z - w|$ we get

$$P(x - \xi, y) = -\frac{2y}{r} K_0'(r)\,; \text{with } r = \sqrt{(x - \xi)^2 + y^2}$$

In special cases 2D; 3D it yields familiar expressions, that could be also produced by the Fourier transform

$$\begin{array}{ll} P = \frac{y}{\pi\left[(x-\xi)^2 + y^2\right]} & \text{2D} \\ P = \frac{y}{2\pi\left[(x-\xi)^2 + y^2\right]^{3/2}} & \text{3D} \\ P = \frac{2(n-2)y}{\omega_{n-1}\left[(x-\xi)^2 + y^2\right]^{n/2}} & \text{nD} \end{array}$$

ω_{n-1}-area of the unit sphere S^{n-1} in $\mathbb{R}^n$.

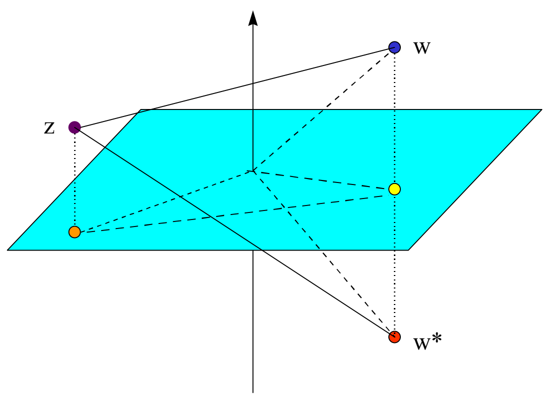


Figure 1: Source and reflected source

Exercise 1 *1. Show that* $P(x-\xi, y) \to \delta(x-\xi)$, *as* $y \to \infty$

2. Derive Poisson propagators for the heat and wave problems

1.1 Examples of Green's functions for the Laplacian.

- The free space Green's function in $\mathbb{R}^3$ is given by the Newton-Coulomb potential (single point charge): $K_0 = \frac{1}{4\pi r}$,
- The half-space Green's function is given by the dipole-potential - a pair of opposite point charges:

$$K(x, y, a) = K_0(x-a, y) - K_0(x+a, y)$$

- The quadrant Green's function is given by the quadrupole potentials,

$$\begin{aligned} & K_N(x, y, \xi, \eta) \\ = \; & K_0(x-\xi, y-\eta) + K_0(x-\xi, y+\eta) + K_0(x+\xi, y-\eta) + K_0(x+\xi, y+\eta) \end{aligned}$$

  corresponds to Dirichlet boundary conditions while Neumann-

$$\begin{aligned} & K_D(x, y, \xi, \eta) \\ = \; & K_0(x-\xi, y-\eta) - K_0(x+\xi, y-\eta) - K_0(x-\xi, y+\eta) + K_0(x+\xi, y+\eta) \end{aligned}$$

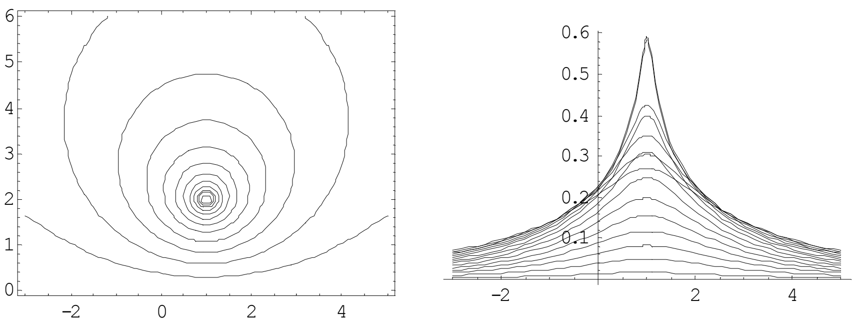


Figure 2: Half-space Green

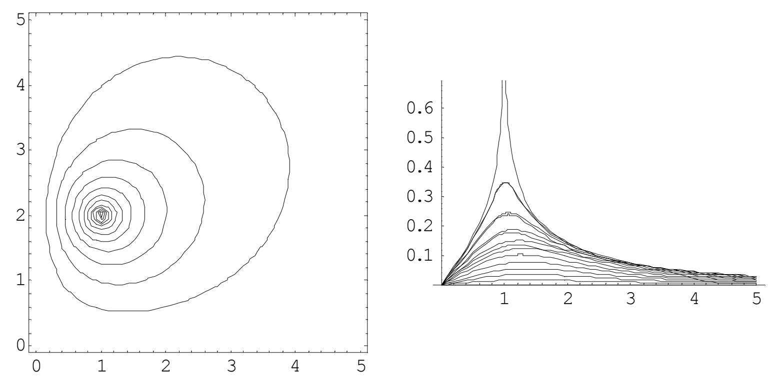


Figure 3: Quadrant (Dirichlet) Green

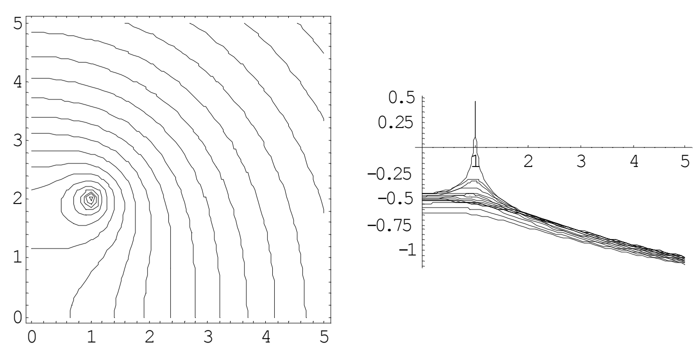


Figure 4: Quadrant Neumann Green

2 Conformal coordinate change in Laplacian

We take map $\phi : x \to y = \phi(x)$ in $\mathbb{R}^n$(or a region in $\mathbb{R}^n$), and denote by $A(x) = \phi'(x)$ its Jacobian matrix, and by $J = \det(A)$ - the Jacobian determinant. The general change of variable formula for the Laplacian has the form

$$\Delta_y = \tfrac{1}{J}\nabla_x \cdot J\left({}^T AA\right)^{-1}\nabla_x \tag{1}$$

-a consequence of the grad and div-transformations

$$\begin{array}{rcl} x & \to & y = \phi(x) \\ \nabla_x & \to & \nabla_y = \left({}^T A\right)^{-1}\nabla_x \\ \nabla_x \cdot ... & \to & \nabla_y \cdot ... = \tfrac{1}{J}\nabla_x\left(JA^{-1}\cdot ...\right) \end{array}$$

The *orthogonal map* ϕ has orthogonal column-vectors in the Jacobian matrix

$$\begin{array}{c} A = \left[a_1\vec{u}_1; a_2\vec{u}_2; ...; a_n\vec{u}_n\right]; \\ \vec{u}_i \cdot \vec{u}_j = \delta_{ij} \end{array}$$

Hence diagonal product $\left({}^T AA\right) = diag\{a_1^2; ...a_n^2\}$ and $J = a_1...a_n$ in (1).

The standard example are spherical coordinates in $\mathbb{R}^2; \mathbb{R}^3$ etc., where

$$\begin{array}{rcl} \phi & : & \begin{pmatrix} r \\ \theta \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \\ A & = & \begin{bmatrix} \cos & -r\sin \\ \sin & r\cos \end{bmatrix}; \left({}^T AA\right) = \begin{bmatrix} 1 & \\ & 1/r^2 \end{bmatrix}; J = r \end{array}$$

hence the resulting spherical Laplacian

$$\frac{1}{r}\begin{pmatrix} \partial_r & \partial_\theta \end{pmatrix}\begin{bmatrix} r & \\ & 1/r \end{bmatrix}\begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix} = \frac{1}{r}\partial_r r\partial_r + \frac{1}{r^2}\partial_\theta^2$$

Conformal maps ϕ have *conformal* Jacobian matrix $A = \rho U$ - "scalar times orthogonal", so they make up a subclass of orthogonal transformations, and their determinant $J = \rho^n$. The resulting Laplacian

$$\Delta = \tfrac{1}{\rho^n}\nabla \cdot \rho^{n-2}\nabla \tag{2}$$

2.1 Radial conformal map and harmonic functions

We are looking for a change of variables (dependent and independent) $u(y) \to \rho^\alpha(u \circ \phi) = v(x)$, that would take any harmonic function $u(x)$ into another harmonic $v(x)$. Substitution into (2) yields,

$$\begin{array}{rl} & \tfrac{1}{\rho^n}\nabla \cdot \rho^{n-2}\nabla\left(\rho^\alpha u \circ \phi\right) \\ = & \rho^{-2+\alpha}\left\{\nabla^2 u + (n-2+2\alpha)\dfrac{\nabla\rho}{\rho}\cdot\nabla u + \right. \\ & \left. +\alpha\left(\nabla\cdot\left(\dfrac{\nabla\rho}{\rho}\right) + (n-2+\alpha)\left|\dfrac{\nabla\rho}{\rho}\right|^2\right)u\right\} \end{array} \tag{3}$$

To get the Laplace's equation for v the first and 0-th order terms of (3) must vanish. Hence, $\alpha = -\frac{n-2}{2}$and the conformal factor ρ solves a nonlinear PDE,

$$\nabla \cdot \left(\frac{\nabla \rho}{\rho}\right) + (n-2+\alpha)\left|\frac{\nabla \rho}{\rho}\right|^2 = 0$$

We rewrite it for the log-derivative of ρ as,

$$\Delta (\ln \rho) + \frac{n-2}{2} |\nabla (\ln \rho)|^2 = 0 \tag{4}$$

The latter could be explicitly solved when ρ, hence $\psi = \ln \rho$ are radial functions $\psi = \psi(r)$. Indeed, (4) becomes a radial ODE for ψ,

$$r^{1-n} \left(r^{n-1}\psi'\right)' + \tfrac{n-2}{2} \left(\psi'\right)^2 = 0 \tag{5}$$

transformed via change of variable $r^{n-1}\frac{d}{dr} = \frac{d}{dz}$ into $\psi'' + \frac{n-2}{2}\left(\psi'\right)^2 = 0$, and solve by separation. The off-short is a general solution of (5) in the form

$$\psi = C_1 \left(C_2 + r^{2-n}\right)^{2/(n-2)} \tag{6}$$

A special family of solutions vanishing at ∞ is given by $\boxed{\psi = \frac{C}{r^2}}$

2.2 Inversion

Let us note that $\rho = \frac{1}{r^2}$ is precisely the conformal factor of the *inversion map*, $\phi : x \to \frac{x}{|x|^2}$. Indeed, its Jacobian-matrix

$$A = \phi' = \tfrac{1}{|x|^2} \left\{\delta_{ij} - 2\frac{x_i x_j}{|x|^2}\right\} = r^{-2} U$$

where U is an orthogonal *reflection*-matrix about the normal hyper-plane to x in $\mathbb{R}^n$ (fig.5). In fact, one could show that a radial map $x \to \phi(r)\, x$ is conformal, iff scalar factor $\phi = \frac{1}{r^2}$, so inversion is the only possibility.

We have thus shown, that any harmonic function $u(x)$ gives rise to another harmonic function $u^*(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right)$.

2.3 Green's function in the ball

Clearly, both u and u^* take on the same value on the unit sphere $S = \{|x| = 1\}$, so inversion ϕ transforms harmonic functions $\{u\}$ in the interior of the unit ball $B = \{|x| \le 1\}$ to harmonic functions $\{u*\}$ in the exterior $\{|x| \ge 1\}$. Applying **??** to the Newton potential $u = K_0(|x-\xi|)$ - the free-space Green's function, we get the requisite harmonic correction to K_0 in the unit ball B,

$$u^* = |x|^{2-n} K_0 \left(|x^* - \xi|\right); \, x^* = \frac{x}{|x|^2} \tag{7}$$

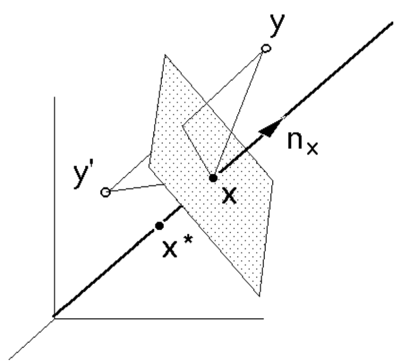


Figure 5: Jacobian matrix of the inversion is the reflection about the normal plane to x

Hence, remembering that $K_0 = \frac{1}{c_n|x-\xi|^{n-2}}$ (constant c_n is the surface area of the unit sphere in $\mathbb{R}^n$) we get the Green's function of the ball

$$\begin{aligned} K(x,\xi) &= K_0(|x-\xi|) - K_0(|(\hat{x} - |x|\,\xi)|) \\ &= K_0(|x-\xi|) - K_0\left(\left|\left(\hat{\xi} - |\xi|\,x\right)\right|\right) \end{aligned} \quad (8)$$

Here $\hat{x} = \frac{x}{|x|}$ and $\hat{\xi} = \frac{\xi}{|\xi|}$ denote normalized (unit) vectors in the direction x and ξ. It is often convenient to rewrite (8) in the polar form, i.e. variables $r = |x|$, $\rho = |\xi|$ and angle θ between x and ξ (see fig.6),

$$K(r,\rho;\theta) = K_0\left(\sqrt{r^2+\rho^2-2r\rho\cos\theta}\right) - K_0\left(\sqrt{1+(r\rho)^2-2r\rho\cos\theta}\right) \quad (9)$$

2.4 Special cases.

2.4.1 2-D case:

Here $K_0 = \frac{1}{2\pi}\ln|z-w|$, complex variables $z, w \in \mathbb{C}$ being used in place of x, ξ. The prefactor $|z|^\alpha = 1$ in (7), hence (8-9) takes the form

$$K(z,w) = \tfrac{1}{2\pi}\ln\left|\frac{z-w}{1-z\bar{w}}\right| = \tfrac{1}{2\pi}\ln\sqrt{\frac{r^2+\rho^2-2r\rho\cos\theta}{1+(r\rho)^2-2r\rho\cos\theta}} \quad (10)$$

in polar coordinates $z = re^{i\theta}$,$w = \rho e^{i\phi}$.Exterior disk Green's function has the same form (10), but this time r, ρ lie outside the unit circle

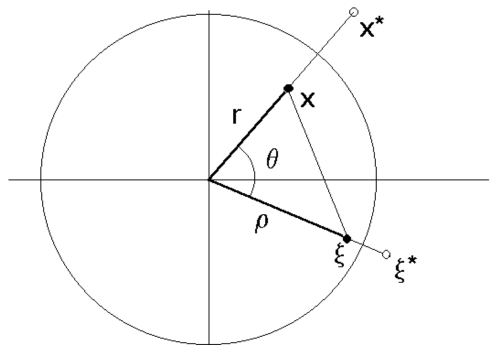


Figure 6:

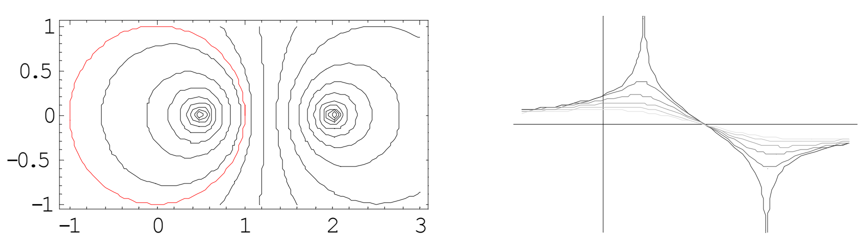


Figure 7: Interior and exterior disk Green's function

2.4.2 3-D case:

Here free-space $K_0 = \frac{1}{4\pi|x-\xi|}$, hence

$$\begin{aligned} K(x;\xi) &= \tfrac{1}{4\pi}\left\{\frac{1}{|x-\xi|} - \frac{1}{|(\hat{x}-|x|\,\xi)|}\right\} \\ &= \tfrac{1}{4\pi}\left\{\frac{1}{\sqrt{r^2+\rho^2-2r\rho\cos\theta}} - \frac{1}{\sqrt{1+(r\rho)^2-2r\rho\cos\theta}}\right\} \end{aligned}$$

2.5 Poisson kernels

For the Dirichlet boundary condition the Poisson kernel is given by normal derivative of the Green's function, $P(x,\xi) = n_\xi \cdot \nabla_\xi K = \frac{\partial}{\partial\rho} K$We observe that on the boundary $|\xi| = 1$, $n_\xi = \xi$. Hence,

$$\begin{aligned} P(x,\xi) &= \frac{(\xi-x)\cdot\xi}{c_n\,|\xi-x|^n} - \frac{(\xi-x^*)\cdot\xi}{c_n\,|x|^{n-2}\,|\xi-x^*|^n} \\ &= \frac{(\xi-x)\cdot\xi}{c_n\,|\xi-x|^n} - \frac{|x|^2 - x\cdot\xi}{c_n\,|(|x|\,\xi-\hat{x})|^n} \end{aligned} \tag{11}$$

It remains to note that both denominators in (11) are equal, so we get Poisson kernel,

$$P = \frac{1-|x|^2}{c_n\,|\xi-x|^n} = \frac{1-r^2}{c_n\,(1+r^2-2r\cos\theta)^{n/2}} \tag{12}$$

From the unit ball one can easily pass to an arbitrary radius a,

$$P = \frac{a^2-r^2}{c_n\,(r^2+a^2-2ra\cos\theta)^{n/2}} \tag{13}$$

Another way to derive (12) is via differentiation of the polar form (9) in variable ρ at $\rho = 1$,

$$P(r,1;\theta) = \left.\frac{\partial K}{\partial\rho}\right|_{\rho=1} = K'\left(\sqrt{1+r^2-2r\cos\theta}\right)\frac{1-r^2}{\sqrt{1+r^2-2r\cos\theta}} = \ldots$$

In special cases we get

Green's functions and Poisson kernels in 2D and 3D balls

	K_0	K	P
2D	$-\frac{1}{2\pi}\ln r$	$\frac{1}{2\pi}\ln\sqrt{\frac{r^2+\rho^2-2r\rho\cos\theta}{1+(r\rho)^2-2r\rho\cos\theta}}$	$\frac{1-r^2}{2\pi(1+r^2-2r\cos\theta)}$
3D	$\frac{1}{4\pi r}$	$\frac{1}{4\pi}\left\{\frac{1}{\sqrt{r^2+\rho^2-2r\rho\cos\theta}} - \frac{1}{\sqrt{1+(r\rho)^2-2r\rho\cos\theta}}\right\}$	$\frac{1-r^2}{4\pi(1+r^2-2r\cos\theta)^{3/2}}$

2.6 Other examples

2.6.1 Half-disk

Green's function and the Poisson kernel in the upper half-disk are obtained by a combination of the reflection and inversion:

$$K(z,\zeta) = \tfrac{1}{2\pi}\left\{\ln\left|\frac{z-\zeta}{1-z\bar{\zeta}}\right| - \ln\left|\frac{z-\bar{\zeta}}{1-z\zeta}\right|\right\}$$

or in polar form $z = re^{i\theta}$, $\zeta = \rho e^{i\phi}$

$$K = \tfrac{1}{2\pi}\left\{\ln\sqrt{\frac{r^2+\rho^2-2r\rho\cos(\theta-\phi)}{1+(r\rho)^2-2r\rho\cos(\theta-\phi)}} - \ln\sqrt{\frac{r^2+\rho^2-2r\rho\cos(\theta+\phi)}{1+(r\rho)^2-2r\rho\cos(\theta+\phi)}}\right\}$$

The corresponding Poisson kernel is also obtained by reflection of (12)

$$P(r;\phi,\theta) = \tfrac{1}{2\pi}\left\{\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} - \frac{1-r^2}{1+r^2-2r\cos(\theta+\phi)}\right\}$$

on the upper (semicircular) boundary and

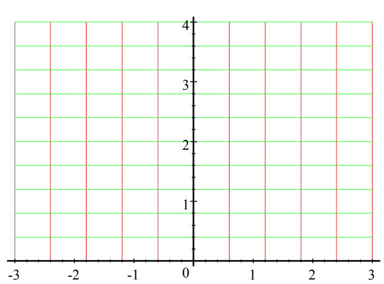
$$P(r;\rho,\theta) = \tfrac{r\sin\theta}{\pi}\left\{\frac{1}{r^2+\rho^2-2r\rho\cos\theta} - \frac{1}{1+(r\rho)^2-2r\rho\cos\theta}\right\}$$

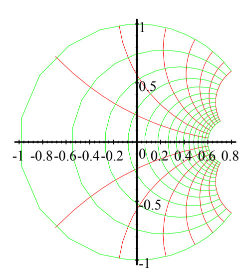
on the lower (flat) boundary.

2.6.2 General 2-D regions in complex plane.

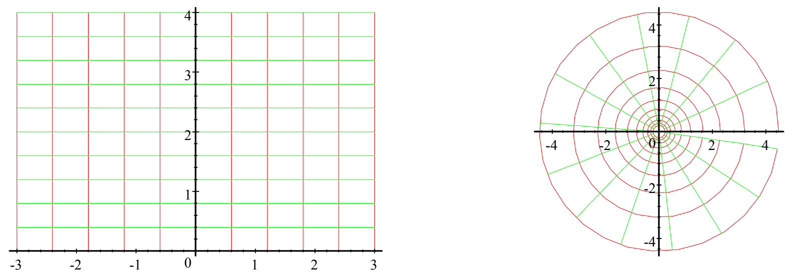
By the general *Riemann mapping Theorem* of complex analysis any planar region $D \subset \mathbb{C}$ can be conformally mapped onto any other such region, e.g. onto the unit disk $\mathbb{D} = \{|z| < 1\}$ or a half-plane $\mathbb{H} = \{\mathrm{Im}(z) > 0\}$.

Example 2 *Map* $\phi : z \to w = \frac{z-i}{z+i}$ *takes a half-plane* $\{\mathrm{Im}(z) < 0\}$ *onto the unit disk* $\{|w| < 1\}$, *and has a conformal factor* $\rho = |\phi'(z)| = \frac{2}{|z+i|^2}z$

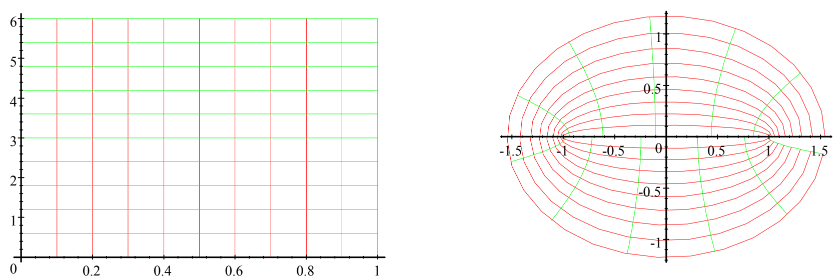




Example 3 *Map $\phi : z \to e^z$ takes a complex strip $\{0 \le \mathrm{Im}(z) \le 2\pi\}$ onto the entire plane $\mathbb{C}$.*



Example 4 *Map $w = \cosh(z)$ takes a coordinate rectangle $\{0 \le x \le a; 0 \le y \le 2\pi\}$ in the z-plane onto an ellipse in the w-plane (elliptical coordinates).*



Conformal maps allow to transfer Green's functions from one region to another. Namely, we let map $z \to w = \phi(z)$ take region D (with the known Green's function $K_D(z, \zeta)$) onto region M, whose Green's function K_M is to be computed. Assuming the boundary of D mapped onto the boundary of M, two Laplacians are related via $\Delta_M [u] \circ \phi = \frac{1}{J} \Delta_D [u \circ \phi]$, where $J = \rho^2$denotes the Jacobian determinant. It follows then that the Green's functions of two regions are related by a change of variables

$$K_M(w, \eta) = K_D\left(\phi^{-1}(w), \phi^{-1}(\eta)\right)$$

In the above examples map $w = \frac{z-i}{z+i}$ from the half-plane $\mathbb{H}$ onto the disk $\mathbb{D}$. The half-space Green's function, computed by the reflection (complex conjugation) has the form: $K = \frac{1}{2\pi} \ln \left| \frac{z-\zeta}{z-\bar{\zeta}} \right|$ Substitution of the inverse map $z = \phi^{-1}(w) = i\frac{w+1}{w-1}$ and $\zeta = \phi^{-1}(\eta) = \ldots$ yields

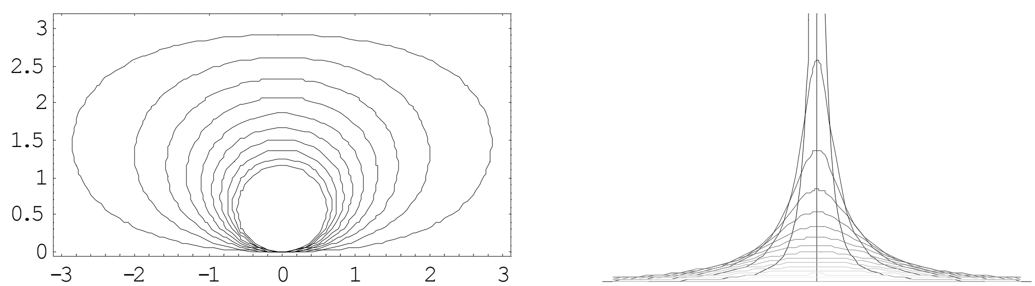
$$K_{\mathbb{D}}(w, \eta) = \frac{1}{2\pi} \ln \left| \frac{w - \eta}{1 - w\bar{\eta}} \right|$$

that was produced earlier by the inversion method.

2.6.3 Green and Poisson kernels on strip

1. . We use conformal map: $z \to e^z$ from strip $\{0 \leq \operatorname{Im} z \leq \pi\}$ onto half-space $\{\operatorname{Im} w \geq 0\}$. Then

$$
\begin{aligned}
G(z,w) &= -\frac{1}{2\pi}\ln\left|\frac{e^z - e^w}{e^z - e^{\bar{w}}}\right| = -\frac{1}{2\pi}\ln\left|\frac{\sinh\left(\frac{z-w}{2}\right)}{\sinh\left(\frac{z-\bar{w}}{2}\right)}\right| \\
&= -\frac{1}{4\pi}\ln\left\{\frac{\cosh^2\left(\frac{x-\xi}{2}\right) - \cos^2\left(\frac{y-\eta}{2}\right)}{\cosh^2\left(\frac{x-\xi}{2}\right) - \cos^2\left(\frac{y+\eta}{2}\right)}\right\} \\
&= -\frac{1}{4\pi}\ln\left\{\frac{\cosh(x-\xi) - \cos(y-\eta)}{\cosh(x-\xi) - \cos(y+\eta)}\right\} \\
P(z,\xi) &= \partial_\eta G(z,w)\big|_{\eta=0} = \frac{1}{2\pi}\frac{\sin y}{\cosh(x-\xi) - \cos(y)}
\end{aligned}
$$



Function $P(x,y)$ along with its contour map near zero, in strip $0 < y < \pi$

2.6.4 Problems

1. Show that complex map $w = \cosh(z)$ takes strip $\mathbb{P} = \{0 \leq \operatorname{Im}(z) \leq 2\pi\}$ onto the upper half-plane $\mathbb{H}$, so that the boundaries of $\mathbb{P}$:$\{\operatorname{Im}(z) = 0\}$ and $\{\operatorname{Im}(z) = 2\pi\}$ go into the boundary of $\mathbb{H}$. Use this map and the known Green's function of $\mathbb{H}$ to construct the Green's function and the Poisson kernel of $\mathbb{P}$.

2. Do the same exercise with the fractional power map $\phi(z) = z^{\alpha/\pi}$ that takes the upper half-plane onto a sector of angle α, and derive the Green's function and the Poisson kernel of the sector.

3 Applications to fluid flows

A 2D ideal (incompressible, inviscid) Euler flow is described by its stream-field ψ. When viewed as a function of complex variable $z = x + iy$, $\psi(z, \bar{z})$, the

Eulerian velocity $\boldsymbol{u} = \left(-\psi_y, \psi_x\right)$ is given by a complex derivative $\boldsymbol{u} = i\partial_{\bar{z}}\psi$. Irrotational flow $\nabla \times \boldsymbol{u} = 0$ corresponds to harmonic stream-field: $\Delta\psi = 0$.

Equivalently $\boldsymbol{u}$ has velocity potential $\boldsymbol{u} = \nabla\phi$. In fact, it could be described by a complex analytic potential $w(z) = \phi + i\psi$, and complex velocity $w'(z) = u - iv$. Indeed, the relations $\boldsymbol{u} = \nabla\phi = \nabla\psi^{\perp}$ are nothing but Cauchy-Riemann equations for w.

Let us also remark that steady Euler flow obeys the Bernoulli relation:

$$\tfrac{\rho}{2}\left|\boldsymbol{u}\right|^2 + p = \text{Const - the hydrostatic pressure} \tag{14}$$

where ρ an p are fluid density and pressure. The latter follows from the momentum conservation

$$\boldsymbol{u}_t + \nabla\left(\tfrac{\rho}{2}\left|\boldsymbol{u}\right|^2 + p\right) - \boldsymbol{u} \times \omega = 0$$

where $\omega = \nabla \times \boldsymbol{u}$ denotes vorticity of $\boldsymbol{u}$. We consider a few examples and basic problems of ideal fluid flows.

3.1 Potential flow passed an obstacle.

We shall study a incompressible potential flow passed an obstacle D, e.g. a cylinder, or sphere of radius a. It produced by a moving parallel flow passed the body, or the motion of the body in a quiescent fluid. In either case potential, laminar flow has stream-field ψ and potential ϕ - both harmonic functions in the exterior of D, $\Delta\psi = \Delta\phi = 0$ in $\mathbb{R}^n \backslash D;\ n = 2, 3$.

Depending on the reference frame one either fixes Γ and certain asymptotic flow at ∞ e.g. $\boldsymbol{u} \approx (U, 0)$ - (body frame), or allows moving Γ and zero ∞-flow (fluid frame). We shall use the latter as it permits non-stationary motion of D.

The body velocity U creates a boundary potential $\phi_0 = -U \cdot \vec{r}|_{\Gamma}$, to maintain the tangential flow along the boundary, i.e. $\psi|_{\Gamma} = \text{Const}$. The latter is extended through the entire $\mathbb{R}^n \backslash D$ via the exterior Poisson kernel $\phi = P_{\Gamma}[\phi_0]$. Any harmonic function ϕ could be expanded at ∞ as

$$\phi \approx \frac{A_0}{r} + \frac{A_1 \cdot \vec{r}}{r^3} + ... + \frac{A_n(\vec{r})}{r^{n+1}} + ... \tag{15}$$

with spherical harmonics $A_n(\vec{r})$ of degree n. For velocity potential ϕ coefficient $A_0 = 0$, since the net outward flux (at large $\vec{r}$) must be 0 (no sources).

Clearly, potential ϕ, and velocity $\boldsymbol{u} = \nabla\phi$ depend linearly on U

$$\begin{aligned} \phi &= A(\vec{r}) \cdot U \\ A_i &= P_{\Gamma}[x_i] \end{aligned} \tag{16}$$

the components A_i- represent harmonic extensions of linear coordinate functions on Γ.

To get the fluid reaction force on the body we compute the total (kinetic) energy of the fluid

$$E = \tfrac{\rho}{2}\iint \boldsymbol{u}^2 = \tfrac{\rho}{2}\sum m_{ij} U_i U_j \tag{17}$$

ρ - fluid density. The coefficients of the quadratic form $E = E(U)$ make up the so-called *virtual mass-tensor*

$$\begin{aligned} m_{ij} &= \rho \iint_{\mathbb{R}^n \setminus D} \nabla A_i(\vec{r}) \cdot \nabla A_j(\vec{r}) = \rho \oint_\Gamma A_i(\partial_n A_j) \\ &= \rho \oint_\Gamma x_i \partial_n P_\Gamma [x_j] = \rho \oint_\Gamma x_j \partial_n P_\Gamma [x_i] \end{aligned} \tag{18}$$

Definition (18) resembles the inertia tensor for a rotating solid of mass-density $\frac{\rho}{r}$ distributed over surface Γ. It's not however, equal $\rho \oint_\Gamma \frac{x_j x_i}{r}$, as Poisson P_Γ makes important difference. Though A_i coinsides with a linear function $x_i|_\Gamma$, its gradient ∇A_i and normal derivative $\partial_n A_j$ on $\Gamma$are not $\boldsymbol{e}_i$ and N_i (the i-the component of $\boldsymbol{n}$) respectively. A simple illustartion would be a disk $\{x^2 + y^2 \leq 1\}$, where $A_1 = \frac{x}{x^2+y^2}$ has $\nabla A_1|_\Gamma = (\cos 2\theta, \sin 2\theta)$ as opposed to $\nabla x|_\Gamma = (\cos\theta, 0)$. Let us also remark that tensor m_{ij} is a geometric invariant of surface Γ, independent of its position in $\mathbb{R}^n$, as one could easily verify, using translational/ rotational symmetries of solution (16), and volume-integral[1] form (18) .

In a similar fashion one computes the total momentum of the induced flow

$$\vec{p} = \rho \iint_{\mathbb{R}^n \setminus D} \boldsymbol{u} = \rho \oint_\Gamma N_i \left(\sum_j U_j x_j \right) = \begin{pmatrix} \cdots \\ \sum m_{ij} U_j \\ \cdots\cdot \end{pmatrix} \tag{19}$$

The rate of the change of the momentum gives the fluid reaction force, exerted on the body. So the resulting equation of motion take the

$$M \frac{dU}{dt} + \frac{d\vec{p}}{dt} = F \text{ -external force}$$

where $M = \iint_D \rho_0$ is the body mass. Remembering the exact form of fluid momentum (19) $\sum_j (M\delta_{ij} + m_{ij}) \frac{dU_j}{dt} = F_i$. Thus the effect of fluid on the moving body is to replace the standard mass by the virtual mass-tensor in the kinetic energy. The components of the reaction force $-\frac{d\vec{p}}{dt}$ along U, and its normal $U^\perp$ gives the *drag* and *buoyancy* (lift) forces

$$F_d = -\sum_j m_{ij} \dot{U}_i U_j; \quad F_b = -\sum_j m_{ij} \dot{U}_i U_j^\perp$$

In the 2D-case and horizontal $U = (U, 0)$ we get $F_d = -m_{11} U \dot{U}$; $F_b = -m_{12} U \dot{U}$.

Exercise 5 *Compute virtual mass-tensor and the drag and buoyancy forces for the cylinder (disk in 2D), half-cylinder ($0 \leq \theta \leq \pi$) , and quarter-cylinder ($0 \leq \theta \leq \pi/2$). Do the same exercise for solid sphere, hemi-sphere and quarter-sphere in 3D.*

[1] Observe, that coefficients m_{ij}, hence the total energy of the induced flow is finite, as functions $A_i(\vec{r}) \approx O(r^{-2})$ decay sufficiently fast at ∞.

Remark 6 *Notice that the reaction force depends on acceleration* $\dot{U}$*, so there is no net drag or lift in a uniform flow (d'Alembert paradox). This is not surprising, particularly for the drag-force, as its presence would require either energy dissipation or a non-zero energy flux to* ∞*. Both are absent in the ideal fluid.*

We shall demonstrate the general principles with two specific examples.

3.1.1 Potential flow passed the cylinder.

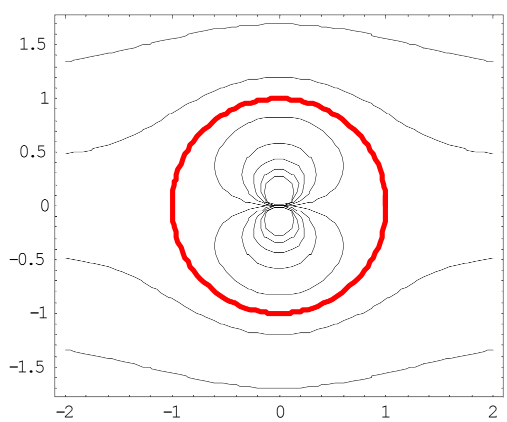
of parallel flow passed the cylinder $r \leq a$. Its stream-field is the sum of the principal (parallel) flow-component $\Psi = -Uy$ and a perturbation ψ', that would make the sum $\psi = \Psi + \psi'$ -constant (e.g. 0) on the boundary. So we solve the Poisson equation for $\psi'(r, \theta)$ in the exterior of disk with boundary value $-Uy$

$$\begin{aligned} \Delta \psi' &= 0 \text{ for } r < a \\ \psi'\big|_{r=a} &= Ua\sin\theta = -\left.\Psi\right|_{r=a} \end{aligned} \tag{20}$$

The solution is obtained via exterior Poisson kernel $P(r,\theta) = \frac{r^2 - a^2}{2\pi(r^2 - 2ar\cos\theta + a^2)}$

$$\psi' = \int_0^{2\pi} P(r, \theta - \tau)\, Ua \sin\tau \tag{21}$$

Direct evaluation of (21) using convolution identity $e^{ik\theta} * e^{im\theta} = 2\pi\delta_{km}$ yields $\psi' = \frac{Ua\sin\theta}{r} = \frac{Uay}{r^2} = U\,\mathrm{Im}\left(\frac{a}{z}\right)$. The latter is also an obvious consequence of $\mathrm{Im}\left(\frac{z}{a} + \frac{a}{z}\right) = 0$ on the circle of radius $|z| = a$. We plot stream-lines of $\psi = Ua\,\mathrm{Im}\left(\frac{z}{a} + \frac{a}{z}\right) = Uy\left(1 - \frac{a^2}{r^2}\right)$



Flow passed cylinder

From the explicit ψ, one computes the velocity $\boldsymbol{u} = \partial_{\bar{z}}\psi$ and the hydrostatic pressure (14) $p = p_\infty - \frac{1}{2}|\nabla\psi|^2$. The reaction forces could be obtained by evaluating pressure gradient along the circle $F = \oint_C \nabla p ds$. Thus we get

$$\begin{aligned}
\psi(r,\theta) &= U\left(r - \frac{a^2}{r}\right)\sin\theta \text{ -stream} \\
p(r,\theta) &= -\tfrac{1}{2}\left\{\left(\frac{\partial\psi(r,\theta)}{\partial r}\right)^2 + r^2\left(\frac{\partial\psi(r,\theta)}{\partial\theta}\right)^2\right\} \text{ -pressure} \\
f(r,\theta) &= r\cos\theta\left(\frac{\partial p(r,\theta)}{\partial r}\right) - \frac{\sin\theta}{r}\left(\frac{\partial p(r,\theta)}{\partial\theta}\right) \text{-horizontal force}
\end{aligned}$$

3.1.2 Point-vortex flow passed cylinder

The point-vortex at ζ of strength Γ creates a flow passed an obstacle D whose stream-field is given by the exterior (Dirichlet) Green's function

$$\begin{aligned}
-\Delta\psi &= \Gamma\delta(z-\zeta) \\
\psi|_{\partial D} &= \text{Const}
\end{aligned}$$

For the cylinder $D = \{|z| < a\}$ we have $\psi(z,\zeta) = \frac{\Gamma}{2\pi}\ln\left|\frac{a(z-\zeta)}{a^2 - z\zeta}\right|$, and the corresponding velocity

$$\boldsymbol{u} = i\partial_{\bar z}\psi = \frac{i\Gamma}{4\pi}\left\{\tfrac{1}{\bar z - \zeta} + \tfrac{\zeta}{a^2 - \bar z\zeta}\right\} = \frac{i\Gamma}{4\pi}\frac{a^2 - |\zeta|^2}{\left(\bar z - \bar\zeta\right)\left(a^2 - \bar z\zeta\right)}$$

Due to rotational symmetry we could place source on the real axis $\zeta = b > 0$ call $z = re^{i\theta}$ and compute pressure

$$p = -\frac{\Gamma^2}{2(4\pi)^2}\frac{\left(a^2 - b^2\right)}{|r - be^{i\theta}|^2\,|br - a^2e^{i\theta}|^2} \tag{22}$$

Clearly, the net force is exerted by the vortex on the cylinder is directed toward the vortex. Its precise form is obtained by evaluating the pressure gradient p_x of (22) along the circle $f = \frac{\Gamma^2}{2(4\pi)^2}\oint_C p_x ds$.