Thm: Regular elliptic op-tor $(\mathcal{L}, B)$ in D has

(i) $\mathcal{L}$ is self-adjoint w.r. to the inner product

$$\langle f | g \rangle = \iint_D f(x)\, g^*(x)\, dV \quad (\text{square-mean, } L^2)$$

$$\langle \mathcal{L}[f] | g \rangle = \langle f | \mathcal{L}[g] \rangle \quad \text{all f-ns } f, g$$

(ii) All eigenvalues $\{\lambda_k\}$ - real & eigenf-ns $\{\varphi_k\}$ orthogonal: $\langle \varphi_k | \varphi_m \rangle = 0$, $\lambda_k \neq \lambda_m$

(iii) System $\{\varphi_k\}$ - complete orthog. basis i.e. any f-n is expanded in a (generalized Fourier) series

$$f = \sum \hat{f}_k\, \varphi_k(x) \quad - (L^2\text{-convergent})$$

with coeff.

$$\hat{f}_k = \frac{\langle f | \varphi_k \rangle}{\| \varphi_k \|^2} = \left(\iint f\, \varphi_k^*\, dV \right) / \left(\iint |\varphi_k|^2 dV \right)$$

(iv) All eigenvalues have finite multiplicity

$$d_k = \dim[\lambda_k\text{-eigenspace}] < \infty$$

(For 1D problems all $d_k = 1$)

(v) Eigenvalues $\lambda_k \to \infty$, all but finite # of λ's are positive

(vi) If $p \geq 0$, $q \geq 0$ operator $\mathcal{L}$ is positive-def.

$\langle \mathcal{L}[f] | f \rangle \geq 0$ all f $\Rightarrow$ all $\lambda \geq 0$

1. Self adjoint on $\{w(x)\}$: $Bw\big|_\Gamma = 0 \Leftrightarrow \frac{\partial_n w}{w}\Big|_\Gamma = -\frac{\alpha}{\beta}$

(1D) (integr. by parts): $\langle L[w] | v\rangle - \langle w | L[v]\rangle$

$$= \int_a^b [-(pw')' + qw]v - [-(pv')' + qv]w =$$

$$= -p(w'v - v'w)\Big|_a^b = -p\left(\underbrace{\frac{w'}{w}}_{} - \underbrace{\frac{v'}{v}}_{}\right)wv\Big|_a^b = 0$$

(nD) (Green's identity)

$$\langle L[w] | v\rangle - \langle w | L[v]\rangle = \iint_D (-\nabla\cdot p\nabla w + qw)v - (-\nabla\cdot p\nabla v + qv)w$$

$$= \oint_\Gamma -p(\partial_n w\, v - w\,\partial_n v) = \oint_\Gamma -p\underbrace{\left(\frac{\partial_n w}{w} - \frac{\partial_n v}{v}\right)}_{\parallel\, 0} wv$$

$\Rightarrow$ λ_k - real ; φ_k - orthog.

2. Positive: $p, q \geq 0$; $\frac{\alpha}{\beta} \geq 0$

(1D) $\int [-(pw')' + qw]w = \underbrace{-pw'w\Big|_a^b}_{p\boxed{\frac{\alpha}{\beta}}w^2\Big|} + \int pw'^2 + qw^2 \geq 0$

(nD) $\iint (-\nabla\cdot p\nabla w + qw)w = \oint_\Gamma -p\underbrace{w_n w} + \iint \underbrace{p|\nabla w|^2 + qw^2}_{\geq 0}$

$$+ \oint_\Gamma p\underbrace{\boxed{\frac{\alpha}{\beta}}}_{\geq 0} w^2$$

$\Rightarrow$ All $\lambda_k \geq 0$