Math. 448 Supplement

Bessel Zoo and harmonic oscillator

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I. Differential equation.

Bessel functions of the 1-st kind and order ν solve the differential equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0 \tag{1}$$

while those of the 2-nd kind (modified) solve

$$y'' + \frac{1}{x}y' - \left(1 + \frac{\nu^2}{x^2}\right)y = 0 \tag{2}$$

Formally the transition from (1) to (2) is by a complex change of variable $x \to ix$.

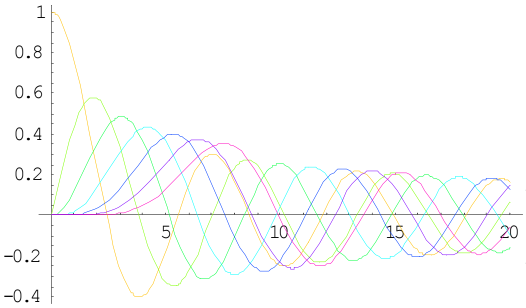
In both case $x = 0$ is a singular point, so by the Frobenius method both have a pair of solutions expanded in the series

$$y_{\pm\nu}(x) = x^{\pm\nu} \sum a_k x^{2k}, \text{ for non-integral } \nu \tag{3}$$

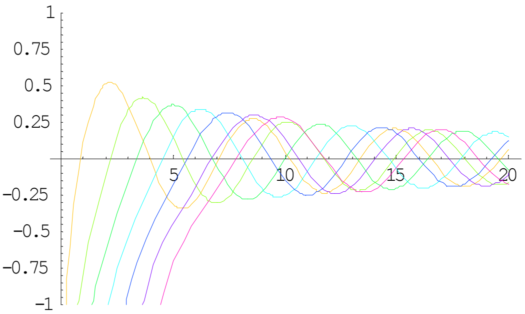
while in the integral case $\nu = n$ two y's are linear dependent $y_{-n} = (-)^n y_n$, and the second independent solution is chosen in the form $Y = \log x + \sum$"regular series".

Solutions of the form (3) are denoted by $J_{\pm\nu}(x)$ for (1), 1-st kind, and $I_{\pm\nu}(x)$ for (2), 2-nd kind. Any other solution of (1-2) is a combination of J's or I's. Two important combinations are the Y- and H- Bessel functions defined as follows

$$Y_\nu = \frac{\cos \pi\nu \, J_\nu - J_{-\nu}}{\sin \pi\nu}; \; H_\nu^{(\pm)}(z) = J_\nu \pm i Y_\nu$$



Bessel: $J_m(x), m = 0, 1, 2, ...$



Bessel: $Y_m(x), m = 0, 1, 2...$

II. Asymptotics

The choice of linear combinations of Bessel solutions (1)-(2) is dictated by their asymptotic behavior at ∞. Namely,

$$J_\nu(x) \sim \sqrt{\frac{\pi}{2x}} \cos\left(x - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right); \; Y_\nu(x) \sim \sqrt{\frac{\pi}{2x}} \sin\left(x - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right)$$

hence

$$H_\nu^{\pm}(x) \sim \sqrt{\frac{\pi}{2x}} \exp\left[\pm i\left(x - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right)\right]$$

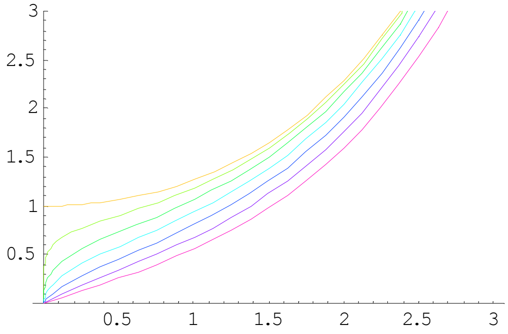
The modified (2-nd kind) functions I and K are related to J and Y via complex change of variable $x \longrightarrow ix$, hence they have exponential asymptotics at ∞. Precisely,

$$I_\nu(x) = e^{-i\frac{\pi}{2}\nu} J_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left\{1 - \frac{4\nu^2 - 1}{8x} + ...\right\}$$

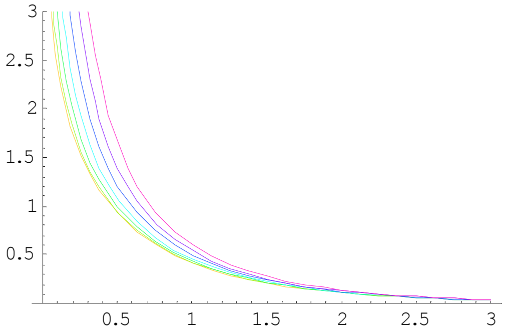
exponentially increasing, while

$$K_\nu = \frac{\pi}{2}\left(\frac{I_{-\nu} - I_\nu}{\sin \pi\nu}\right) = \frac{\pi i}{2} e^{i\frac{\pi}{2}\nu} H_\nu^+(x) \sim \sqrt{\frac{\pi}{2x}} \, e^{-x} \{1 + ...\}$$

exponentially decreasing



Bessel: $I_m(x), 0 \leq m \leq 1$



Bessel: $K_m(x), 0 \leq m \leq 1$

III. Integral representations

There many integral representations of Bessel functions but most of them fall in two types: Poisson type (finite integral) and the Mehler-Sonine type (infinite integral). The first type is exemplified by

$$\Gamma\left(\tfrac{1}{2}+\nu\right) J_\nu(x) = \frac{1}{\sqrt{\pi}}\left(\frac{x}{2}\right)^\nu \int_0^\pi e^{ix\cos\theta}\sin^{2\nu}\theta\, d\theta = \left(\frac{x}{2}\right)^\nu \int_{-1}^{1} \cos xu\ \left(1-u^2\right)^{\nu-1/2} du$$

The second (Mehler-Sonine) type gives

$$\Gamma\left(\tfrac{1}{2}-\nu\right) J_\nu(x) = \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{-\nu} \int_1^\infty \sin\, xt \frac{dt}{\left(t^2-1\right)^{\nu+1/2}} \tag{4}$$

and similarly

$$\Gamma\left(\tfrac{1}{2}-\nu\right) Y_\nu(x) = -\frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{-\nu} \int_1^\infty \cos\, xt \frac{dt}{\left(t^2-1\right)^{\nu+1/2}}$$

IV. Fundamental solutions

The Laplacian plus constant (Hemholtz operator) $L = -\Delta \pm m^2$ on $\mathbb{R}^n$ has fundamental solutions (Green's functions) $G(r) = G(|x-\xi|)$of the following form

$$-\Delta + m^2 \Longrightarrow G(r) = r^{\frac{2-n}{2}} K_{\frac{n-2}{2}}(mr) \sim \frac{e^{-mr}}{r^{\frac{n-1}{2}}}$$

with exponentially decaying Kelvin function of order $\nu = \frac{n-2}{2}$. In the $-m^2$ case we get

$$-\Delta - m^2 \Longrightarrow G(r) = r^{\frac{2-n}{2}} H^{\pm}_{\frac{n-2}{2}}(mr) \sim \frac{e^{\pm imr}}{r^{\frac{n-1}{2}}}$$

with oscillating $H^{\pm}$-function, corresponding to the incoming/outgoing radiation condition at ∞.

The wave operator: $\Box = \partial_t^2 - \Delta$ and the Klein-Gordon: $\Box + m^2 = \partial_t^2 + L$, where $L = -\Delta + m^2$ have fundamental solutions of the type $K = \frac{\sin t\sqrt{L}}{\sqrt{L}}$; $M = \cos t\sqrt{L}$. Both integral kernels are functions of $r = |x - \xi|$ and t. The Fourier transform $\mathcal{F}_{x\to k}$ yields a representation of K in terms of the Bessel function J_ν of order $\nu = \frac{n-2}{2}$

$$K(r;t) = c_{n-1} \int_0^\infty \frac{\sin t\rho}{\rho} (r\rho)^{-\nu} J_\nu(r\rho)\, \rho^{n-1} d\rho \text{ (for wave)}$$

and

$$K(r;t) = c_{n-1} \int_0^\infty \frac{\sin t\sqrt{\rho^2 + m^2}}{\sqrt{\rho^2 + m^2}} (r\rho)^{-\nu} J_\nu(r\rho)\, \rho^{n-1} d\rho \text{ (for KG)} \qquad (5)$$

In $\mathbb{R}^3$ Bessel $J_{1/2} = \frac{\sin x}{\sqrt{x}}$ is an elementary function, so (5) becomes

$$K = \frac{1}{4\pi^2} \int_0^\infty \frac{\sin t\sqrt{\rho^2 + m^2}}{\sqrt{\rho^2 + m^2}} \frac{\sin r\rho}{r\rho} \rho^2 d\rho = \frac{1}{r} \int_0^\infty \frac{\cos\left(t\sqrt{\cdots} - r\rho\right) - \cos\left(t\sqrt{\cdots} + r\rho\right)}{2\sqrt{\cdots}} \rho d\rho$$

We take the hyperbolic radius $R = \sqrt{t^2 - r^2}$ and replace polar variables $\{r;\rho\}$ by a new pair $\{u;v\}$, where

$$\begin{aligned} \rho &= R \sinh u \\ r &= R \sinh v \end{aligned}$$

Then K becomes

$$K = \frac{\sinh v}{R} \int\limits_{-\infty}^{\infty} \cos\left(mR \cosh u\right) \cosh u \, du = \frac{1}{R} \int\limits_{1}^{\infty} \cos\left(mR\, s\right) \frac{ds}{\sqrt{s^2 - 1}}$$

(after another change: $u \to s = \cosh u$). The latter is recognized as the derivative of J_0 in the Mehler-Sonine form (4)

$$K\left(R\right) = \frac{1}{R} \frac{d}{dR} \left\{ \int\limits_{1}^{\infty} \sin\left(mR\, s\right) \frac{ds}{\sqrt{s^2 - 1}} \right\} = \frac{1}{R} \frac{d}{dR} \left\{ J_0\left(mR\right) H\left(R\right) \right\}$$

with Heaviside function H. Hence follows the final result for the 3-D Klein-Gordon propagator

$$K\left(R\right) = \frac{1}{R} \left[\delta\left(R\right) - \frac{1}{m} J_1\left(mR\right) \right] \tag{6}$$

in terms of hyperbolic radius $R = \sqrt{t^2 - \left|x - \xi\right|^2}$.

Remark 1 *Formula (6) shows K to depend on the hyperbolic radius R only, and thus gives another demonstration of the Huygens principle (causality) for solutions of the wave and K-G equations.*

V. Harmonic oscillator

Differential operator of the form $h = -\partial^2 + x^2$ on $\mathbb{R}$, known as (quantum) harmonic oscillator plays fundamental role in Mathematics and Physics. Its spectrum (eigenvalues/eigenfunctions) can be easily derived from commutation relations for a special pair of 1-st order operators know in Mathematics as *raising/lowering* operators and in Physics as *creation/annihilation* pair $a = \partial + x; a^\dagger = -\partial + x$. The latter is the formal adjoint of a with respect to the L^2-inner product $\langle u|v\rangle = \int\limits_{-\infty}^{\infty} u\, \bar{v}\, dx$. Operator h factors into the product of $a; a^\dagger$, namely

$$h = a^\dagger a + 1 = a a^\dagger - 1 \tag{7}$$

and the triple $\left\{a; a^{\dagger}; h\right\}$ is easily shown to obey the following commutation relations

$$\begin{array}{l} \left[a^{\dagger}; a\right] = -2 \\ \left[h; a\right] = \left[a^{\dagger}; a\right] a = -2a \\ \left[h; a^{\dagger}\right] = \left[a; a^{\dagger}\right] a^{\dagger} = 2a^{\dagger} \end{array} \tag{8}$$

Here $[a; b] = ab - ba$ denotes the commutator of two operators. In second and third lines we used the product-rule for commutators: $[a; bc] = [a; b]\, c + [a; c]\, b$, for any triple $a; b; c$.

Relations (8), particularly the second and the third ones have immediate application to spectrum h. Indeed, if ψ is an eigenfunction $h\,[\psi] = \lambda\psi$ then raising or lowering ψ gives another eigenfunction: $\psi_+ = a^{\dagger}\,[\psi]$ and $\psi_- = a\,[\psi]$ are also eigenfunctions of h of eigenvalues $\lambda + 2$ and $\lambda - 2$. Indeed,

$$h\,[\psi_+] = ha^{\dagger}\,[\psi] = \left\{a^{\dagger}h + \left[h; a^{\dagger}\right]\right\}[\psi] = (\lambda + 2)\, a^{\dagger}\,[\psi] = (\lambda + 2)\,\psi_+$$

and similar one shows $h\,[\psi_-] = ... = (\lambda - 2)\,\psi_-$for ψ_-. Here we used the obvious relation $ab = ba + [a; b]$. So operator $a^{\dagger}$ raises each eigenvalue of h by 2, while a lowers it by 2, whence the terminology.

We proceed to compute the eigenvalues. Take the lowest eigenvalue λ_0 of h and its eigenfunction ψ_0. Such eigen always exists as operator h is positive-definite: $\langle f\,|h|\,f\rangle \geq 0$ for any square-integrable function $f \in L^2$. Clearly, the lowering operator a takes ψ_0 to 0, so it solves an ode

$$a\,[\psi] = \psi' + x\psi = 0 \implies \psi_0 = e^{-x^2/2} \text{ -the Gaussian}$$

We plug ψ_0 in (7) and find the lowest eigenvalue $\lambda_0 = 1$. Next we apply the m-th iterate of the raising operator $a^{\dagger}$ to ψ_0

$$\psi_m\,(x) = \left(a^{\dagger}\right)^m [\psi_0] = (-\partial + x)^m \left[e^{-x^2/2}\right] \tag{9}$$

The result is a new eigenfunction of h of eigenvalue $\lambda_m = 1 + 2m$. Thus spectrum of h contains all odd integers with eigenfunctions given by (9). To bring $\{\psi_m\}$ to the standard form of the m-th Hermite function we use another commutation relation $-\partial + x = e^{x^2/2}\,(-\partial)\,e^{-x^2/2}$,whence follows

$$\psi_m\,(x) = e^{x^2/2}\,(-\partial)^m \left[e^{-x^2/2}\,e^{-x^2/2}\right] = e^{x^2/2}\,(-\partial)^m \left[e^{-x^2}\right] \tag{10}$$

Clearly, ψ_m is a product of Gaussian $e^{-x^2/2}$ and a Hermite polynomial of degree m, $\psi_m = e^{-x^2/2} H_m(x)$. Those can be computed via (10)

$$H_0 = 1;\ H_1 = 2x;\ H_2 = 4x^2 - 2;\ H_3 = 8x^3 - 12x$$

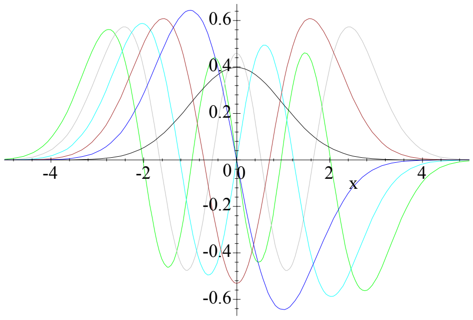
Polynomials form a complete set of functions on any finite interval of $\mathbb{R}$ (any f can be approximated by polynomials), hence Gaussian×polynomials approximate all functions on $\mathbb{R}$ that vanish at infnity. Thus we get a complete *eigenvalue spectrum* of the harmonic oscillator (Hermite operator) h

$$\boxed{\begin{array}{l}\text{eigenvalues} = \{1 + 2m : m = 0; 1; 2; ...\} \\ \text{eigenfunctions} = \{\text{Hermite } \psi_m\}\end{array}}$$

Below we compute and plot the first 7 Hermite polynomials $m = 0, 1, ..., 6$, $H_m(x) = e^{x^2} \frac{d^m}{dx^m}\left(e^{-x^2}\right)$ the corresponding (normalized) Hermite functions

$$h_m(x) = e^{-x^2/2} H_m(x) \Bigg/ \left(\int\limits_{-\infty}^{\infty} e^{-x^2} H_m(x)^2\, dx\right)^{1/2}$$

$H_m(x)$	$h_m(x)$
1	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$
$-2x$	$\frac{\sqrt{2}}{\sqrt[4]{\pi}} e^{-\frac{1}{2}x^2} x$
$-2 + 4x^2$	$\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2}\,\sqrt[4]{\pi}} \left(-1 + 2x^2\right)$
$-4x\left(-3 + 2x^2\right)$	$\frac{e^{-\frac{1}{2}x^2}}{\sqrt{3}\,\sqrt[4]{\pi}} \left(3 - 2x^2\right) x$
$12 - 48x^2 + 16x^4$	$\frac{e^{-\frac{1}{2}x^2}}{2\sqrt{6}\,\sqrt[4]{\pi}} \left(3 - 12x^2 + 4x^4\right)$
$-8x\left(15 - 20x^2 + 4x^4\right)$	$\frac{e^{-\frac{1}{2}x^2}}{2\sqrt{15}\,\sqrt[4]{\pi}} \left(15 - 20x^2 + 4x^4\right) x$
$-120 + 720x^2 - 480x^4 + 64x^6$	$\frac{e^{-\frac{1}{2}x^2}}{12\sqrt{5}\,\sqrt[4]{\pi}} \left(-15 + 90x^2 - 60x^4 + 8x^6\right)$



Hermite functions: $h_m(x), m = 0, 1, ...5$