Basic hydrodynamics

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1 Newtonian fluids: Euler and Navier-Stokes equations

The basic hydrodynamic equations in the Eulerian form consist of conservation of mass, momentum and energy. We denote by $\mathbf{u} = \mathbf{u}(x;t)$ velocity field, and ρ - density field, with $D_t = \partial_t + \mathbf{u} \cdot \nabla$ - derivative (of any material quantity) along the flow direction.

1.1 Mass conservation

The continuity (conservation) of mass requires the mass transfer rate along the flow $D_t\rho$ be balanced by its change due to the volume compression/expansion by the flow $\nabla \cdot \mathbf{u}$,

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = D_t\rho + (\nabla \cdot \mathbf{u})\,\rho = 0 \tag{1}$$

If density ρ is constant, (1) turns into incompressibility condition

$$\nabla \cdot \mathbf{u} = 0$$

1.2 Momentum conservation

The change of momentum $\rho D_t u$ of a Newtonian fluid is balanced by a combination of *body forces* F_b (distributed through the volume) and *surface forces* F_s. The former include gravity, Coriolis (rotation), Coulomb, Lorentz (for electrically conducting fluids) etc. The latter have a general form

$$F_s = \sigma \cdot N dS$$

where σ is the *stress-tensor*, N -normal to the surface element dS. The stresses is made of two components: *pressure* p and *viscous* (frictional) forces, depending on the rate of (velocity) deformation: $\partial \mathbf{u} = (\partial_i u_j)$. The latter is made of the antisymmetric (rotation) component: $\nabla \times \mathbf{u} = \left(\frac{\partial_i u_j - \partial_j u_i}{2}\right)$, and the symmetric *rate of strain*-tensor $s_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Only the latter (strain) would contribute to the viscous stresses.

Assuming isotropy (rotational symmetry) one can derives the following representation of the combined pressure + viscosity stresses [1]

$$\sigma_{ij} = -p\delta_{ij} + \tilde{\nu} s_{ij} + \frac{\tilde{\mu} - \tilde{\nu}}{3} \operatorname{tr}(s)\, \delta_{ij} \tag{3}$$

Lame coefficients $\tilde{\nu}, \tilde{\mu}$ measure the *dynamic viscosity*, while $\operatorname{tr}(s) = \nabla \cdot \mathbf{u}$ gives divergence. Now the momentum conservation: $\partial_t (\rho \mathbf{u}) + \nabla \cdot [\mathbf{u}(\rho \mathbf{u})] = \nabla \cdot \sigma$ turns into the *Navier-Stokes equation*

$$\begin{aligned} \partial_t (\rho u_i) + \nabla \cdot [\mathbf{u}(\rho u_i)] &= -\partial_i p + \nabla \cdot \tilde{\nu} \nabla u_i + \partial_i \left[\left(\frac{\tilde{\mu} - \tilde{\nu}}{3} \right) \nabla \cdot \mathbf{u} \right] \text{, or} \qquad (4) \\ &= \partial_i p + \tilde{\nu} \nabla^2 u_i + \left(\frac{\tilde{\mu} - \tilde{\nu}}{3} \right) \partial_i (\nabla \cdot \mathbf{u}) \text{, for const } \tilde{\nu}, \tilde{\mu} \end{aligned}$$

written here for the i-th component of momentum. Dividing by ρ and using mass conservation (1) it takes the form

$$D_t \mathbf{u} = -\frac{\nabla p}{\rho} + \nu \Delta \mathbf{u} + \left(\tfrac{\mu - \nu}{3} \right) \nabla (\nabla \cdot \mathbf{u}) \tag{5}$$

where $\nu = \frac{\tilde{\nu}}{\rho}$, $\mu = \frac{\tilde{\mu}}{\rho}$ are the kinematic viscosities.

Two other simplifications arise for incompressible fluids (second viscous term drops)

$$D_t (\mathbf{u}) = -\frac{\nabla p}{\rho} + \nu \Delta \mathbf{u}$$

and the ideal Euler fluid (no viscosity)

$$D_t \mathbf{u} = \mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\nabla p}{\rho} = 0 \tag{6}$$

The Euler momentum equation can be rewritten via (15) as

$$\mathbf{u}_t + \nabla \left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) - \mathbf{u} \times (\nabla \times \mathbf{u}) = 0 \tag{7}$$

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[1] Relation (3) results from the decomposition of a symmetric $n \times n$-matrix s into its *scalar* and *traceless* components $s_0 + s'$

$$s = \underbrace{\frac{\operatorname{tr}(s)}{n} I}_{s_0} + \underbrace{\left(s - \frac{\operatorname{tr}(s)}{n} I \right)}_{s'} \tag{2}$$

If $\mathcal{P}_n$ denotes the space of all such matrices, the isotropy of the linear strain-to-stress map $\Lambda : s \to \sigma$ implies that Λ commutes with the natural action of Lie group $\mathbb{SO}(n)$ on space $\mathcal{P}_n$: $s \to U^* s U$ (conjugation). Any such map Λ of $\mathcal{P}_n$ ($\mathbb{SO}(n)$-invariant) is well known to be of the form

$$s_0 + s' \to \lambda s_0 + \mu s'$$

So Λ is determined by a pair of scalars λ, μ - *Lame coefficients*. In other words, Λ respects the "scalar+traceless" decomposition (2), hence follows (3).

In particular, steady-state potential flow ($\nabla \times \mathbf{u} = \mathbf{0}$) has Bernoulli form, $W = \dfrac{\mathbf{u}^2}{2} + \frac{p}{\rho}$ conserved along flow-lines

$$\begin{array}{c} \mathbf{u}_t = \mathbf{0} \\ \boldsymbol{\omega} = \nabla \times \mathbf{u} = \mathbf{0} \end{array} \Rightarrow \frac{\mathbf{u}^2}{2} + \frac{p}{\rho} = \text{Const}$$

Taking $di\mathbf{v}$ of (7) we get

$$\partial_t \left(\nabla \cdot \mathbf{u} \right) + \Delta \left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) - \left| \nabla \times \mathbf{u} \right|^2 - \mathbf{u} \cdot \Delta \mathbf{u} = 0$$

Hence the ideal fluid pressure is a quadratic functional of velocity $\mathbf{u}$

$$\frac{p}{\rho} = \Delta^{-1} \left\{ \left| \nabla \times \mathbf{u} \right|^2 + \mathbf{u} \cdot \Delta \mathbf{u} \right\} - \frac{\mathbf{u}^2}{2} = \Delta^{-1} \left\{ \boldsymbol{\zeta}^2 + \mathbf{u} \cdot \Delta \mathbf{u} \right\} - \frac{\mathbf{u}^2}{2} \tag{8}$$

Taking $di\mathbf{v}$ of (7) we get the Euler equation for the vorticity field

$$\boldsymbol{\zeta}_t - \nabla \times \left(\mathbf{u} \times \boldsymbol{\zeta} \right) = \boldsymbol{\zeta}_t + \left(\mathbf{u} \cdot \nabla \right) \boldsymbol{\zeta} - \left(\boldsymbol{\zeta} \cdot \nabla \right) \mathbf{u} = 0 \tag{9}$$

2 Integral conservation laws

Several integral conservation laws are easily verified for Euler fluids:
Momentum:

$$\rho \iiint \mathbf{u} \, dx = const$$

is conserved due to $\nabla \cdot \mathbf{u} = 0$
Energy:

$$E = \rho \iiint \tfrac{1}{2} \mathbf{u}^2 dx = const$$

Indeed,

$$\partial_t \left(\frac{\mathbf{u}^2}{2} \right) = \mathbf{u} \cdot \mathbf{u}_t = -\mathbf{u} \cdot \nabla \left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) + \mathbf{u} \cdot \left(\mathbf{u} \times \boldsymbol{\zeta} \right) = -\nabla \cdot \left(\left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) \mathbf{u} \right)$$

-complete divergence.
Total vorticity

$$\iiint \boldsymbol{\zeta} \, dx = const$$

Indeed, incompressibility of fields $\mathbf{u}; \boldsymbol{\zeta}$ implies $\left(\mathbf{u} \cdot \nabla \right) f = \nabla \cdot \left(f \mathbf{u} \right)$ and $\left(\boldsymbol{\zeta} \cdot \nabla \right) f = \nabla \cdot \left(f \boldsymbol{\zeta} \right)$ for any f. Hence each component $\boldsymbol{\zeta}_k$ of $\boldsymbol{\zeta}$ obeys by (9)

$$\partial_t \left(\boldsymbol{\zeta}_k \right) = \nabla \cdot \left(\mathbf{u}_k \boldsymbol{\zeta} - \boldsymbol{\zeta}_k \mathbf{u} \right) \text{ - complete divergence}$$

Circulation (Kelvin): for any loop γ_t that moves along the flow

$$C(t) = \oint_{\gamma_t} \mathbf{u} \cdot ds = const$$

We use the transport along the flow

$$\frac{d}{dt} C = \oint_{\gamma_t} D_t \mathbf{u} \cdot ds = \oint_{\gamma_t} \left[\nabla \left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) - \mathbf{u} \times \boldsymbol{\zeta} \right] \cdot ds$$

The grad-term gives 0 along any closed loop, while the $\oint (\mathbf{u} \times \boldsymbol{\zeta}) = \int \int_{\Sigma_t} \nabla \times (\mathbf{u} \times \boldsymbol{\zeta}) \cdot N \, dS = 0$, due to the zero integrand (16).

Helicity (Mofatt):

$$\iiint (\mathbf{u} \cdot \boldsymbol{\zeta}) \, dx = const$$

One writes

$$\begin{gathered} \mathbf{u}_t + \nabla \left(\tfrac{\mathbf{u}^2}{2} + \tfrac{p}{\rho} \right) - \mathbf{u} \times \boldsymbol{\zeta} = 0 \\ \boldsymbol{\zeta}_t - \nabla \times (\mathbf{u} \times \boldsymbol{\zeta}) = 0 \end{gathered}$$

multiplies the first equation by $\cdot \boldsymbol{\zeta}$ the second one by $\cdot \mathbf{u}$ and sums together

$$(\boldsymbol{\zeta} \cdot \mathbf{u})_t = -\nabla \cdot \left[\left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) \boldsymbol{\zeta} \right] + \boldsymbol{\zeta} \cdot (\mathbf{u} \times \boldsymbol{\zeta}) - \mathbf{u} \cdot \{ \nabla \times (\mathbf{u} \times \boldsymbol{\zeta}) \}$$

The 2-nd term in the r.h.s. is 0, while the 3-rd is transformed into

$$\mathbf{u} \cdot (\nabla \times R) = \nabla \cdot (R \times \mathbf{u}) - R \cdot (\nabla \times \mathbf{u}) = \nabla \cdot (R \times \mathbf{u})$$

where $R = \mathbf{u} \times \boldsymbol{\zeta}$ and $R \cdot (\nabla \times \mathbf{u}) = R \cdot \boldsymbol{\zeta} = 0$ vanishes. Hence

$$(\boldsymbol{\zeta} \cdot \mathbf{u})_t = -\nabla \cdot \left[\left(\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} \right) \boldsymbol{\zeta} - R \times \mathbf{u} \right] \text{ - complete divergence}$$

3 System of conservation laws

The basic Euler gas-fluid dynamics written as mass-momentum conservation for $(\rho, \mathbf{u}, p)$ is incomplete unless an equation of state is imposed, that would link pressure p to $\rho, \mathbf{u}$, and/or other parameters of state, s.a. entropy, temperature, etc.

The simplest case is isentropic (barotropic) gas, where $p = p(\rho)$ - a given function of ρ, e.g. $C\rho^\gamma$. One example is provided by the *shallow water* equations for $(\mathbf{u}, h)$, viewed as 2D compressible fluid of density equal to height h and pressure $p = \frac{1}{2} g h^2$,

$$\begin{gathered} D_t \mathbf{u} = -g \nabla h \\ \partial_t h + \nabla \cdot (\mathbf{u} h) = 0 \end{gathered} \tag{10}$$

More generally the internal state is described by *specific entropy* (per unit mass) s, that obeys the conservation law

$$\partial_t (\rho s) + \nabla \cdot (\mathbf{u} \rho s) = 0 \tag{11}$$

-active tracer, and $p = p(\rho, s)$ - a function of pressure and density.

The combined system for $(\rho, \mathbf{u}, s)$ has the *conservation law* form

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho s \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u s \end{pmatrix} + \partial_y \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v s \end{pmatrix} = \mathbf{0} \tag{12}$$

for 2D, and similar expression for 3D. In the isentropic case the last equation drops, so the system order is reduced by 1.

Euler system (12) has another conserved quantity, the total energy E, made of the kinetic one $K = \frac{1}{2}\rho \mathbf{u}^2$, and the potential (internal) one $\mathcal{E}(\rho, s)$. Indeed,

$$\begin{aligned} \partial_t K &= -\nabla \cdot \left[\mathbf{u} \left(\frac{\rho \mathbf{u}^2}{2} + p \right) \right] + p (\nabla \cdot \mathbf{u}) \\ \partial_t \mathcal{E} &= -\nabla \cdot [\mathbf{u} \mathcal{E}] + (\nabla \cdot \mathbf{u}) (\mathcal{E} - \rho \mathcal{E}_\rho - s \mathcal{E}_s) \end{aligned} \tag{13}$$

Hence, the total energy

$$\partial_t (K + \mathcal{E}) + \nabla \cdot [\mathbf{u} (K + \mathcal{E} + p)] = (\nabla \cdot \mathbf{u}) (p + \mathcal{E} - \rho \mathcal{E}_\rho - s \mathcal{E}_s) \tag{14}$$

-conserved provided the r.h.s. is zero.

For incompressible fluids $\nabla \cdot \mathbf{u} = 0$, the kinetic energy is conserved, along with any function $\mathcal{E}(\rho, s)$, as ρ, s are passive scalars. In the compressible case we get the differential relation between p and $\mathcal{E}$

$$p = (\rho \partial_\rho + s \partial_s) \mathcal{E} - \mathcal{E}$$

In particular,$\mathcal{E} = \rho^\alpha s^\beta$ yields $p = (\alpha + \beta - 1) \rho^\alpha s^\beta$. For shallow water, $\mathcal{E} = \frac{1}{2} g h^2$ yields $p = \frac{1}{2} g h^2$.

4 Appendix: vector algebra

Commonly used vector identities for scalar/vector fields:

$$\begin{array}{ll} \text{I} & \nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}) \Rightarrow \\ & \quad \nabla \cdot (\mathbf{a} \times \mathbf{u}) = -\mathbf{a} \cdot (\nabla \times \mathbf{u}), \text{ for } \mathbf{a} \text{ - const} \\ \text{II} & \nabla \times (f \mathbf{u}) = f \nabla \times \mathbf{u} + \nabla f \times \mathbf{u} \\ \text{III} & \nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{v} + \mathbf{u} (\nabla \cdot \mathbf{v}) - \mathbf{v} (\nabla \cdot \mathbf{u}) \Rightarrow \\ & \quad \nabla \times (\mathbf{a} \times \mathbf{v}) = +\mathbf{a} (\nabla \cdot \mathbf{v}) - (\mathbf{a} \cdot \nabla) \mathbf{v}, \text{ for } \mathbf{a} \text{ - const} \\ \text{I}\mathbf{V} & \nabla \times (\nabla \times \mathbf{u}) = \nabla (\nabla \cdot \mathbf{u}) - \Delta \mathbf{u} \\ \mathbf{V} & \mathbf{u} \times (\nabla \times \mathbf{v}) = \mathbf{u} \cdot (\nabla \mathbf{v})^t - (\mathbf{u} \cdot \nabla) \mathbf{v} \\ \mathbf{V}\text{I} & \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) = \nabla (\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{u} \\ \mathbf{V}\text{II} & (\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left(\frac{\mathbf{u}^2}{2} \right) - \mathbf{u} \times (\nabla \times \mathbf{u}) \end{array} \tag{15}$$

Here $\nabla \mathbf{v} = (\partial_i \mathbf{v}_j)$ denotes the complete derivative (Jacobian matrix) of v.f. $\mathbf{v}$.

Relation (I) follows from the formal determinant expansion by swapping rows

$$\begin{vmatrix} \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} v_1 & v_2 & v_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix} - \begin{vmatrix} u_1 & u_2 & u_3 \\ \partial_1 & \partial_2 & \partial_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

To check (III) we call the components of fields $\mathbf{u} = (a, b, c)\,; \mathbf{v} = (f, g, h)$. Then

$$\mathbf{u} \times \mathbf{v} = (bh - cg, cf - ah, ag - bf)$$

and the first component of the curl becomes

$$(a_y g - b_y f) - (c_z f - a_z h) - (f_y b - g_y a) + (h_z a - f_z c)$$

Here the second pair of brackets is obtained by interchanging fields $\mathbf{u}, \mathbf{v}$ in the anti-symmetric expression $\nabla \times (\mathbf{u} \times \mathbf{v})$. Adding and subtracting af_x the first pair is brought into the form

$$fa_x + ga_y + ha_z - (a_x + b_y + c_z)\, f = \text{1-st comp. "} (\mathbf{v} \cdot \nabla)\, \mathbf{u} - (\nabla \cdot \mathbf{u})\, \mathbf{v} \text{"}$$

The second pair gives a similar expression with $\mathbf{u}, \mathbf{v}$ interchanged, Q.E.D.

Relations (VI-VII) are consequences of (V). To verify the latter we write $\mathbf{u} = (a; b; c)$ and $\mathbf{v} = (f; g; h)$. then $\mathbf{u} \times (\nabla \times \mathbf{v})$ is equal to

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ h_y - g_z & f_z - h_x & g_x - f_y \end{vmatrix} = (b\,(g_x - f_y) - c\,(f_z - h_x)\,; ...; ...)$$

Adding and subtracting af_x we arrange the 1-st component to be

$$af_x + bg_x + ch_x - (af_x + bf_y + cf_z) = \mathbf{u} \cdot \partial_x \mathbf{v} - (\mathbf{u} \cdot \nabla)\, f$$

By the same pattern the 2-nd and 3-rd components are

$$\begin{gathered} \mathbf{u} \cdot \partial_y \mathbf{v} - (\mathbf{u} \cdot \nabla)\, g \\ \mathbf{u} \cdot \partial_z \mathbf{v} - (\mathbf{u} \cdot \nabla)\, h \end{gathered}$$

whence follows (V).

Corollary 1 *If* $\mathbf{u}$ *denotes incompressible velocity and* $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$ *-vorticity, then one has*

$$\begin{gathered} \nabla \cdot (\mathbf{u} \times \boldsymbol{\zeta}) = \boldsymbol{\zeta}^2 + \mathbf{u} \cdot \Delta \mathbf{u} \\ \nabla \times (\mathbf{u} \times \boldsymbol{\zeta}) = (\boldsymbol{\zeta} \cdot \nabla)\, \mathbf{u} - (\mathbf{u} \cdot \nabla)\, \boldsymbol{\zeta} \end{gathered} \tag{16}$$