D. GURARIE

Laplace transform method

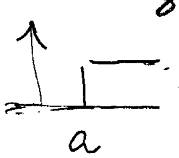
1) **Goal:** * transform DE to AE (alg. eq-ns)

* Solve AE; tranform bak

2) **Definition:** $\boxed{f(t)} \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} \boxed{F(s)} = \int_0^\infty f(t) e^{-st} dt$

Tables

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
t^n	$n!/s^{n+1}$
$e^{\alpha t}$	$1/s-\alpha$
$\cos\beta t$	$s/s^2+\beta^2$
$\sin\beta t$	$\beta/s^2+\beta^2$
$u_a(t)$	e^{-as}/s



Properties

f	F
Shift: $f(t-a)u_a$	$e^{-as}F(s)$
Multipl: $e^{at}f(t)$	$F(s-a)$
$t\, f(t)$	$-\frac{d}{ds}F(s)$
$f' = \frac{d}{dt}f$	$sF(s) - f(0)$
f''	$s^2F(s) - f'(0) - sf(0)$
...	...

3) **DE solutions:** partial fractions, inversion

Take diff. op-tor $L = D + a$; $D^2 + aD + b$; ...

1^0 IVP $\boxed{\begin{cases} L[y] = f(t) \\ y(0) = y_0 \\ \ldots \end{cases}}$ $\xrightarrow{\mathcal{L}}$ $\boxed{\underbrace{p(s)}_{\text{char. polyn.}} Y(s) = F(s) + \underbrace{\ldots}_{\text{initial}}}$

DE $\quad$ AE

2^0 $Y(s) = F(s)/p(s) + \ldots/p(s)$

3^0 **Inversion:** $Y(t) \xrightarrow{\mathcal{L}^{-1}} y(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{p(s)}\right] + \mathcal{L}^{-1}\left[\frac{\ldots}{p(s)}\right]$

Examples:

1) $\left\{ \begin{array}{l} y' - 2y = 3e^{-t} \\ y(0) = 4 \end{array} \right]$ linear growth

Char. polyn. $p(\lambda) = \lambda - 2$

$\Downarrow \mathcal{L}$

1° $(s-2)Y(s) = \frac{3}{s+1} + 4$

2° $Y(s) = \frac{3}{(s+1)(s-2)} + \frac{4}{s-2}$

$\downarrow$ partial fractions

$-\frac{1}{s+1} + \frac{1}{s-2} + \frac{4}{s-2}$

3° $y(t) = \underbrace{\boxed{-e^{-t} + e^{2t}}}_{y_p} + \underbrace{\boxed{4e^{2t}}}_{y_h}$ (particular + homog.)

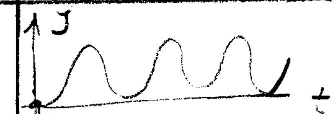
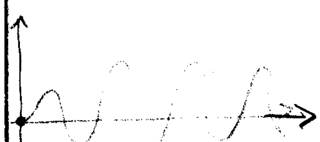
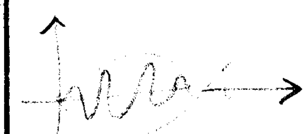
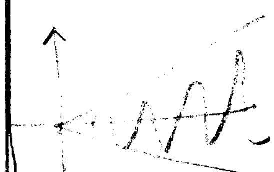
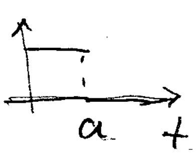
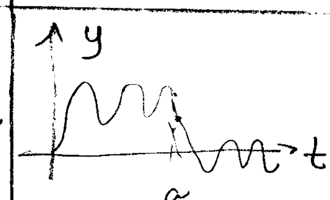
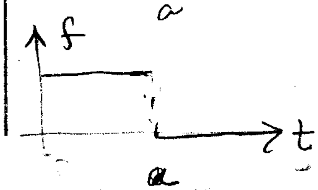
Comparison

* Undetermined: $\boxed{y(t) = \frac{3e^{-t}}{p(-1)} + c e^{2t}}$ (IV) $\Rightarrow \boxed{c=5}$

* multiplier: $y(t) = ce^{2t} + \int_0^t e^{2(t-\tau)} 3e^{-\tau} d\tau = \dots -e^{-t} + 4e^{2t}$

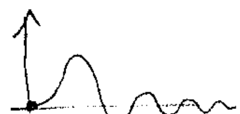
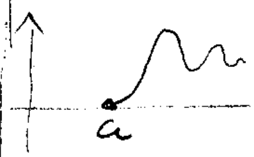
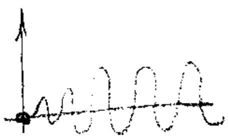
2)

$$\begin{cases} y''+4y=f(t) \\ y(0)=a \\ y'(0)=b \end{cases} \xrightarrow{\mathcal{L}} Y(s)=\frac{F(s)}{s^2+4}+\frac{b+as}{s^2+4} \xrightarrow{\mathcal{L}^{-1}} y_p(t)+\underbrace{a\cos 2t+\frac{b}{2}\sin 2t}_{y_h(t)}$$

f	$Y_p(s)$	$y_p(t)$	
1	$\frac{1}{s(s^2+4)}=\frac{1}{4}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$	$\frac{1-\cos 2t}{4}$	
e^{-at}	$\frac{1}{(s+a)(s^2+4)}=\frac{1}{4+a^2}\left(\frac{1}{s+a}-\frac{s+a}{s^2+4}\right)$	$\frac{e^{-at}-\cos 2t+\frac{a}{2}\sin 2t}{4+a^2}$	
$\cos\omega t$	$\frac{s}{(s^2+\omega^2)(s^2+4)}=\left(\frac{s}{s^2+\omega^2}-\frac{s}{s^2+4}\right)\frac{1}{4-\omega^2}$	$\frac{\cos\omega t-\cos 2t}{4-\omega^2}$; $(\omega\neq 2)$	
	$\frac{s}{(s^2+4)^2}$ $(\omega=2)$	$-t\sin 2t$ $(\omega=2)$	
$1-u_a(t)$ 	$\frac{1-e^{-as}}{s(s^2+4)}=\frac{1}{4}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)(1-e^{-as})$	$\frac{1-\cos 2t}{4}-\frac{1-\cos 2(t-a)}{4}u_a$	 

$$\begin{cases} y'' + 2y' + 5y = f(t) \\ y(0) = A \\ y'(0) = B \end{cases} \longrightarrow Y = \frac{F(s)}{s^2+2s+5} + \frac{As + (B+2A)}{s^2+2s+5} = \frac{A(s+1) + (B+A)}{(s+1)^2+4}$$

$$\frac{F(s)}{s^2+2s+5} \downarrow y_p \qquad \frac{As + (B+2A)}{s^2+2s+5} \downarrow e^{-t}\left(A \cos 2t + \frac{B+A}{2} \sin 2t\right)$$

f	Y_p	y_p	
1	$\frac{1}{s(s^2+2s+5)} = \frac{1}{5}\left(\frac{1}{s} - \frac{s+2}{(s+1)^2+4}\right)$	$\frac{1 - e^{-t}(\cos 2t + \sin 2t)}{5} = z(t)$	
e^{-3t}	$\frac{1}{(s+3)(s^2+\dots)} = \frac{1}{8}\left(\frac{1}{s+3} - \frac{s-1}{(s+1)^2+4}\right)$	$\frac{e^{-3t} - e^{-t}(\cos 2t - \frac{1}{2}\sin 2t)}{8}$	
u_a	$\frac{e^{-as}}{s(s^2+\dots)} = \frac{e^{-as}}{5}\left(\frac{1}{s} - \frac{s+2}{(s+1)^2+4}\right)$	$z(t-a)u_a$ - shifted z	
$\cos \omega t$	$\frac{1}{(s^2+\omega^2)(s^2+2s+5)} = \frac{1}{(\omega^4 - 6\omega^2 + 25)}\left[\frac{2s + (\omega^2-5)}{s^2+\omega^2} + \frac{2s + (\omega^2-1)}{(s+1)^2+4}\right]$	$\frac{\left(2\cos \omega t + \frac{\omega^2-5}{\omega}\sin \omega t\right)}{\omega^4 - 6\omega^2 + 25} + \frac{e^{-t}\left(2\cos 2t + \frac{\omega^2-1}{2}\sin 2t\right)}{\omega^4 - 6\omega^2 + 25}$	

Convolution & fundam. solution of IVP

Convolution: $(f * g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

Basic property:

$$\boxed{f * g \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)\, G(s)}$$

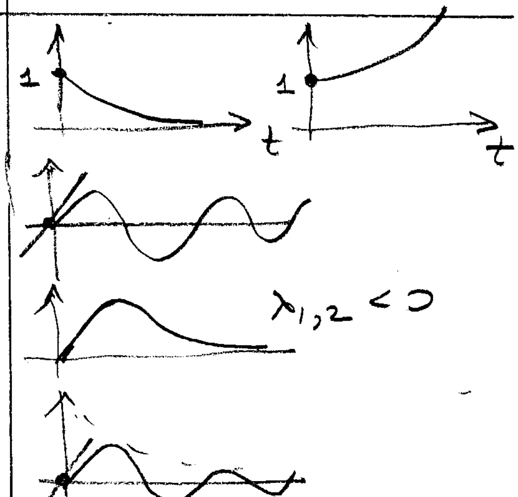
Application to IVP:

$$\begin{cases} L[y] = f(t) \\ y(0) = 0 \\ \dots \quad \dots \end{cases} \longrightarrow Y(s) = \frac{1}{p(s)} F(s) \xrightarrow{\mathcal{L}^{-1}} \boxed{y(t) = (K * f)(t)}$$

Fundamental solution of IVP:

$$K(t) = \mathcal{L}^{-1}\left(\frac{1}{p(s)}\right) = \dots$$

L	$p(\lambda)$	K
$D + a$	$\lambda + a$	e^{-at}
$D^2 + a^2$	$\lambda^2 + a^2$	$\frac{\sin at}{a}$
$D^2 + aD + b$	$\lambda^2 + a\lambda + b$ $\lambda_{1,2}$ - real	$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}$
	$\lambda = -\alpha \pm i\beta$	$e^{-\alpha t} \frac{\sin \beta t}{\beta}$



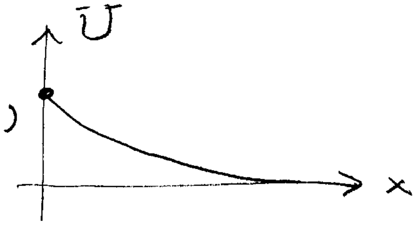
II. B-dary source problems via Laplace transform

1. Heat-diffusion on half-line:

a) (Dir) $\begin{cases} u_t - K u_{xx} = 0 ; & 0 < x < \infty \\ u\big|_{x=0} = \delta(t) \text{ or } b(t) ; & u\big|_{t=0} = f \end{cases}$

$$\Big\downarrow \mathcal{L}_{t \to s}$$

$\begin{cases} s\, U(x,s) - K\, U_{xx} = 0 ; & x > 0 \\ U\big|_{x=0} = 1 \text{ or } B(s) \end{cases}$



Solution: $U = \underbrace{e^{-x\sqrt{s/K}}}\, B(s)$

$$\Big\downarrow \mathcal{L}^{-1}_{s \to t}$$

$$\boxed{P(x,t) = \frac{x\, e^{-x^2/4Kt}}{2\sqrt{\pi K}\; t^{3/2}}} \quad \text{Poisson kernel}$$

For source $b(t) \Rightarrow$ $\left| \; u(x,t) = \int_0^t P(x, t-\tau)\, b(\tau)\, d\tau \right.$

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b) (Neuman) $\begin{cases} v_t - K v_{xx} = 0 \\ v\big|_{x=0} = 1 \text{ or } b(t); \dots \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} sV - K V_{xx} = 0 \\ V_x\big|_{x=0} = 1 \end{cases}$

$$\Rightarrow V = -\frac{e^{-x\sqrt{s/K}}}{\sqrt{s/K}} \xrightarrow{\mathcal{L}^{-1}} \boxed{P = \frac{e^{-x^2/4Kt}}{\sqrt{\pi K t}}} = 2 \times \text{Gaussian}$$

Relation between U & V

$$\boxed{\partial_x V = U} \quad \text{or} \quad \boxed{\partial_x v = u}$$

B-dary Wave source on half-line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u\big|_{x=0} = b(t); \quad u\big|_{t=0} = 0 \end{cases} \qquad \begin{cases} u_{tt} - c^2 u_{xx} + m^2 u = 0 \\ u\big|_{x=0} = b(t); \ \ldots \end{cases}$$

$$\Big\downarrow \mathcal{L}_{t \to s} \qquad\qquad \Big\downarrow$$

$$\begin{cases} s^2 U - c^2 U_{xx} = 0 \\ U\big|_{x=0} = B(s) \end{cases} \qquad \begin{cases} (s^2 + m^2) U - c^2 U_{xx} = 0 \\ U\big|_{x=0} = B(s) \end{cases}$$

$$U = e^{-\frac{s}{c}x} B \qquad\qquad U = e^{-\frac{x}{c}\sqrt{s^2+m^2}}$$

$$\Big\downarrow \mathcal{L}^{-1} \qquad\qquad \Big\downarrow \mathcal{L}^{-1}$$

$$\boxed{u = b(t - |x|/c)} \qquad\qquad c\,\partial_x \left[J_0\left(m\sqrt{t^2 - \left(\tfrac{x}{c}\right)^2}\right) \right]$$