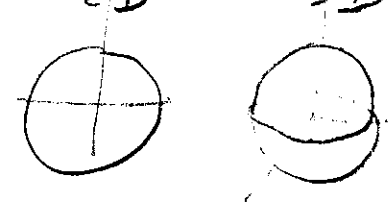
Applications: disk, ball

2D 3D



i) Harmonic f-ns: $\begin{cases} \Delta u = 0 \\ u|_\Gamma = f \end{cases}$

ii) Stationary source. $\{-\Delta u = F;\ u|_\Gamma = f$

iii) Eigenmodes: $\{\Delta\psi + \lambda\psi = 0,\ \psi|_\Gamma = 0$

iv) Heat/wave: $\{u_t - \nu\Delta u = F;\ u|_\Gamma = \cdots \quad \{u_{tt} - c^2\Delta u = F$

Method: separation → angular harmonics → radial ODE

$$\boxed{u = \sum u_m(r)\, e^{im\theta}}$$

Radial ODE for (i)-(ii)

$$\begin{cases} \left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{m^2}{r^2}\right)u_m = \begin{cases} 0 \\ \hat{F}_m \end{cases} \\ u_m|_0 = \dots \quad u_m|_a = \hat{f}_m \end{cases}$$

Radial ODE for (iii)

$$\left[\partial_r^2 + \frac{1}{r}\partial_r + \left(\lambda - \frac{m^2}{r^2}\right)\right]\psi = 0$$

$$\boxed{u = \sum u_{lm}(r)\, Y_l^m(\theta,\phi)}$$

$$\begin{cases} \left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2}\right]u_{lm} = \begin{cases} 0; \\ \hat{F}_{lm} \end{cases} \\ u_{lm}|_0 = \dots \quad u_{lm}|_a = \hat{f}_{lm} \end{cases}$$

$$\left[\partial_r^2 + \frac{2}{r}\partial_r + \left(\lambda - \frac{l(l+1)}{r^2}\right)\right]\psi = 0$$

BV problem (i): harmonic f-rs (2

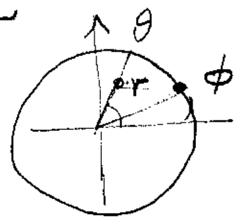
$$u = \begin{cases} c_1 + c_2 \ln r, & m=0 \\ c_1 r^m + c_2 r^{-m}, & m \neq 0 \end{cases}$$

$$u = \sum_{-\infty}^{\infty} \left(\frac{r}{a}\right)^{|m|} \hat{f}_m e^{im\theta} \text{ – } \underline{\text{F. series}}$$

$$\hat{f}_m = \frac{1}{2\pi} \int f e^{-im\phi}$$

Poisson kernel

$$u(r,\theta) = \frac{1}{2\pi} \int \underbrace{P(r, \theta - \phi)} f(\phi)\, d\phi$$



For $a = 1$

$$P(r,\theta) = 2\,\mathrm{Re}\left(\sum_{0}^{\infty} r^m e^{im\theta}\right) - 1 = \frac{1-r^2}{(1 - 2r\cos\theta + r^2)}$$

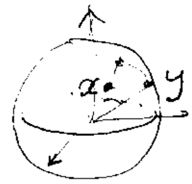
Complex: $z = re^{i\theta}$; $w = ae^{i\phi}$

$$P(z,w) = \frac{a^2 - |z|^2}{2\pi\, |z-w|^2}$$

$$u = c_1 r^l + c_2 r^{-l-1}; \quad \{r^l, r^{-l-1}\}$$

$$u = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} r^l \hat{f}_{lm} Y_l^m$$

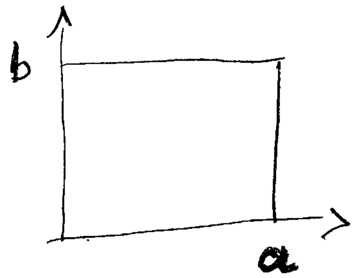
$$\hat{f}_{lm} = \frac{\langle f | Y_l^m \rangle}{\|Y_l^m\|^2} = \frac{1}{2\pi \|P_l^m\|^2} \iint f P_l^m(\theta) e^{-im\phi}$$

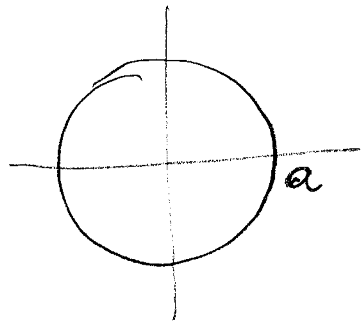


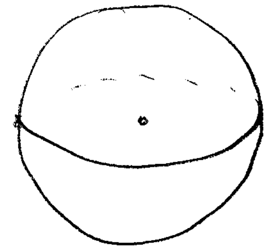
$$u(x) = \iint P(x,y) f(y)\, dS$$

$$P(x,y) = \frac{1 - |x|^2}{4\pi |x-y|^3}$$

Stationary source problem: $\begin{cases} -\Delta u = F \\ Bu|_{\Gamma} = f \end{cases}$







I. Expansion method (via eigenmodes $\{\psi_k(x)\}$)

1) $$\boxed{u(x) = \sum \frac{\hat{F}_k}{\lambda_k} \psi_k(x)}$$ (for zero B.C.) $f = 0$

2) Converting b-dary source f into cont. source via auxiliary harmonic W

$$\begin{cases} -\Delta W = 0 \\ BW|_{\Gamma} = f \end{cases} \Rightarrow \boxed{u(x) = W(x) + \sum \frac{\hat{F}_k}{\lambda_k} \psi_k(x)}$$

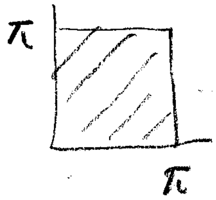
II. Green's f-n or direct solution:

$$u(x) = \iint_D K(x,\xi) F(\xi) + \oint_{\Gamma} P(x,\xi) f(\xi)$$

Examples:

1) Uniform source in rectangle/square: $F = 1$

$u|_{\Gamma} = 0$.

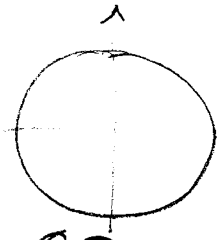


Eigen modes: $\{\psi_{km} = \sin kx \sin my;\ \lambda_{km} = k^2 + m^2\}$

Expansion: $F = \left(\frac{4}{\pi}\right)^2 \sum_{k,m-\text{odd}} \frac{\sin kx \sin my}{km}$

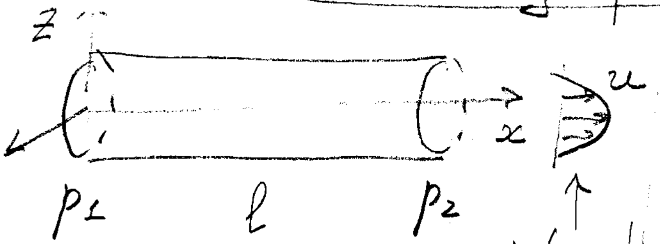
Solution: $$\boxed{u = \left(\frac{4}{\pi}\right)^2 \sum_{\text{odd}} \frac{\psi_{km}}{km(k^2+m^2)}}$$

2) Uniform disk:



$$\begin{cases} -\Delta u = 1 \\ u|_{r=a} = 0 \end{cases}$$

Also stationary pipe (Poiseuille) flow

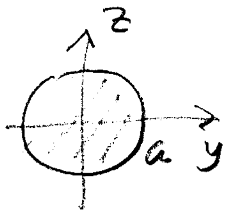


$$\boxed{-\frac{p_x}{\rho} + \nu \Delta_H u = 0}$$

velocity profile $u(y,z)$ — cross-sectional Laplace

Pressure drop: $p_x = \boxed{\frac{p_2 - p_1}{\ell}} = \frac{\delta p}{\ell} < 0$

Direct solution: $u = u(r)$



$$\begin{cases} u_{rr} + \frac{1}{r} u_r = \boxed{\frac{\delta p}{\ell \rho \nu}} = -A \\ u|_{r=a} = 0, \quad u|_{r=0} \text{ - regular} \end{cases} \Rightarrow \boxed{u = -\frac{A}{4} r^2 + c_1 + c_2 \ln r}$$

B.C. $\Rightarrow$ $\boxed{u(r) = \frac{A}{4}(a^2 - r^2)}$

Expansion: $u(r) = A \sum_{k=1}^{\infty} \hat{F}_k \; J_0\left(z_{0k} \frac{r}{a}\right)$; $\quad F_k = \dfrac{\int_0^a J_0(\cdot\, r) r\,dr}{\| J_0(\;)\|^2}$

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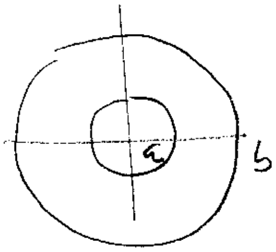
3) Green - Poisson: $\left[\begin{cases} -\Delta u = F \\ u|_{r=a} = f \end{cases}\right]$

$$u(z) = \iint_D \underbrace{K(z,w)} F(w)\, d^2 w + \oint_\Gamma \underbrace{P(z,w)} f(w)\, ds(w)$$

$$K(z,w) = \frac{1}{2\pi} \ln \left| \frac{z-w}{1 - z\bar{w}} \right| ; \quad P(z,w) = \frac{1-r^2}{2\pi(1 - 2r\cos(\phi - \theta) + r^2)}$$

$z = re^{i\phi}$; $w = e^{i\theta}$

Stationary source in annulus (6.1:6)



$\boxed{a < r < b}$

$$\begin{cases} -\Delta u = F \\ u|_{r=a} = A, \quad u|_{r=b} = B \end{cases}$$

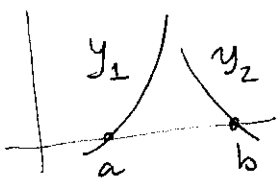
I. Radial source & b-dary cond:

ODE $\begin{cases} u_{rr} + \frac{1}{r} u_r = -F(r); \\ u|_a = A \quad u|_b = B \end{cases}$ Take $F = 1$

(i) Undet. coeff. (for any $F = r^{\alpha} \Rightarrow u_{part} = c\, r^{\alpha+2}$)

$$\boxed{u = -\frac{r^2}{4} + c_1 + c_2 \ln r} \Rightarrow \begin{matrix} c_1 = \dots \\ c_2 = \dots \end{matrix}$$

(ii) Green's f-n fund. pair $\{\ln \frac{r}{a}, \ln \frac{r}{b}\}$; $W = \frac{\ln b/a}{r}$



$$K(r,\rho) = \frac{\rho}{\ln b/a} \begin{cases} \ln \frac{r}{a} \ln \frac{\rho}{b}; & r < \rho \\ \ln \frac{\rho}{a} \ln \frac{r}{b}; & r > \rho \end{cases}$$

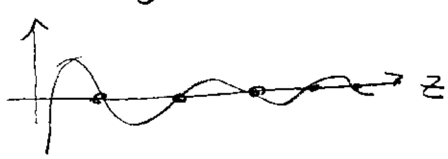
$$u(r) = \int_a^b K(r,\rho) \underbrace{1}_{F(r)} d\rho = \frac{1}{\ln b/a} \underbrace{\left\{ \ln \frac{r}{a} \int_a^r \rho \ln \frac{\rho}{b} + \ln \frac{r}{b} \int_r^b \rho \ln \rho/a \right\}} = -\frac{r^2}{4} + \dots$$

(iii) Bessel eigen modes:

$$u = \sum_k \frac{\langle 1 | \psi_k \rangle}{\lambda_k \| \psi_k \|^2} \psi_k(r) + c_1 + c_2 \ln r; \quad \text{determined by}$$

$$\psi(za, zr) = Y_0(za) J_0(zr) - J_0(za) Y_0(zr)$$

$$\begin{cases} \lambda_k = z_k^2 \quad \boxed{\psi(za, zb) = 0}; \\ \psi_k(r) = \psi(z_k a, z_k r) \end{cases}$$



6.1: 4, 8

Stationary source problem

$$\begin{cases} \left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{m^2}{r^2}\right) u = F \\ u\big|_{r=0} = \quad u\big|_{r=a} = \end{cases}$$

$$\begin{cases} \left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{\ell(\ell+1)}{r^2}\right] u = F \\ - \end{cases}$$

$$u = \underbrace{u_p}_{\text{particular}} + \underbrace{\begin{cases} c_1 + c_2 \ln r \\ c_1 r^m + c_2 r^{-m} \end{cases}}_{\text{homog}}$$

$$u = \underbrace{u_p}_{\text{part.}} + \underbrace{c_1 r^\ell + c_2 r^{-\ell-1}}_{\text{homog.}}$$

(I) Undetermined coeff. ($F = \text{const}$ or r^α)

$$u_p = \frac{r^{\alpha+2}}{\alpha^2 + m^2}$$

$$u_p = \frac{r^{\alpha+2}}{\alpha(\alpha+1) - \ell(\ell+1)}$$

(II) Green's fn: $K(r,\rho) = \frac{1}{W(y_1, y_2; \rho)} \begin{cases} y_1(r) y_2(\rho); & r<\rho \\ y_1(\rho) y_2(r); & r>\rho \end{cases}$

$y_1\ y_2$	W	K
$\{1,\ \ln r/a\}$	$1/r$	$\rho \begin{cases} \ln \rho/a; & r<\rho \\ \ln r/a; & r>\rho \end{cases}$
$\left\{r^m,\ \left(\frac{r}{a}\right)^m - \left(\frac{a}{r}\right)^m\right\}$	$\frac{2ma^{2m}}{r}$	$\frac{\rho}{2m}\left\{\left(\frac{r\rho}{a^2}\right)^m - \min\left(\frac{r}{\rho}, \frac{\rho}{r}\right)^m\right\}$
$\{1,\ 1 - a/r\}$	$W = \frac{a}{r^2}$	$\frac{\rho^2}{a} - \rho \begin{cases} 1; & r<\rho \\ \rho/r; & r>\rho \end{cases}$
$\left\{r^\ell;\ \left(\frac{r}{a}\right)^\ell - \left(\frac{a}{r}\right)^{\ell+1}\right\}$	$W = \frac{(2\ell+1)a^{\ell+1}}{r^2}$	$\cdots\cdots$

$$\boxed{u_p = \int_0^a K(r,\rho) F(\rho)\, \rho\, d\rho}$$

(III) Eigen (Bessel-Fourier) expansion:

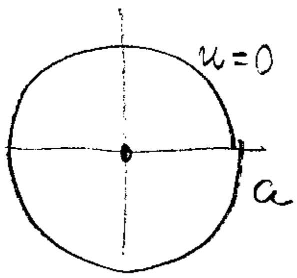
$$\psi_k = \begin{cases} J_0\left(z_k \frac{r}{a}\right), & m=0 \\ J_m\left(z_{mk} \frac{r}{a}\right) & m \neq 0 \\ \frac{1}{r} J_{\ell+\frac{1}{2}}\left(z_k \frac{r}{a}\right); & \ell \end{cases}$$

$$F = \sum_k \hat{F}_k \psi_k(r) \Rightarrow \boxed{u_p(r) = \sum \frac{\hat{F}_k}{\lambda_k} \psi_k(r)}$$

$$\boxed{\lambda_k = (z_{mk}/a)^2}$$

M445 Examples:

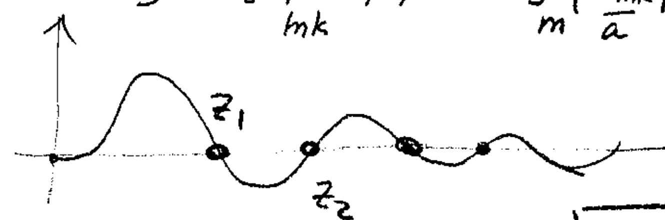
J-Bessel expansion for δ-source in the disk



$$(*)\ \begin{cases} u_t - \Delta u = F = \delta_2(r) = \frac{1}{2\pi r}\delta(r) \\ u|_{r=a} = 0 \end{cases}$$

Complete eigenexpansion: $L = -\Delta|_D \to \begin{cases} \lambda_{mk} = \left(\frac{z_{mk}}{a}\right)^2 \\ \psi_{mk}(r,\theta) = e^{\pm im\theta} J_m\left(\frac{z_{mk} r}{a}\right) \end{cases}$

$\{z_{mk}\}_{k=1}^{\infty}$ - zeros of $J_m(z) = 0$



For radial sourse need only radial modes $\boxed{m=0}$

$$\begin{cases} \psi_k(r) = J_0\left(\frac{z_{0k}}{a} r\right); & \langle \psi_k | \psi_l \rangle = 2\pi\int_0^a J_0(\ldots) J_0(\ldots) r\,dr = \frac{[a J_1(z_{0k})]^2}{2} \delta_{kl} \\ \lambda_k = \left(\frac{z_{0k}}{a}\right)^2 \end{cases}$$

orthog. relations

Solution (*)

$$u = \sum_1^{\infty} \frac{1 - e^{-\lambda_k t}}{\lambda_k} \hat{F}_k \psi_k; \quad \boxed{\hat{F}_k = \frac{2}{[a J_1(z_{0k})]^2} \int_0^a F(r) J_0(z_{0k} r) r\,dr}$$

J-Bessel coeff.

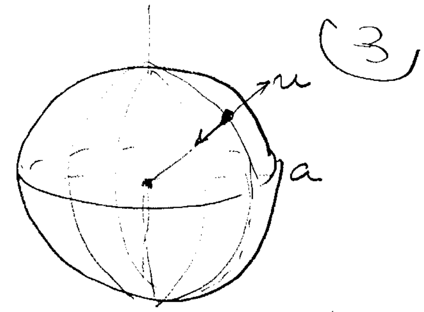
For $F = \delta_2(r) \Rightarrow \hat{F}_k = \frac{2 J_0(0)}{[a J_1(\ldots)]^2}$

Equilibr.: $V(r) = 2\sum_1^{\infty} \frac{1}{[z_{0k} J_1(z_{0k})]^2} J_0\left(z_{0k}\frac{r}{a}\right) = \boxed{-\frac{1}{2\pi}\ln\frac{r}{a}}$ - from ODE $\begin{cases} -(rV_r)_r = r\delta(r) \\ V|_{r=a} = 0 \end{cases}$

Solut. (*)

$$\boxed{u(r,t) = -\frac{1}{2\pi}\ln\frac{r}{a} + 2\sum_1^{\infty} \frac{e^{-(z_{0k}/a)^2 t} J_0\left(\frac{z_{0k}}{a} r\right)}{[z_{0k} J_1(z_{0k})]^2}}$$

2. Vibrating spherical shell:



→ radial displacement: $r = a + u(\phi, \theta, t)$

→ Energy: $P = \iint_{r=a} T\left(\sqrt{|\nabla u|^2 + 1} - 1\right) dS' \approx \frac{1}{2} \iint T |\nabla u|^2 dS'$

spherical Laplace

(E-L) $\begin{cases} \rho u_{tt} - \frac{T}{a^2} \boxed{\Delta_S} u = \dots \\ u|_{t=0} = f; \quad u_t|_{t=0} = g \end{cases}$

Solution: $u = \sum_{k=1}^{\infty} \underbrace{\left[\sum_{m=-k}^{k} A_{km} \cos(\omega_k t - \phi_{km}) Y_k^m \right]}_{\text{Single frequency modes: } \omega_k = \frac{c}{a}\sqrt{k(k+1)}}$

Sound spectrum: $\sqrt{2}, \sqrt{6}, \sqrt{12} \dots$

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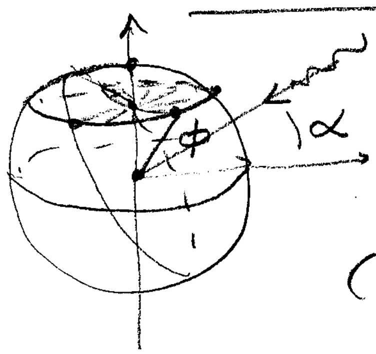
3. Radiating heat source:



$I(\phi)$ is obtained by averaging radiation flux over illuminated latitudinal band $0 < \phi < \pi$. (complicated f-n of ϕ).

Stationary heat distribution inside the sphere is harmonic f-n:

(D) $\begin{cases} \Delta u = 0 \\ u|_{r=a} = I \end{cases}$ or (N) $\begin{cases} \Delta u = 0 \\ u_r|_{r=a} = I \end{cases}$

Expand: $I = \sum_0^{\infty} \hat{I}_k Y_k^0(\phi) \implies$

$$\boxed{u(r, \phi) = \sum_0^{\infty} \hat{I}_k \left(\frac{r}{a}\right)^k Y_k^0(\phi)}$$