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Thermostat feedback control problem

Setup: long (1D) room with thermostat at the left end & heater at the right

$u(x,t)$ – temperat.

Processes: 1) Heat diffusion along the room $= D u_{xx}$

2) Lateral radiation/absorption $= a(u-U)$

U – outside temperature

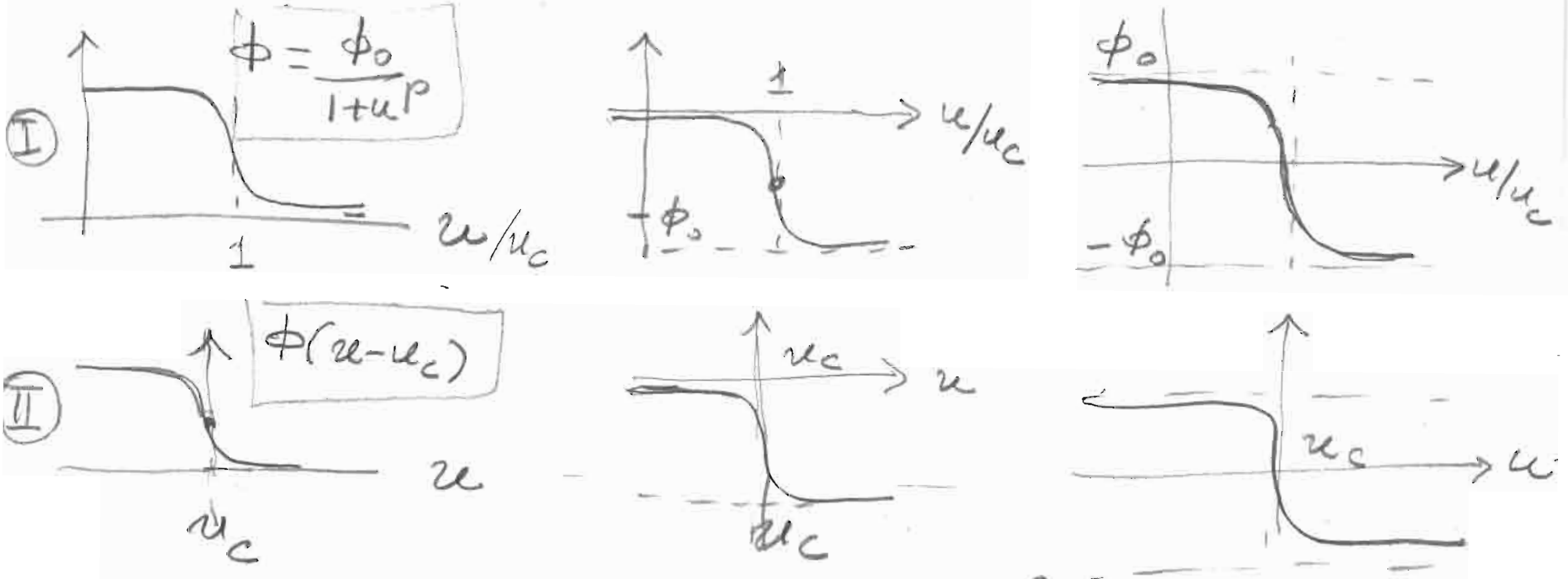
DE model:

$$(1)\quad \begin{cases} u_t - D u_{xx} + a(u-U) = 0; & 0<x<\ell \\ u_x\big|_{x=0} = 0 & \text{(insulated left end)} \\ u_x\big|_{x=\ell} = \Phi\left(\frac{u(0,t)}{u_c}\right) & \text{– control f-n} \end{cases}$$

Thermostat control f-n Φ compares temperature $u(0,t)$ at $x=0$ with "desired threshold" u_c and turns heat-flux at $x=\ell$ "on"/"off".

Sigmoid (step) control f-ns

Heater	Cooler (Air-cond)	Combined



Choices of
$$\begin{cases} \phi(u) = \frac{1}{1+u^p} \text{ or } \frac{u^p}{1+u^p}\,; \quad \frac{1-u^p}{1+u^p} & \text{(I)} \\ \phi(u) = \frac{1-\tanh u}{2}\,; \quad -\frac{1+\tanh u}{2}\,; \quad -\tanh u & \text{(II)} \end{cases}$$

Equilibrium temperature:

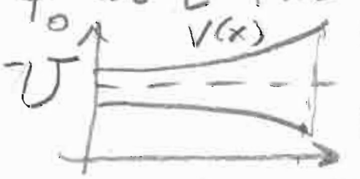
Solution DE + BC$|_0$

$$\begin{cases} V_{xx} - \frac{a}{D}(V - U) = 0 \\ V_x|_0 = 0; \quad V_x|_\ell = \phi(V(0)) \end{cases} \Longrightarrow \boxed{\begin{array}{c} V(x) = U + A \operatorname{ch} \alpha x \\ \alpha = \sqrt{a/D} \end{array}} \quad (2)$$

To find undetermined A use BC at $x = \ell$

Case 1°: steady heat-flux ϕ_0 at ℓ (no thermostat)

$$\Rightarrow \boxed{A = \frac{\phi_0}{\alpha \operatorname{sh} \alpha \ell}}$$

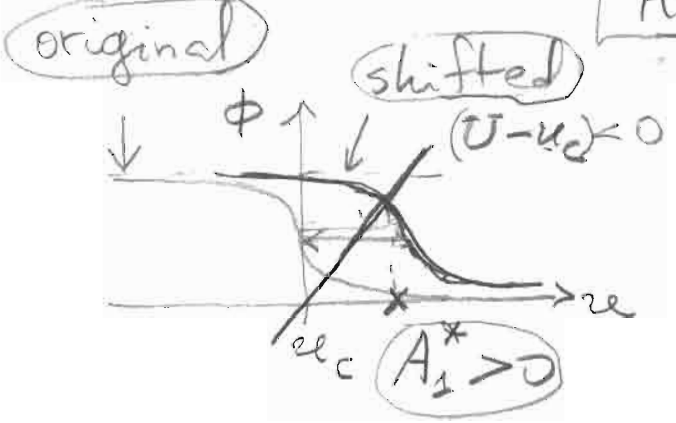


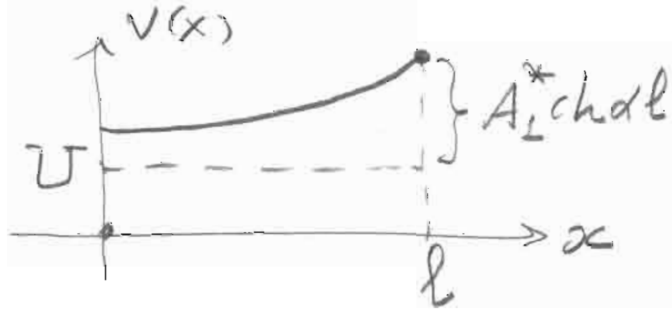
Problem 1:
Show that equilibrium (2) is always stable by solving E-V problem for (2)

Case 2°: Thermostat equilibrium eq-n

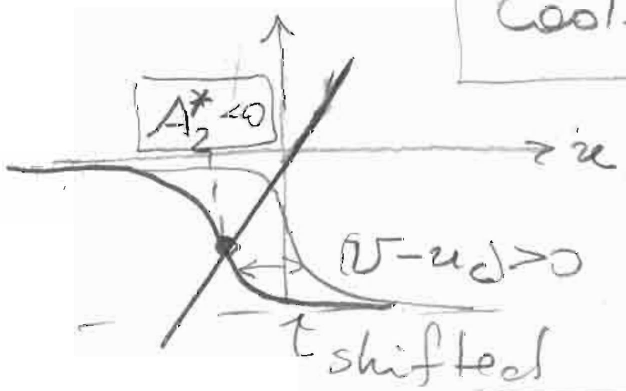
$$BC|_{\ell} \Rightarrow \boxed{(\underbrace{\alpha\, sh\, \alpha \ell}_{\text{linear}})\, A = \underbrace{\phi(A+U)}_{\text{shifted } \phi}} \Rightarrow A^*(\ldots)$$

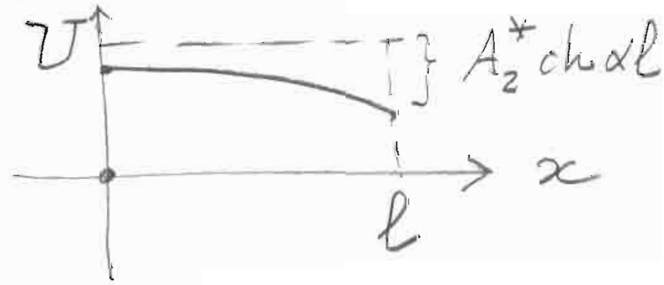
Heater: $U < u_c$



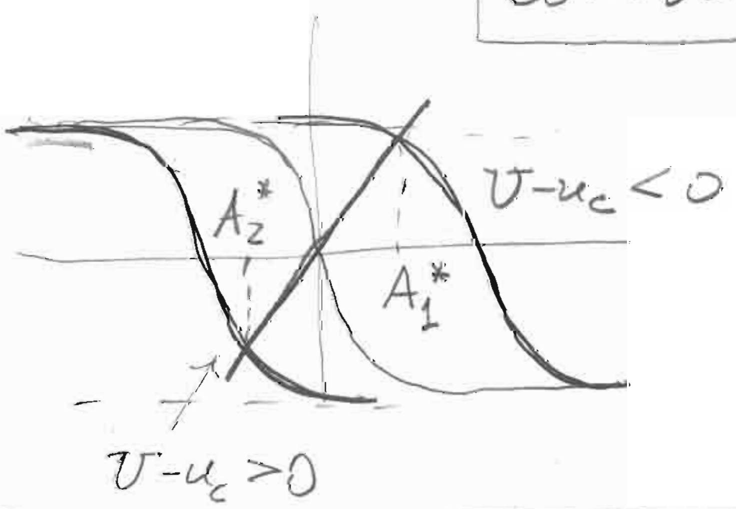


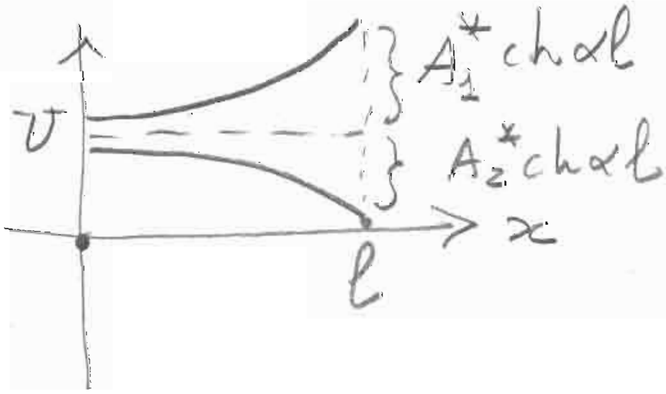
Cooler: $U > u_c$





Combined





Problem 2: Is mean temperature $V_0 = \frac{1}{\ell}\int_0^{\ell} V\,dx$ above or below u_c in each case?

Stability analysis

Linearized problem about equilibrium

$u \to V(x) + u(x,t)$ ← perturbation

Linear diffusion-radiation eq-n

$$(2)\begin{cases} u_t - D u_{xx} + a u = 0; \\ u_x\big|_{x=0} = 0; \quad u_x\big|_{\ell} = \dfrac{\phi'(V(0))}{u_c}\, u\big|_0 = -b u\big|_0 \end{cases}$$

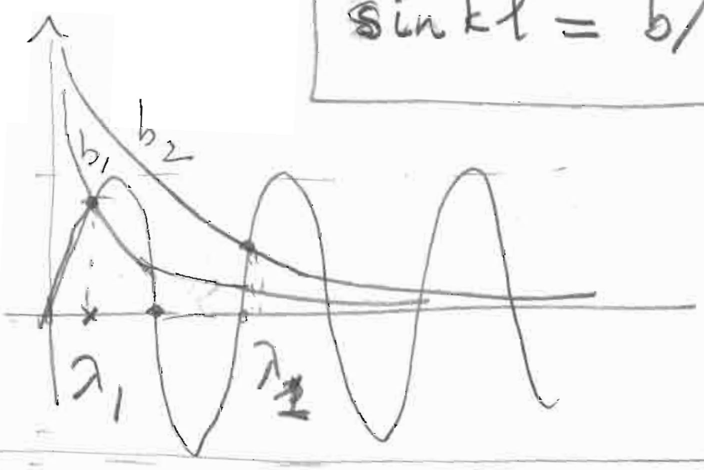
Show: coefficient $b < 0$

Sturm-Liouville problem (2) has eigenvalues: $\begin{cases} \lambda_k = Dk^2 + a \\ \psi_k(x) = \cos kx \end{cases}$ with charact. eq-n for k

$$\boxed{\sin k\ell = b/k}$$

⇒ 1) All solutions $u(x,t) \to V(x)$ equilibrium

2) Relaxation rate $\lambda_1(b)$ can be large $\boxed{\lambda_1 > \frac{2\pi}{\ell}}$ (for b_2) or small $\boxed{\lambda_1 < \pi/\ell}$ (- b_1)



Projects: 1) Develop solution methods for (1) with time-dependent (periodic) $U(t)$

2) Study the effect of time-lag in thermostat control.