445

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Quasi-linear characteristics

"linear in u_x, u_t"

DE: $u_t + c(u) u_x = S(u, x, t)$ ← source

IVP $\left\{ u|_{t=0} = f(x) \right.$

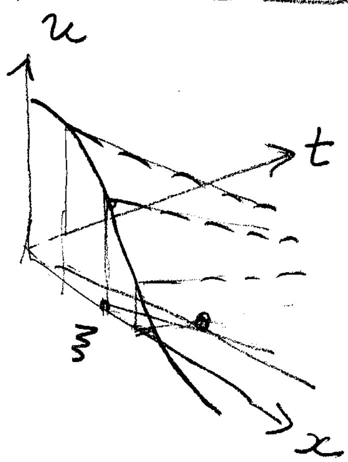
Charact. flow is a coupled system

$$\left\{ \begin{array}{l} \frac{dx}{dt} = c(u) \\ \frac{du}{dt} = S \end{array} \right. \quad \left. \begin{pmatrix} x \\ u \end{pmatrix} \right|_{t=0} = \begin{pmatrix} \xi \\ f(\xi) \end{pmatrix} \Rightarrow \boxed{\begin{array}{c} x(\xi, t) \\ u(\xi, t) \end{array}}$$

Solution

depends on IC + BC

Case $S = 0$: $\left\{ \begin{array}{l} u = f(\xi) \text{ - const} \\ x = \xi + c(u) t \end{array} \right.$



Implicit solution:

$$\boxed{u = f(\underbrace{x - c(u)t}_{})} \overset{?}{\Rightarrow} u(x, t)$$

* multivalued charact. surface?
* Blow up time - ?

For blow-up t ask: $u_x = \frac{f'(\xi)}{1 + t\,(c \circ f)'(\xi)} = \infty$

$$\Rightarrow \quad \boxed{t_{blow} = -\frac{1}{(c \circ f)'(\xi)}}$$

Examples: 1) $\begin{array}{l} c = u \\ f = \cos x \end{array} \Rightarrow t_{bl} = -\frac{1}{f'(\xi)} = \frac{1}{\sin \xi} \Rightarrow \boxed{\xi = \pm \frac{\pi}{2};\ t = 1}$

2) $\begin{array}{l} c = u^2 \\ f = \cos x \end{array} \Rightarrow t_{bl} = -\frac{1}{2 f f'(\xi)} = \frac{1}{\sin 2\xi} \Rightarrow \boxed{\xi = \pm \frac{\pi}{4};\ t = 1}$

Weak solutions & shock propagation for conservation laws in 1D

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A conservation law in 1D is given by

$$(*) \quad \boxed{u_t + [C(u)]_x = 0} \quad c(u) = C'(u) \text{ - speed of propag.}$$

A concept of weak solution applies to discontinuous solutions of DE $u(x,t)$

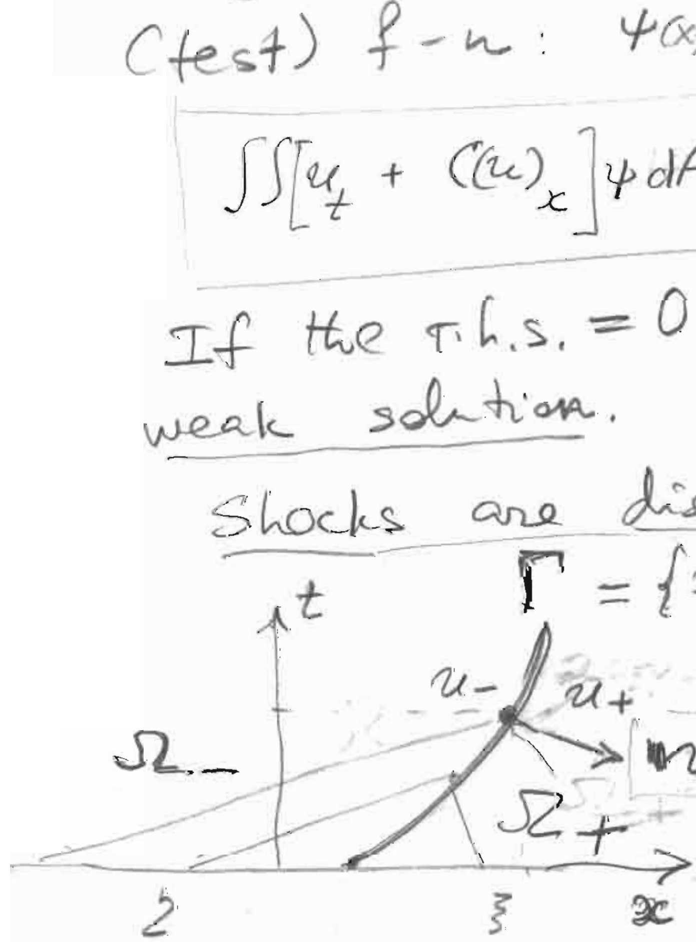
We replace (*) by an equivalent integral eq-n w. an arbitrary probbing (test) f-n: $\psi(x,t)$

$$\boxed{\iint [u_t + (C(u))_x]\psi \, dA = -\iint [\psi_t u + \psi_x C(u)] dA = 0}$$

If the r.h.s. $= 0$ for any ψ we call $u(x,t)$ weak solution.

Shocks are discont. weak solutions of (*):

$\Gamma = \{x(t)\}$ - shock position/path



$\boxed{n\, ds} = \boxed{(1, -x'(t))dt}$

n = unit normal to shock path Γ - ?

Apply divergence Thm. to discont. vector field $\mathbb{F} = \psi(C(u), u) = \begin{cases} \mathbb{F}_+ & \text{in } \Omega_+ \\ \mathbb{F}_- & \text{in } \Omega_- \end{cases}$

$$\iint_{\Omega_-} \nabla\cdot\mathbb{F}\,dA = \iint \left[\psi_x\, C(u) + \psi_t\, u\right] + \psi\,[\underbrace{u_t + C(u)_x}_{=0}] = \oint_{\Gamma} \mathbb{F}_-\cdot n\,ds$$

$$\iint_{\Omega_+} \nabla\cdot\mathbb{F}\,dA = \dots = -\oint_{\Gamma} \mathbb{F}_+\cdot n\,ds;$$

Use $n\,ds = (1, -\frac{dx}{dt})\,dt$

$$0 = \iint_{\Omega_+\cup\Omega_-} \nabla\cdot\mathbb{F}\,dA = \oint_{\Gamma} \psi \underbrace{\left[C(u) - u\frac{dx}{dt}\right]}\Big|_{\oplus}^{\ominus} dt \qquad (2)$$

Designate jump-across Γ by $[\dots]$

Since (2) holds ~~for~~ any Γ $\Longrightarrow$

$$(\text{R-H}) \quad \boxed{\frac{dx}{dt} = \frac{[C(u)]}{[u]} = \frac{C(u)_+ - C(u)_-}{u_+ - u_-}}$$

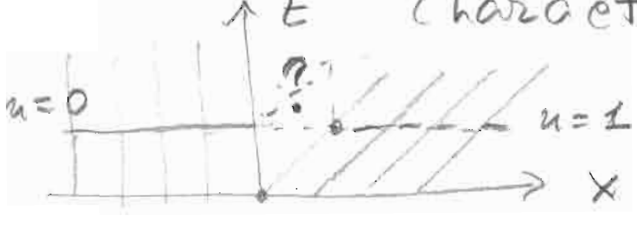
Rankine-Hugoniot shock eq-n

Example 1: Inviscid Burgers: $\begin{cases} u_t + u u_x = 0 \\ u|_{t=0} = g(x) \end{cases}$

Conserv. form $C(u) = u^2/2$ $\Rightarrow$ R-H $\boxed{\frac{dx}{dt} = \frac{u_+^2 - u_-^2}{2(u_+ - u_-)} = \frac{u_+ + u_-}{2}}$

(I) Rarefaction wave;

charact.



$$u(x,t) = \begin{cases} 0; & x < 0 \\ x/t; & 0 < x < t \\ 1; & x > t \end{cases} \qquad (3)$$

Explanation of rarefaction wave (3)

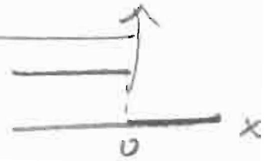
Problem: 1) check that any Burgers solution linear in x, $\boxed{u = p(t)\,x}$ obeys ODE $p' + p^2 = 0 \Rightarrow \boxed{p = \dfrac{p_0}{1 + p_0 t}}$ or $1/t$.

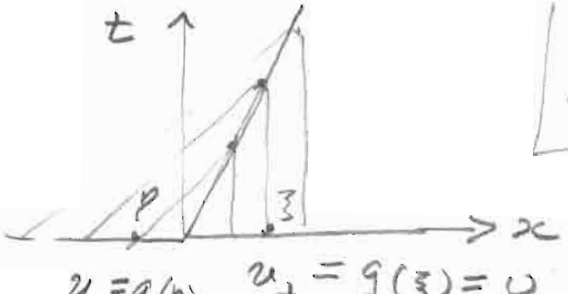
2) More general separable solution:

$$u = p(t)\,q(x) \Rightarrow \underbrace{\frac{p'}{p^2}}_{-m} + \underbrace{q'(x)}_{m} = 0 \qquad m - \text{const}$$

$$\Rightarrow \boxed{u = \frac{(q_0 + m x)\,p_0}{1 + m p_0 t}}$$

(II) Shock waves for step $g(x) =$





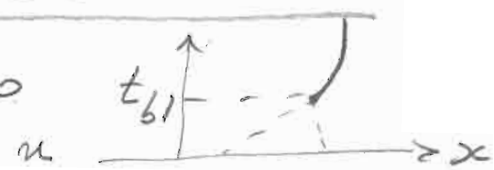
$u_- = g(\eta) = 1 \quad u_+ = g(\xi) = 0$

$$\boxed{\frac{dx}{dt} = \frac{u_+ + u_-}{2} = \frac{1}{2}} \quad R-H$$

Entropy condition: $c(u_+) \le \dfrac{dx}{dt} \le c(u_-)$

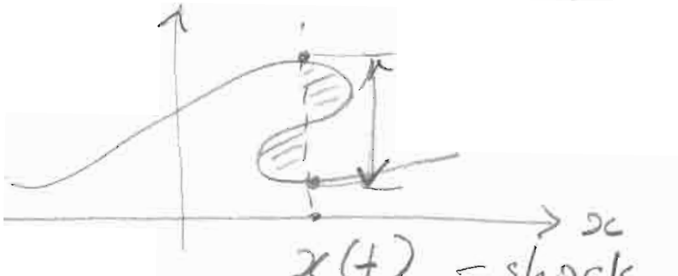
Shock formation & structure

1) Initiation at 1st blowup t_{bl}



2) Equal-area rule:

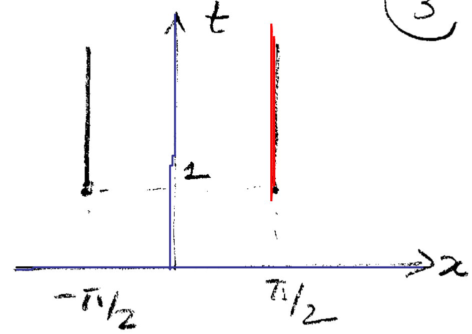
(from mass conservation $\displaystyle\int_{-\infty}^{\infty} u\,dx = \text{const}$)



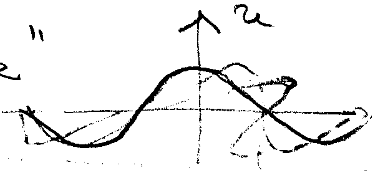
$x(t)$ – shock position

Examples:

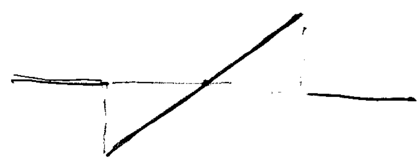
1) $\left\{\begin{array}{l} u_t + u u_x = 0 \\ u|_{t=0} = \cos x \end{array}\right] \Rightarrow$ Charact $\boxed{x = \xi + t\, f(\xi)}$, $t_{bl} = -\frac{1}{\sin\xi}$



Shocks start at $\{\pm\pi/2\}$ at $t_{bl} = 1$, but they don't propagate by the "area rule"

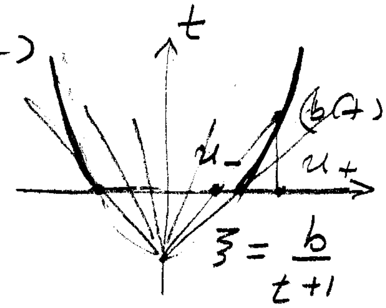


2) N-wave: $f(x) =$



$= \left\{\begin{array}{ll} x; & |x| < 1 \\ 0, & |x| > 1 \end{array}\right.$

Solution: $u(x,t) = \left\{\begin{array}{ll} x/t; & |x| \le b(t) \text{ - shock} \\ 0; & |x| > b(t) \end{array}\right.$

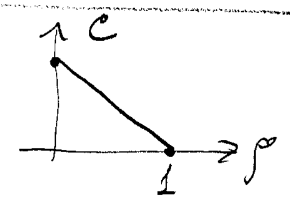


Charact: $x = \xi + t f(\xi) = \xi(1+t)$

R-G: $\frac{db}{dt} = \frac{u_+ + u_-}{2}$; $\left\{\begin{array}{l} u_+ = 0 \\ u_- = \frac{b}{t+1} \end{array}\right.$

Shock eq-n: $\frac{db}{dt} = \frac{b}{2(t+1)} \Longrightarrow \boxed{b = \pm\sqrt{t+1}}$ parabola

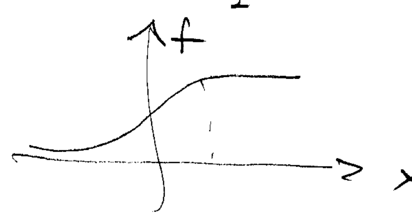
3) Traffic: $\rho_t + (1-\rho)\rho_x = 0$



Cons. form: $C(\rho) = \rho(1 - \rho/2)$;

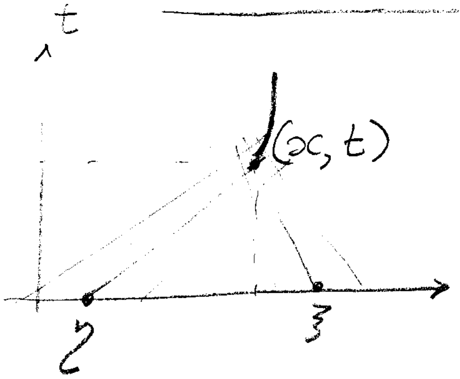
Charact: $x = \xi + (1 - f(\xi))t$

Blow up: $t_{bl} = \frac{1}{f'(\xi)}$



R-G: $\boxed{\frac{dx}{dt} = 1 - \frac{u_+ + u_-}{2}}$?

Characteristic coordinates for $u_t + [C(u)]_x = 0$



Any initial profile $g(\xi)$ defines local characteristic coordinates, where two families of characteristics meet (speed $c = f'(u)$)

Eq-ns: $\quad x = \eta + t\, c \circ g(\eta) = \xi + t \cdot (c \circ g)(\xi) \Rightarrow$

$$(1) \begin{cases} t = \dfrac{\xi - \eta}{(c\circ g)(\eta) - (c \circ g)(\xi)} = t(\xi, \eta) \\[2ex] x = \eta + \dfrac{(\xi - \eta)\ (c\circ g)(\eta)}{(c\circ g)(\eta) - (c\circ g)(\xi)} = x(\xi, \eta) \end{cases}$$

Shock evolution could be described either in (x,t)-variables $\quad x = x(t)$ or in (ξ, η)-variables: $\quad \xi = \xi(\eta)$

(Rankin-Hugonio) $\quad \boxed{\dfrac{dx}{dt} = \dfrac{C(g(\xi)) - C(g(\eta))}{g(\xi) - g(\eta)}} = V(\xi, \eta)$

mixes $\dfrac{dx}{dt}$ (left) with $V(\xi, \eta)$ (right).

To get (ξ, η)-equation for shock, write

$$\frac{dx}{dt} = \frac{x_\xi \frac{d\xi}{d\eta} + x_\eta}{t_\xi \frac{d\xi}{d\eta} + t_\eta} = V \quad \text{and solve for } \frac{d\xi}{d\eta}$$

$$\Rightarrow (2) \quad \boxed{\frac{d\xi}{d\eta} = \frac{t_\eta V - x_\eta}{-x_\xi V + t_\xi} = F(\xi, \eta)}$$

Rankin-Hugonio in char. coord.

Example: $\boxed{u_t + u u_x = 0}$ speed $c = u$; $C = u^2/2$

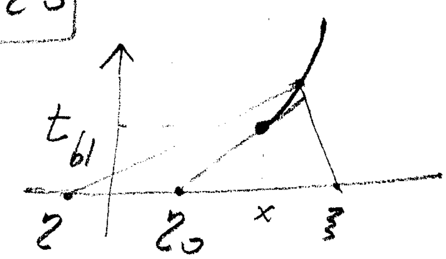
$$V = \frac{g(\xi)+g(\eta)}{2}; \quad t = \frac{\xi-\eta}{g(\eta)-g(\xi)}; \quad x = \eta + \frac{\xi-\eta}{g(\eta)-g(\xi)}\, g(\xi)$$

$$\Rightarrow \boxed{F(\xi,\eta) = \frac{g(\xi)-g(\eta)-(\xi-\eta)g'(\eta)}{-g(\xi)+g(\eta)+(\xi-\eta)g'(\xi)}} \approx 1 - \frac{g'''(\eta)}{3g''(\eta)}(\xi-\eta)$$

for $\xi \approx \eta$

Shock evolution:

(i) Solve $\boxed{\frac{d\xi}{d\eta} = F(\xi,\eta); \quad \xi(\eta_0) = \eta_0}$; η_0 = first blowup



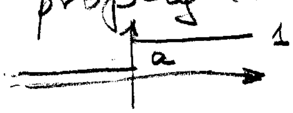
(ii) Parametric shock path:

$$\begin{pmatrix} x(\xi(\eta),\eta) \\ t(\xi(\eta),\eta) \end{pmatrix}$$

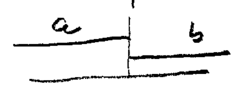
(iii) Jump: $[u(x,t)] = g(\eta) - g(\xi)$

Problem: (traffic flow)

(i) Find shock propagation for traffic flow if

(a) $g =$ 

$(g_- = a < 1;\ g_+ = 1)$

(b) $g =$ 

$(g_- = a > g_+ = b > 0)$

(ii) Same problem for Gaussian bump $g(x) = e^{-x^2/2}$ using the shock eq-n (2) and Mathematica.