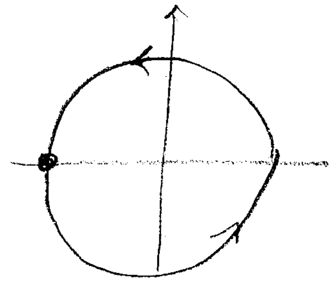
Periodic B.C.

$L = -p\partial_x^2 + q$ on $[-\pi, \pi]$ =



P.B.C.: $\psi\big|_{-\pi} = \psi\big|_{\pi}$; $\psi'\big|_{-\pi} = \psi'\big|_{\pi}$

E-V problem: $\{\psi_k = e^{ikx}\}_{-\infty}^{\infty}$; $\boxed{\lambda_k = pk^2 + q}$

F. harmonics

$$\left.\begin{array}{l}
\text{1) Elliptic problem:} \\
L^{-1} = K(x-\xi) = \frac{1}{2\pi}\sum_{-\infty}^{\infty}{}' \frac{e^{ik(x-\xi)}}{\lambda_k} \\
\text{2) Heat:} \\
e^{-tL} = G(x-\xi, t) = \frac{1}{2\pi}\sum_{-\infty}^{\infty}{}' e^{-\lambda_k t} e^{ik(x-\xi)} \\
\text{3) Wave:} \\
\frac{\sin\sqrt{\lambda_k}\, t}{\sqrt{\lambda_k}} = S(x-\xi, t) = \sum_k{}' \frac{\sin\sqrt{\lambda_k}\, t}{\sqrt{\lambda_k}} e^{ik(x-\xi)}
\end{array}\right\} \text{convolution kernels in } x$$

$\sum{}' \ldots$ summation over all $\lambda_k \neq 0$

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* Series 1) – 3) are slowly converging
* Accuracy may require "large truncation" ($\sum_{-N}^{N} \ldots : N \gg 1$)!

Basis Fourier transform

	continuous analogues
Direct: $f(x) \xmapsto{\mathcal{F}_{x\to k}} \hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$	$\underbrace{\langle f \mid e^{ikx}\rangle}_{\|\cdots\|^2}$
Inverse: $\hat{f}(k) \xrightarrow{\mathcal{F}^{-1}_{k\to x}} f(x) = \int \hat{f}(k)\, e^{ikx}\, dx$	$\sum \hat{f}_k\, e^{ikx}$

Other normalizations:

$0 \le a \le 1$

$$f(x) \to \frac{1}{(2\pi)^a}\int \dots$$

$$\hat{f}(k) \to \frac{1}{(2a)^{1-a}}\int \dots$$

Choise $(a=\frac{1}{2})$ make $\mathcal{F}$ - a "unitary map"

$$\int |f|^2 dx = \boxed{\|f\|^2 = \|\hat{f}\|^2} = \int |\hat{f}|^2 dk$$

Examples:

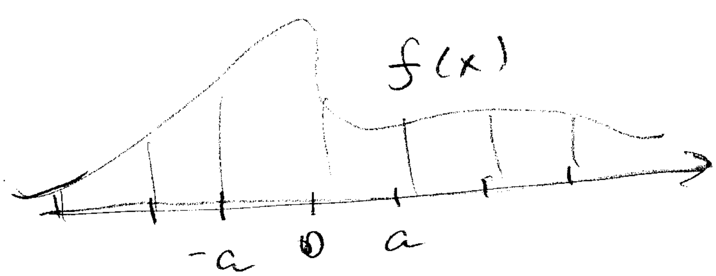
$f(x)$	$\hat{f}(k)$
$\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$	$e^{-tk^2/2}$
$e^{-a\lvert x\rvert}$	$\frac{a}{\pi(k^2+a^2)}$
$\delta(x)$	1
e^{iax}	$\frac{1}{2\pi}\delta(k-a)$

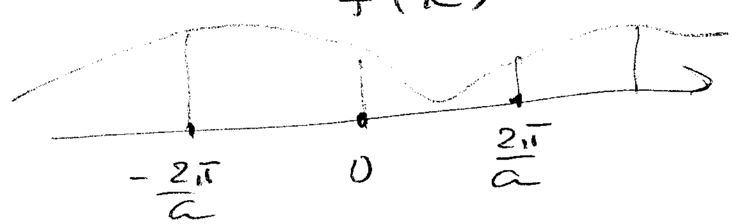
Poisson summation

For any "nice" (smooth, decaying) f-ns: $f(x)$ on lattice $\{ma\}_{-\infty}^{\infty}$

$\hat{f}(k)$ on dual lattice $\left\{ \frac{2\pi}{a} k \right\}_{k=-\infty}^{\infty}$

$$\boxed{\sum_{m=-\infty}^{\infty} f(ma) = \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{2\pi}{a} k\right)}$$

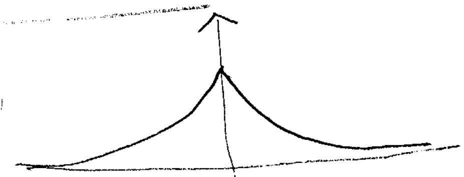




Applications (Periodic B.C.):

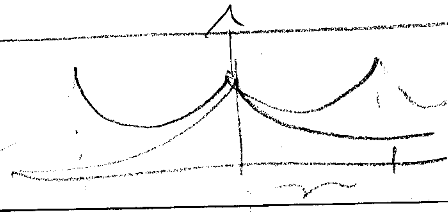
1. $L = -\partial_x^2 + b^2$ (KG) on $\mathbb{R}$, or circle

(ℝ) $K_0(x) = \dfrac{e^{-b|x|}}{2b}$



(Circle) $K(x) = \dfrac{1}{2\pi} \displaystyle\sum_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+b^2} = \sum_{m=-\infty}^{\infty} \frac{e^{-b|x-2\pi m|}}{2b}$

$$K(x) \approx \frac{1}{2b}\left(e^{-b|x|} + e^{-b|x-2\pi|} + e^{-b|x+2\pi|} \right)$$



3-term approximation

2. <u>Heat:</u> $(\partial_t - D\partial_x^2)u = \cdots$ on $\mathbb{R}$, circle

($\mathbb{R}$) $G_0(x,t) = \dfrac{e^{-x^2/4Dt}}{\sqrt{4\pi D t}}$;

(Circle): $G(x,t) = \displaystyle\sum_{m=-\infty}^{\infty} G_0(x-2\pi m, t) \approx \sum_{m=-1}^{1} G_0(\ldots)$