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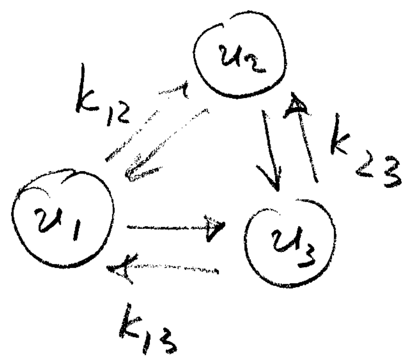
Eigenf-n expansion for ODS

(1) $\boxed{B\dot{u} + A[u] = F}$ (2) $\boxed{B\ddot{u} + A[u] = F}$

Heat exchange/migration

(1)



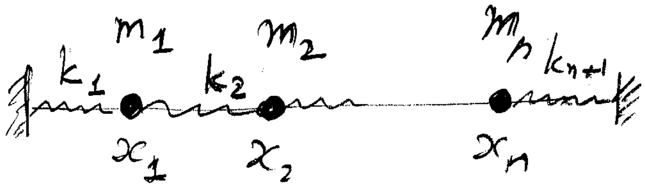
u_i - temp.

ρ_i - spec. heat

k_{ij} - conduct./exchange

$$\frac{d}{dt}\begin{pmatrix}\rho_1 u_1\\ \rho_2 u_2\\ \rho_3 u_3\end{pmatrix} = \begin{bmatrix}-(k_{12}+k_{13}) & k_{12} & k_{13}\\ k_{21} & -(k_{21}+k_{23}) & k_{23}\\ k_{31} & k_{32} & -(k_{31}+k_{32})\end{bmatrix}\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix} + \begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix}$$

(2)



Oscillator

$$\frac{d}{dt}\begin{pmatrix}m_1 x_1\\ m_2 x_2\\ \vdots\end{pmatrix} + \begin{bmatrix}-(k_1+k_2) & k_2 & 0 & \cdots & 0\\ k_2 & -(k_2+k_3) & k_3 & & 0\\ 0 & k_3 & -(k_3+k_4) & & k_4\\ 0 & 0 & k_4 & & -(k_4+k_5)\end{bmatrix}\begin{pmatrix}x_1\\ x_2\\ \vdots\\ \vdots\end{pmatrix} = F$$

Generalized E-V problem:

1) $\boxed{(\lambda B - A)X = \vec{0}} \Rightarrow \left\{ \begin{matrix} \lambda_1 \dots \lambda_n \\ X_1 \dots X_n \end{matrix} \right\}$ - real orthog. basis

$\{e^{\lambda_1 t} X_1 \dots e^{\lambda_n t} X_n\}$ - fund. solution

2) $\boxed{(\omega^2 B + A)X = \vec{0}} \Rightarrow \left\{ \begin{matrix} \omega_1 \dots \omega_n \\ X_1 \dots X_n \end{matrix} \right\}$

$\{e^{\pm i\omega_1 t} X_1 \dots e^{\pm i\omega_n t} X_n\}$

General solution: expansion

1) $\left\{ \begin{aligned} B\dot{\vec{u}} + A\vec{u} &= F(t) \\ u|_{t=0} &= f \end{aligned} \right.$

$$\boxed{\begin{aligned} \vec{u} &= \sum u_k(t) X_k \\ F &= \sum F_k(t) X_k \\ f &= \sum f_k X_k \end{aligned}}$$

Reduced ODEs for coord. f-ns

$$\underbrace{(\dot{u}_k B + u_k A)\boxed{X_k}}_{} = F_k \boxed{X_k}$$

$$\left\{ \begin{aligned} \dot{u}_k + \lambda_k u_k &= F_k(t) \\ u_k(0) &= f_k \end{aligned} \right.$$

$$\boxed{u_k(t) = e^{-\lambda_k t} f_k + \int_0^t e^{-\lambda_k(t-\tau)} F_k(\tau)}$$

2) $$\begin{cases} B\ddot{\vec{u}} + A\vec{u} = F \\ \vec{u}\big|_{t=0} = f \; ; \; \dot{\vec{u}}\big|_0 = g \end{cases} \; ; \qquad \vec{u} = \sum u_k(t) X_k$$

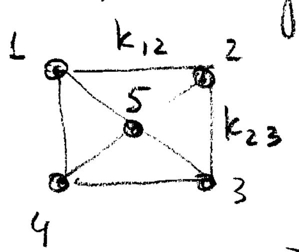
$$u_k(t) = \cos\omega_k t \, f_k + \frac{\sin\omega_k t}{\omega_k} g_k + \int_0^t \frac{\sin\omega_k(t-\tau)}{\omega_k} F_k(\tau)$$

Problem:

1. a) Show that heat exchange problem (1) for n bodies has an equilibrium solution iff total source: $\sum f_i = 0$ (Hint: matrix A is singular with null-vector $X = \ldots$)

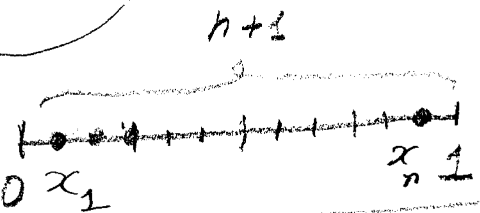
b) Show that all other eigenvalues of exchange matrix A are negative (Hint: $AX \cdot X = -\sum_{i<j} k_{ij} \cdot (x_i - x_j)^2$, for any vector $X = (x_1 \ldots x_n)$)

c) Compute equilibrium for 5 bodies arrange in a square with exchange coefficients inversely proportional to the length of sides ($k_{12} = 1$; $k_{15} = \sqrt{2}$, ...) the initial state $\vec{u}_0 = (0,0,0,0,1)$ and the source $F = (1,-1,1,-1,0)$ (Use computer).



Band matrix vs. S-L problem

$$A = \begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & & \\ & & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}; \qquad \frac{1}{(\Delta x)^2}A = \text{discretized } L = \frac{d^2}{dx^2} \text{ on } [0,1]$$



$$\{x_i = \tfrac{i}{n+1}\}_{i=1}^{n} \qquad \Delta x = \frac{1}{n+1}$$

Check: Eigenvalues: $\mu_k = 2\left(1-\cos\frac{\pi k}{n+1}\right)$ ← Finite

Eigenvectors: $X_k = \begin{pmatrix} \sin\frac{\pi k}{n+1} \\ \vdots \\ \sin\frac{\pi k j}{n+1} \\ \vdots \\ \sin\frac{\pi k n}{n+1} \end{pmatrix}$ ← Fourier modes

$k = 1, \dots, n$

Compare to S-L problem: $\begin{cases} w'' + \lambda w = 0 \\ w|_{0,1} = 0 \end{cases}$

$$\boxed{\begin{aligned} \lambda_k &= (\pi k)^2 \\ w_k &= \sin \pi k x \end{aligned}}$$

Continuous

$$\begin{cases} \dfrac{\mu_k}{(\Delta x)^2} = \left[4\sin^2\dfrac{\pi k}{2(n+1)}\right](n+1)^2 \approx (\pi k)^2 \\ X_k^j = w_k(x_j) \end{cases}$$

discrete