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Laplacian & harmonic f-ns

Topics:

1. Typical problems
2. Harmonic & analytic f-ns:
   i) Cauchy-Riemann eq-ns
   ii) Cauchy formula
   iii) Poisson f-la
3. Radial solutions & Potentials
4. Green's identities:
   i) Max principle
   ii) Dirichlet principle
   iii) Mean value
5. Special methods: eigenf-n expansion
   Application to rectangle, disk, annulus

Laplacian & harmonic f-ns:

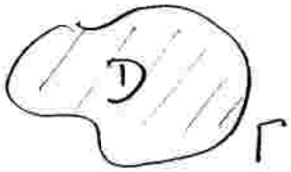
solutions of $\boxed{\Delta u = 0}$

Applications:

* Cauchy-Riemann eq-ns & complex analysis (analytic f-ns). Harmonic analysis.
* Steady temperature distributions (w/o sources)
* Electrostatics, potential theory
* Fluids: incompressible, irrotational flows

Typical problems:

(I) BVP (Dirichlet, Neumann): interior/exterior



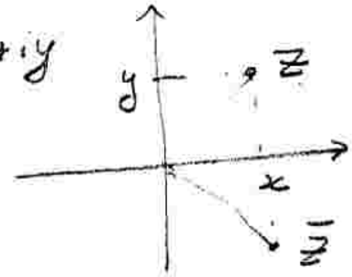
$$\begin{cases} \Delta u = 0 \text{ in } D \\ u|_{\Gamma} = b \end{cases} ; \quad \begin{cases} \cdots \\ \partial_n u|_{\Gamma} = b \end{cases}$$

(II) Inhomogeneous (Poisson) eq-n:

$$\begin{cases} -\Delta u = F \\ \text{B.C.} \end{cases} \quad \cdots$$

Harmonic & analytic f-ns in 2D

Use complex variable: $z = x + iy$



Any f-n $f(x,y)$ can be written as $f(z, \bar{z}) = \underset{\overset{\shortparallel}{Re(f)}}{u} + i \underset{\overset{\shortparallel}{Im(f)}}{V}$

Change of variables: $(x,y) \rightleftarrows (z, \bar{z})$ gives complex derivatives:

$$\partial_z = \frac{1}{2}(\partial_x + i\partial_y)$$

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x - i\partial_y)$$

Function f is <u>analytic</u> if $\boxed{\partial_{\bar{z}} f = 0}$

The latter is equivalent to a system of <u>Cauchy-Riemann</u> (C-R) eq-ns for $u = Re(f)$, $V = Im(f)$

$$\boxed{\begin{aligned} u_x + V_y &= 0 \\ u_y - V_x &= 0 \end{aligned}} \qquad (C\text{-}R)$$

Any <u>analytic</u> f-n $f(z)$ (or anty-analytic $f(\bar{z})$) can be <u>expanded</u> into a covergent Taylor series: $\boxed{f(z) = \sum_0^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k}$ or $\{(\bar{z}-a)^k\}$

Connection between analytic & harmonic f-ns

Laplacian: $\Delta = \partial_x^2 + \partial_y^2 = (\partial_x + i\partial_y)(\partial_x - i\partial_y)$

$$\boxed{\Delta = 4\,\partial_z \partial_{\bar z}}$$ "characteristics"

Corollary: If $f(z)$ analytic

then $u = Re(f)$, $v = Im(f)$ - harmonic

(check: $\boxed{\partial_z \bar f = \overline{\partial_{\bar z} f}}$, for any f. So for analytic $f(z)$, $\overline{f(z)}$ is anti-analyt.

Examples of 2D harmonic f-ns constructed from analytic f-ns (Taylor series)

1. Harmonic polynomials: $Re(z^n)$; $Im(z^n)$

$Re\,(x+iy)^n = x^n - \binom{n}{2} x^{n-2} y^2 + \ldots$

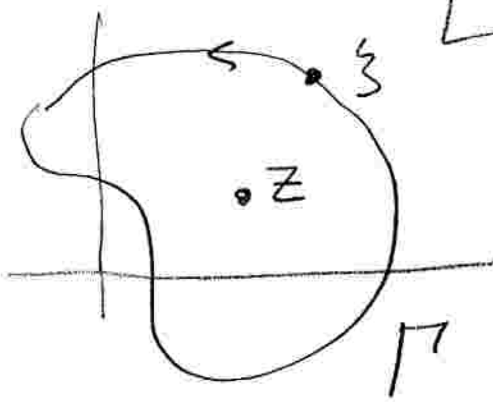
$Im(\quad)^n = n x^{n-1} y - \binom{n}{3} x^{n-3} y^3 + \ldots$

2. $e^{az} = e^{ax}(\cos ay + i \sin ay)$

3. $\ln z = \boxed{\frac{1}{2} \ln(x^2+y^2)} + i \boxed{\tan^{-1}(y/x)}$

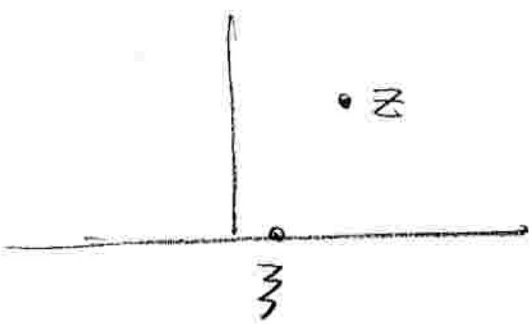
4. $\frac{1}{z-a} = \frac{x-a}{(x-a)^2+y^2} - i\,\frac{y}{(x-a)^2+y^2}$ — Cauchy learned

Cauchy f-la: $\boxed{f(z) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(\zeta)}{\zeta - z}\,d\zeta}$



Any analytic function f(z) in the region bounded by contour G can be represented by its boundary values through the Cauchy integral

Application to half-plane



$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - z}\,d\xi$$

For $u = Re(f)$ & $v = Im(f)$

get Poisson formula:

$$\boxed{u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2+y^2}\,d\xi}$$

— solution of the Laplace eq-n BVP

$$\begin{cases} \Delta u = 0; \quad u|_{y=0} = f(x) \end{cases}$$

Radial solutions $u = u(r)$

2D: $\Delta = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 \Rightarrow \frac{1}{r}(ru')' = 0 \Rightarrow \boxed{u = C_0 + C_1 \ln r}$

3D: $\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_S \Rightarrow \frac{1}{r^2}(r^2u')' = 0 \Rightarrow \boxed{u = C_0 + \frac{C_1}{r}}$

Green's identities

(I) $\boxed{\iint_D u\,\Delta u + |\nabla u|^2 = \oint_\Gamma u\,\partial_n u; \quad \text{any } u, D, \Gamma}$

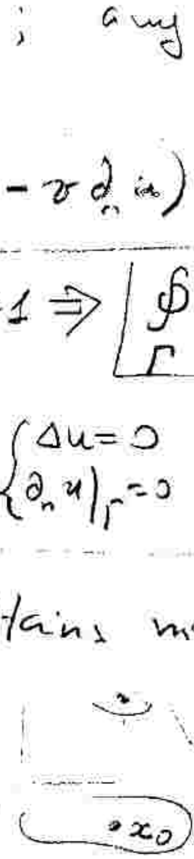
(II) $\boxed{\iint_D u\,\Delta v - v\,\Delta u = \oint_\Gamma (u\,\partial_n v - v\,\partial_n u); \quad \text{any } u, v}$

Corollaries: (1) Green II with harm. u & $v = 1 \Rightarrow \boxed{\oint_\Gamma \partial_n u = 0}$

(2) $\begin{cases}\Delta u = 0\\ u|_\Gamma = 0\end{cases} \Rightarrow \boxed{u = 0}$ - uniqueness; $\begin{cases}\Delta u = 0\\ \partial_n u|_\Gamma = 0\end{cases} \Rightarrow u = \text{const}$

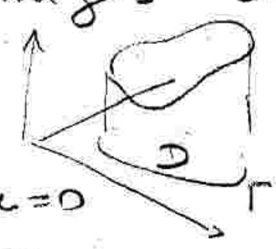
Max. Principle: Any harm $u(x)$ attains max/min on b-dary Γ

Proof: if x_0 - local max $\Rightarrow$ $\Delta u|_{x_0} < 0$



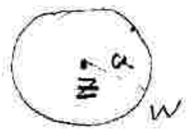
Dirichlet principle: Harmonic u with fixed b-dary values $u|_\Gamma = f$ minimizes Dirichlet Q.-form: $Q[u] = \frac{1}{2}\iint_D |\nabla u|^2$

Proof: (E-L) $\frac{\delta Q}{\delta u} = \nabla \cdot \nabla u = \Delta u = 0$



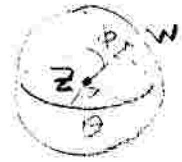
Mean value Thm: Any harm. u & disk/ball $\{|z-w|=a\}$

(i) 2D



$$u(z)=\frac{1}{2\pi a}\oint u(\underbrace{z+ae^{i\theta}}_{w})\,ds = \frac{1}{2\pi}\int_0^{2\pi} u(z+ae^{i\theta})\,d\theta$$

(ii) 3D



$$u(z)=\frac{1}{4\pi a^2}\oiint_{|w-z|=a} u(w)\,dS(w) = \frac{1}{4\pi}\iint \sin\phi\, d\phi\, d\theta\, u(\ldots)$$

Proof: Use Green II with $v = \begin{cases}\frac{1}{2\pi}\ln r\,; & 2D\\ \frac{1}{4\pi r}\,; & 3D\end{cases}$

and annular

$$\iint u\underbrace{\Delta v}_{=0} - v\underbrace{\Delta u}_{=0} = \underbrace{\left(\oint_{r=a} u\,\frac{1}{2\pi r}\,ds - \frac{\ln a}{2\pi}\underbrace{\oint\partial_n u}_{=0}\right)}_{\text{outer}} - \underbrace{\left(\oint_{r=\varepsilon} u\,\frac{1}{2\pi r} - \frac{\ln\varepsilon}{2\pi}\underbrace{\oint\partial_n u}_{=0}\right)}_{\text{inner}}$$

$$\Rightarrow \frac{1}{2\pi a}\oint_{r=a} u\,ds_a = \frac{1}{2\pi\varepsilon}\int_{r=\varepsilon} u\,ds_\varepsilon$$

← Mean value indep. of radius ⇒ $\boxed{=u(0)}$

$$\underline{3D}\quad \iiint_{\varepsilon\le|z|\le a}(\underbrace{u\Delta v - v\Delta u}_{=0}) = \left(\oiint_{r=a} u\,\frac{1}{4\pi r^2} - \frac{1}{4\pi a}\underbrace{\oiint\partial_n u}_{=0}\right) - \left(\iint_{r=\varepsilon} u\,\frac{1}{4\pi r^2} - \frac{1}{4\pi\varepsilon}\underbrace{\oiint}_{=0}\right)$$

$$\Rightarrow \boxed{\frac{1}{4\pi a^2}\oiint_{|z|=a} u\,dS_a = \frac{1}{4\pi\varepsilon^2}\oiint_{|z|=\varepsilon} u\,dS_\varepsilon = u(0)}\quad q.e.d.$$