Green’s functions

D. Gurarie

Green’s functions (fundamental solutions) of stationary and non stationary problems

$$\begin{aligned} &L[u]=F\\ &u_t+L[u]=F\\ &u_{tt}+L[u]=F \end{aligned} \qquad (1)$$

for matrix/operator L, or elliptic BVP $L[u]=\begin{cases}-\nabla\cdot p\nabla u+qu\\ B[u]|_\Gamma=0\end{cases}$. Eigendata of L $\{\lambda_k;\psi_k\}$

Expansion

Problem	Formal Solution	Fund. Solution	Expansion
$L[u]=F$	$u=L^{-1}[F]$	$L^{-1}=K(x,\xi)$	$\sum_k \frac{1}{\lambda_k}\frac{\lvert\psi_k\rangle\langle\bar\psi_k\rvert}{\lVert\psi_k\rVert^2}$
$u_t+L[u]=F$	$u(t)=e^{-tL}*F$	$e^{-tL}=G(x,\xi;t)$	$\sum_k e^{-t\lambda_k}\frac{\lvert\psi_k\rangle\langle\bar\psi_k\rvert}{\lVert\psi_k\rVert^2}$
$u_{tt}+L[u]=F$	$u(t)=\frac{\sin\left(t\sqrt{L}\right)}{\sqrt{L}}*F$	$\frac{\sin\left(t\sqrt{L}\right)}{\sqrt{L}}=S(x,\xi;t)$	$\sum_k \frac{\sin\left(\sqrt{\lambda_k}\,t\right)}{\sqrt{\lambda_k}}\frac{\lvert\psi_k\rangle\langle\bar\psi_k\rvert}{\lVert\psi_k\rVert^2}$

Bra-ket notation for orthogonal projection $\frac{\lvert\psi\rangle\langle\bar\psi\rvert}{\lVert\psi\rVert^2}$

1) for vector $\psi=(\psi_1,\dots,\psi_m,\dots)$, -> matrix

$$\frac{1}{\lVert\psi\rVert^2}\begin{bmatrix}\cdots\\ \psi_i\\ \cdots\end{bmatrix}\cdot\begin{bmatrix}\cdots & \bar\psi_j & \cdots\end{bmatrix}=\frac{1}{\lVert\psi\rVert^2}\begin{bmatrix} & \cdots & \\ \vdots & \psi_i\bar\psi_j & \vdots\\ & \cdots & \end{bmatrix} \qquad (2)$$

2) for function $\psi(x)$, -> integral kernel

$$\frac{\psi(x)\bar{\psi}(\xi)}{\|\psi\|^2}$$

Alternative representation in 1D for L^{-1}

For $L=-\partial_x p\partial_x+q$ on $[0,l]$ with BC $-u'+a_0u|_0=0;u'+a_1u|_l=0;$ find 2 solutions:

$$\text{Left BC: }\begin{cases}L[y_1]=0\\B[y_1]|_0=0\end{cases};\text{Right BC: }\begin{cases}L[y_2]=0;\\B[y_2]|_l=0\end{cases}\tag{3}$$

Then

$$K(x,\xi)=\frac{1}{p(\xi)W(\xi)}\begin{cases}y_1(x)y_2(\xi);x<\xi\\y_2(x)y_1(\xi);x>\xi\end{cases}\tag{4}$$

with Wronskian $W(x)=\begin{vmatrix}y_1(x) & y_2(x)\\y_1'(x) & y_2'(x)\end{vmatrix}$, provided $\{y_1(x),y_2(x)\}$ - independent pair: $y_1\neq cy_2$, or $\{y_i\}$ are 'zero eigenmodes' of L.

Examples

L, BC=(D)	Expansion of L^{-1}	Pair $\{y_1;y_2\}$; W	Kernel $K(x,\xi)$
$-\partial_x{}^2$ on $[0,l];$	$\frac{2}{l}\sum_{k=1}^{\infty}\frac{\sin(\pi kx/l)\sin(\pi k\xi/l)}{(\pi k/l)^2}$	$\{x,l-x\}$ $W=l$	$\frac{1}{l}\begin{cases}x(l-\xi);x<\xi\\\xi(l-x);x>\xi\end{cases}$
$\partial_x{}^2+a^2$ on $[0,l]$	$\frac{2}{l}\sum_{k=1}^{\infty}\frac{\sin(\pi kx/l)\sin(\pi k\xi/l)}{(\pi k/l)^2-a^2}$ $a\neq\frac{\pi k}{l}$	$\{\sin(ax),\sin[a(l-x)]\}$ $W=a\sin(al)$	$\begin{cases}\frac{\sin(ax)\sin[a(l-\xi)]}{a\sin(al)}\\\ldots\end{cases}$
$\partial_x{}^2-a^2$ on $[0,l]$	$\frac{2}{l}\sum_{k=1}^{\infty}\frac{\sin(\pi kx/l)\sin(\pi k\xi/l)}{(\pi k/l)^2+a^2}$	$\{\sinh(ax),\sinh[a(l-x)$ $W=a\sinh(al)$	$\begin{cases}\frac{\sinh(ax)\sinh[a(l-\xi)]}{a\sinh(al)}\\\ldots\end{cases}$

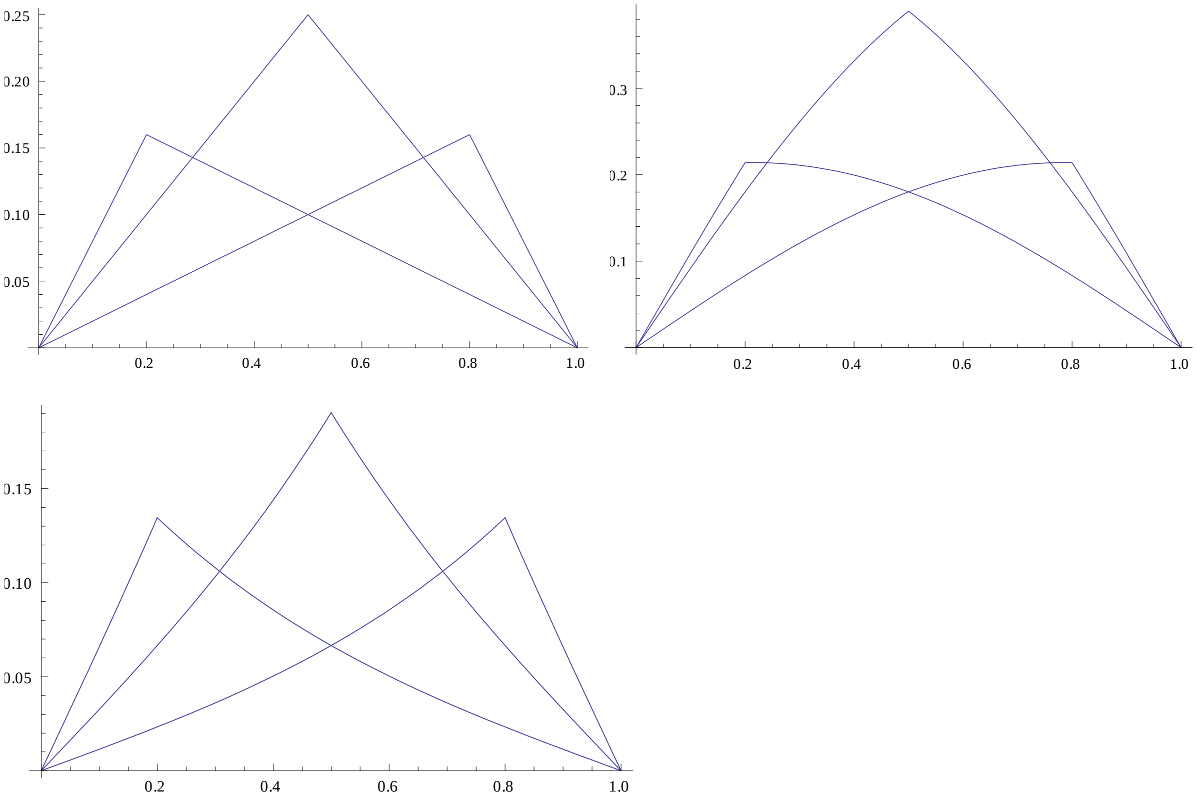

Figure 1: K(x,y$_j$) for cases 1,2,3

Example 4: $L = \partial_r{}^2 + \frac{1}{r}\partial_r - \frac{m^2}{r^2}$ on $0 < r < a$ (singular boundary r=0).

E-V problem:

i) Eigenmodes = Bessel modes: $\psi_{m,k}(r) = \{J_m(z_k r/l), Y_m(z_k r/l)\}, \{z_k\}$ = Bessel roots);

ii) Eigenvalues: $\lambda_k = \left(\frac{z_k}{l}\right)^2$

Fundamental pair: $\left\{r^m; (r/a)^m - (a/r)^m\right\}$ (m>0), or $\{1, \log(r/a)\}$ (m=0)