Differential equations; Green functions and Fourier analysis
Math. 448: Supplement

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Basic models of linear differential equations can be written in terms of an elliptic differential operator $L = -\nabla \cdot p\nabla + q$ on $\mathbb{R}^n$ or region $\Omega \subset \mathbb{R}^n$, whose leading coefficients $p = (p_{ij})$ form a positive symmetric metric tensor (or positive scalar), and potential q - real. Differential operator L is usually supplemented by suitable boundary conditions. Those could be regular Dirichlet, Neumann, or more general mixed-type boundary condition given by a differential expression $B = \alpha + \beta\partial_n$ on smooth boundary Γ of Ω. In some cases the boundary condition could be *singular* (vanishing coefficients, "sources"), or the boundary itself is *degenerate*, e.g. isolated "boundary" points, curves, low-D surfaces, $\{\infty\}$ etc.

1. Three basic *pde* types and their formal solutions.

There are three basic types of differential equations that appear in various applications.

1. **Parabolic** (heat-diffusion)

$$\begin{cases} u_t + L[u] = F(x,t) \\ B[u]|_\Gamma = b \\ u|_{t=0} = u_0 \end{cases} \tag{1.1}$$

in a space-time cylinder $\Omega \times [0;\infty)$.

2. **Hyperbolic** (Wave):

$$\begin{cases} u_{tt} + L[u] = F(x,t) \\ B[u]|_\Gamma = b \\ u|_{t=0} = u_0;\ u_t|_{t=0} = u_1 \end{cases} \tag{1.2}$$

3. **Elliptic**:

$$\begin{array}{l} L[u] = F(x) \\ B[u]|_{\Gamma} = b \end{array} \quad \text{in } \Omega \tag{1.3}$$

or elliptic problems in cylindrical regions $\Omega \times [0;\infty)\,;\Omega \times [0;T]$

$$(a)\left\{\begin{array}{l} u_{tt} - L[u] = F(x,t) \\ B[u]|_{\Gamma} = b \\ u|_{t=0} = u_0;\ u|_{\infty} = \text{regular} \end{array}\right. \quad \text{or}\ \ (b)\left\{\begin{array}{l} u_{tt} - L[u] = F(x,t) \\ B[u]|_{\Gamma} = b \\ u|_{t=0} = u_0;\ u|_{T} = u_1 \end{array}\right. \tag{1.4}$$

Problems 1,2 and 3(a) are the *initial-value* ones, while 3(b) constitutes a 2-point boundary-value problem in variable t.

1.1. Formal solutions

Solutions of all three basic types can be written formally as "functions of L". Namely, for (1.1) the corresponding "function" is given by exponential of L

$$\boxed{u = e^{-tL}[u_0] + \int_0^t e^{-(t-s)L}[F(.;s)]\,ds} \tag{1.5}$$

For hyperbolic problem (1.2) solution is expressed in terms of the pair $\left\{\cos\left(t\sqrt{L}\right);\frac{\sin\left(t\sqrt{L}\right)}{\sqrt{L}}\right\}$

$$\boxed{u = \cos\left(t\sqrt{L}\right)[u_0] + \frac{\sin\left(t\sqrt{L}\right)}{\sqrt{L}}[u_1] + \int_0^t \frac{\sin(t-s)\sqrt{L}}{\sqrt{L}}[F(.;s)]\,ds} \tag{1.6}$$

Solutions of elliptic problems (1.3) are given either by the inverse of L

$$u = L^{-1}[F]$$

or in terms of exponential/hyperbolic trig. functions of L

$$\boxed{u = e^{-t\sqrt{L}}[u_0] + \int_0^t e^{-(t-s)\sqrt{L}}[F(.;s)]\,ds} \text{ on } [0;\infty) \tag{1.7}$$

$$\boxed{u = \frac{\sinh(T-t)\sqrt{L}[u_0] + \sinh t\sqrt{L}[u_0]}{\sinh T\sqrt{L}} + \int_0^t \mathcal{K}_{\sqrt{L}}(t;s)[F(.;s)]\,ds} \text{ on } [0;T] \tag{1.8}$$

where $\mathcal{K}_a\left(t;s\right)$ is the Green's function of the ODE (Sturm-Liouville) operator $\partial^2 - a^2$ on $[0;T]$

$$\mathcal{K}_a\left(t;s\right) = \frac{1}{\sinh\left(aT\right)}\begin{cases} \sinh at \, \sinh a\left(T-s\right); \; t \le s \\ \sinh as \, \sinh a\left(T-t\right); \; t \ge s \end{cases}$$

The expansion and 4 (e.g. Fourier analysis) gives the meaning to "functions of L", like $\exp(tL)$, $\cos\left(tL\right)$, etc. on $\mathbb{R}^n$and other domains. The latter are typically given by integral kernels, $\mathcal{K}(x;y)$ or $\mathcal{G}(x,y;t,)$, etc. and represent the so called *fundamental solutions* or *Green's functions* of the corresponding problems.

2. Green's functions

We denote integral kernels (possibly distributional) that represent Green's functions by $\mathcal{K}\left(x;y\right)$ - for elliptic operator L, by $\mathcal{G}\left(x,y;t,s\right)$ - for parabolic $\partial_t - L$, and $\mathcal{N}\left(x,y;t,s\right)$- for (sin-type) hyperbolic $\partial^2 \pm L$. A Green's function satisfies a differential equation (or its adjoint) with the point (Dirac δ) source in the r.h.s., and the appropriate homogeneous boundary condition. Specifically,

- **For parabolic problem** $\mathcal{G} = \mathcal{G}\left(x,y;t-s\right)$ obeys the direct and adjoint equations

  $$\begin{cases} \left(\partial_t + L_x\right)\left[\mathcal{G}\right] = \delta\left(x-y\right)\delta\left(t-s\right) \\ B_x\left[\mathcal{G}\right]|_\Gamma = 0 \\ \mathcal{G}|_{t=0} = 0 \end{cases} \quad \begin{cases} \left(-\partial_s + L_y\right)\left[\mathcal{G}\right] = \delta\left(x-y\right)\delta\left(t-s\right) \\ B_y\left[\mathcal{G}\right]|_\Gamma = 0 \\ \mathcal{G}|_{t=0} = 0 \end{cases}$$

  The latter could be simplified for the reduced Green's kernel $\mathcal{G}\left(x,y;t\right)$

  $$\underset{(x;t)}{\text{direct}}\begin{cases} \left(\partial_t + L_x\right)\left[\mathcal{G}\right] = 0; \; t > 0 \\ B_x\left[\mathcal{G}\right]|_\Gamma = 0 \\ \mathcal{G}|_{t=0} = \delta\left(x-y\right) \end{cases} \quad \underset{(y;t)}{\text{adjoint}}\begin{cases} \left(-\partial_t + L_y\right)\left[\mathcal{G}\right] = 0; \; t < 0 \\ B_y\left[\mathcal{G}\right]|_\Gamma = 0 \\ \mathcal{G}|_{t=0} = \delta\left(x-y\right) \end{cases}$$

  Here integral kernel

  $$\mathcal{G} : u \to \int \mathcal{G}\left(x,y;...\right)u\left(y\right)dy$$

  represents exponential of L, $\mathcal{G} = e^{-tL}$.

- **For hyperbolic problem** (1.2) the sin-type Green's $\mathcal{N} = \frac{\sin\left(t\sqrt{L}\right)}{\sqrt{L}}$ obeys

$$\begin{cases} (\partial_{tt} + L_x)\,[\mathcal{N}] = \delta\,(x-y)\,\delta\,(t-s)\,;\; t > 0 \\ B_x\,[\mathcal{N}]|_\Gamma = 0 \\ \mathcal{N}|_{t=0} = \delta\,(x-y) \end{cases}$$

as well as the adjoint problem in$(y; s)$

- **For elliptic problems** (1.3) Green's kernel $\mathcal{K}\,(x, \xi)$ obeys

$$\begin{cases} L_x\,[\mathcal{K}] = \delta\,(x-y)\,; \\ B_x\,[\mathcal{K}]|_\Gamma = 0 \end{cases} \quad \text{and its adjoint in } y$$

Subscripts $x; y$ with operators $L; B$ indicate variables of differentiation.

2.1. Poisson kernel

Solutions of problems (1.1)-(1.3) can be written in terms of the corresponding *Green's functions*. Precisely, u will typically contain two terms: the Green's term $\mathcal{K}\,[F]$, $\mathcal{G}\,[F]$ etc., representing contribution of "continuously distributed sources" F, and the Poisson term $\mathcal{P}\,[b]$, representing the "boundary/initial sources".

Thus for parabolic problem (1.1) it takes the form

$$\begin{aligned} u = &\underbrace{\int_0^t \int \mathcal{G}\,(x, y; t-\tau)\,[F\,(.;\tau)]\,dy\,d\tau +}_{\text{Green}} \\ &\underbrace{+ \int_0^t \oint_\Gamma \mathcal{P}\,(x, y; t-\tau)\,u_0(y) dS d\tau + \int_\Omega \mathcal{G}\,(x, y; t)\,u_0(y) dy}_{\text{Poisson}} \end{aligned} \tag{2.1}$$

Similarly for the wave equation (1.2)

$$u = \int_0^t \left\{ \underbrace{\int \mathcal{N}\left(x, y; t-\tau\right) \left[F\left(.;\tau\right)\right] dy}_{\text{Green}} + \underbrace{\oint_\Gamma \mathcal{P}\left(x, y; t-\tau\right) u_0(y) dS}_{\text{Poisson}} \right\} d\tau + \\ + \underbrace{\int_\Omega \mathcal{N}_t\left(x, y; t\right) u_0(y) dy + \int_\Omega \mathcal{N}\left(x, y; t\right) u_1(y) dy}_{\text{Poisson}} \tag{2.2}$$

In other words $\mathcal{N}(x, y; t)$ represents the $\sin(tL)$-operator, while its derivative $\mathcal{N}_t$ in 2-nd integral (2.2) corresponds to $\cos(tL)$.

The elliptic problem for $L; B$ is solved by

$$u = \int_\Omega \mathcal{K}\left(x, y\right) F(y) dy + \oint_\Gamma \mathcal{P}\left(x, y\right) b(y) dS \tag{2.3}$$

i.e. $\mathcal{K}$ gives the inverse operator L^{-1}, while solution of (1.4) has the same form as (2.1) with $\mathcal{K}$ representing the exponential $\exp\left(t\sqrt{L}\right)$.

In all 3 cases the Poisson kernel $\mathcal{P}$ can be explicitly calculated in terms of the Green's kernel $\mathcal{K}$ and the boundary condition B

$$\mathcal{P}\left(x, y; ...\right) = \begin{cases} \frac{p}{\beta}\mathcal{K}\left(x, y; ...\right) \text{ if } \beta \neq 0 \\ -\frac{p}{\alpha}\partial_n\mathcal{K}\left(x, y; ...\right) \text{ if } \alpha \neq 0 \end{cases} \tag{2.4}$$

Two special cases are the *Dirichlet* and *Neumann* Poisson kernels

$$\mathcal{P} = -\left(\partial_n\mathcal{K}\right)\left(x, y; ...\right) \text{ and } \mathcal{P} = \mathcal{K}\left(x, y; ...\right); \; x \in \Omega; y \in \Gamma$$

Solution formulae (2.1)-(2.3) including the Poisson term are derived by the standard Green's identity (integration by parts), applied to functions $u(y)$ and $\mathcal{K}(x, y)$

$$\int_\Omega \mathcal{K}\left(x, y\right) L_y\left[u\right] dy = \int_\Omega L_y\left[\mathcal{K}\left(x, y\right)\right] u dy + \oint_\Gamma p\left(\partial_n\mathcal{K}\, u - \mathcal{K}\, \partial_n u\right) dS$$

The first integral in the r.h.s. gives $u(x)$. Replacing function $\mathcal{K}$, or $\partial_n\mathcal{K}$ in the Γ-integral by $-\frac{\beta}{\alpha}\partial_n\mathcal{K}$ (if $\alpha \neq 0$), or by $-\frac{\alpha}{\beta}\partial_n\mathcal{K}$ (if $\beta \neq 0$), via the boundary condition, the Γ-integrand assumes the form $\mathcal{P}(x, y)\left(\alpha u + \beta u_n\right)$, which yields solution formulae (2.1)-(2.3) with the Poisson kernel (2.4).

3. Construction of Green's functions.

Different methods can be used to construct Green's functions in $\mathbb{R}^n$ and other regions. The first one is based on the *3.1* and separation of variables to certain ODE problems, the second one employs expansion-transform methods (e.g. Fourier analysis) and distribution theory. For bounded regions one can use the eigenfunction expansion of the corresponding problem to write $\mathcal{K}$ as a "generalized Fourier series" (the utility of such expansion depends largely on the knowledge of the eigendata). Finally for special regions the "Method of Images" applies to express $\mathcal{K}$ in terms of the free space Green's function of various "reflected sources". We shall outline all 4 methods.

3.1. Symmetry reduction.

Green's functions are formally represented as functions "$f(L)$", hence they inherit naturally all basic symmetries of the original problem (operator $L; B$). That allows to reduce the number of variables in $\mathcal{K}$.

Thus symmetry (real or Hermitian) of operator $L \sim (L; B)$ with respect to the L^2-inner product $\langle u | v \rangle = \int_\Omega u(x)\, \bar{v}(x)\, dx$, namely

$$\langle L[u] | v \rangle = \langle u | L[v] \rangle \text{ for all functions } u, v \text{ satisfying boundary condition } B[u]|_\Gamma = 0$$

implies symmetry (real/Hermitian) of the integral kernel $\mathcal{K}$

$$\mathcal{K}(x, y) = \mathcal{K}(y, x) \text{ (or } \mathcal{K}^*(y, x) \text{ - complex conjugate) for all } x; y \in \Omega$$

Time independence of $L; B$ means that kernel $\mathcal{K}$ is convolution in t, $\mathcal{K} = \mathcal{K}(x, y; t - s)$.

Similarly, constant coefficients in x on $\mathbb{R}^n$ (i.e. space translational symmetry of the problem) results in a translation-invariant (convolution) kernel $\mathcal{K} = \mathcal{K}(x - y; ...)$.

Yet larger symmetry groups appear for special operators L, like orthogonal rotations $\mathbb{SO}(n)$ for the Laplacian Δ on $\mathbb{R}^n$, or *7.1* $\mathbb{SO}(1; n)$ for the wave operator (d'Alambertian) $\Box = \partial_t^2 - c^2\Delta$ (see Appendix 7.1). The corresponding Green's functions $\mathcal{K}$ have the same symmetry, which allows to reduce $\mathcal{K}(x, \xi)$ to a single variable function $\mathcal{K}(r)$ of the Euclidean radius: $r = |x|$, or hyperbolic (Minkowski) radius: $\rho^2 = (ct^2) - |x|^2$.

In both cases the resulting radial function $\mathcal{K}$ solves the reduced (Laplace, d'Alambert) equation

$$\mathcal{K}_{rr} + \frac{n-1}{r}\mathcal{K}_r \pm m^2\mathcal{K} = 0 \tag{3.1}$$

for operator $L = \Delta \pm m^2$ or $\Box \pm m^2$. It also requires an appropriate boundary condition at singular point $\{0\}$ and $\{\infty\}$.

3.2. Source condition.

In the case of Laplacian (Helmholtz-type operator) $L = -p\Delta + q$ the boundary condition at $\{0\}$ represents the "point source"

$$r^{n-1}\,\mathcal{K}'\,(r)\big|_{r=0} = \frac{1}{p\omega_{n-1}} \tag{3.2}$$

ω_{n-1} = surface area of the unit sphere in $\mathbb{R}^n$. For derivation of the 7.2 see Appendix 7.2.

Reduced equation (3.1) for $q = 0$ takes the form

$$\mathcal{K}_{rr} + \frac{n-1}{r}\mathcal{K}_r = 0 \tag{3.3}$$

It has two solutions (regular and singular): $1; \ln r$ (2-D), and $\{1; r^{2-n}\}$ in n-D ($n \geq 3$). General solution of (3.3) is a combination of two $\mathcal{K} = c_1 + c_2 r^{2-n}$ (or $\ln r$). Constant c_1 is eliminated via boundary condition at $\{\infty\}$, while the source condition (3.2) yields the proper normalization of c_2 , whence follows the familiar Newton/Gauss potential, the Green's function of the Laplacian,

$$\mathcal{K} = \begin{cases} -\frac{1}{2\pi}\ln|x-y|\,; \text{ for } n=2 \\ \frac{1}{\omega_{n-1}|x-y|^{n-2}}; \text{ for } n \geq 3 \end{cases}$$

Similar treatment applies in the case $q \neq 0$. Then equation (3.1) becomes *7.3.1* type (see Appendix 7.3)

$$\mathcal{K}'' + \frac{n-1}{r}\mathcal{K}' \pm m^2\mathcal{K} = 0 \text{ with } m^2 = \frac{q}{p}$$

It is easily reduced to the standard (1-st or 2-nd kind) Bessel equation

$$Y'' + \frac{1}{r}Y' \pm \left(1 - \frac{\nu^2}{r^2}\right)Y = 0$$

of order $\nu = \frac{n-2}{2}$ by the change

$$\mathcal{K}(r) = r^{-\nu} Y_\nu (mr)$$

(Appendix 7.3). Thus solution of (3.1) with positive m^2 becomes a combination of the first kind Bessel functions

$$\mathcal{K} = r^{-\nu} \left\{ c_1 J_\nu (mr) + c_2 Y_\nu (mr) \right\}, \text{ of order } \nu = \tfrac{n-2}{2}$$

The source condition at $\{0\}$ would be provided by a nonvanishing coefficient c_2. One also need a suitable condition at $\{\infty\}$, which could be chosen an "*incoming*" or "*outgoing*" *radiation condition*

$$\mathcal{K} \sim r^{-\left(\frac{n-1}{2}\right)} e^{\pm imr} \tag{3.4}$$

The latter along with a proper normalization at $\{0\}$ uniquely defines $\mathcal{K}$ in terms of the *MacDonald function* $H_\nu^\pm = C\left(J_\nu \pm iY_\nu\right)$, so

$$\mathcal{K} = \left(\Delta + m^2\right)^{-1} = Cr^{-v} H_\nu (mr) \sim \begin{cases} C\,|x-y|^{2-n}; & \text{at small } r \\ r^{-\left(\frac{n-1}{2}\right)} e^{\pm imr}; & \text{ar large } r \end{cases}$$

Similarly, for operator $\Delta - m^2$ the reduced equation (3.1) is changed into the modified 7.4

$$\mathcal{K}'' + \frac{n-1}{r}\mathcal{K}' - m^2 \mathcal{K} = 0 \text{ with } m^2 = \frac{q}{p}$$

whose solutions are $I_v \sim r^{\left(\frac{1-n}{2}\right)} e^{mr}$ and $\mathcal{K}_v \sim r^{\left(\frac{1-n}{2}\right)} e^{-mr}$. Clearly, we choose the exponentially decaying *Kelvin function* K_ν , normalized to satisfy the source condition at $\{0\}$. We list a few low-D cases

dim	$\mathcal{K}$
1	$\frac{1}{2}e^{-mr}$
2	$K_0 (mr)$
3	$\frac{e^{-mr}}{4\pi r}$

Unlike elliptic problems above it becomes more difficult to introduce the proper source condition for hyperbolic equations, which is partly due to more complicated geometry of the singular set: "light cone" $\rho = 0$ instead of a single point $r = 0$. In the next section we shall develop the Fourier-transform methods for hyperbolic problems.

3.3. Heat-diffusion and scaling symmetry.

Green's function of the heat problem is easily found by the Fourier transform $\mathcal{F}: u(x) \to \hat{u}(\xi)$

$$\begin{cases} G_t - \Delta G = 0, \text{ for } t > 0 \\ G|_{t=0} = \delta \end{cases} \Longrightarrow \hat{G}(\xi; t) = e^{-t|\xi|^2}$$

whence follows

$$G(x;t) = \mathcal{F}^{-1}\left[e^{-t|\xi|^2}\right] = \tfrac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \tag{3.5}$$

the familiar Gaussian.

The same form (3.5) can be also derived by the symmetry reductions, as we shall now demonstrate. The relevant coordinate transformations are nonisotropic dilations in space-time

$$(x;t) \to \left(\alpha x; \alpha^2 t\right); \; \alpha > 0$$

If function $u(x,t)$ solves the homogeneous heat equation, then obviously $u^* = u(\alpha x; \alpha^2 t)$ does it. So the dilated Green's $G^* = G(\alpha x; \alpha^2 t)$ solves the heat problem with the initial state

$$\begin{aligned} G^*(x;0) &= \delta(\alpha x) = \alpha^n \delta(x) \Rightarrow \\ G\left(\alpha x; \alpha^2 t\right) &= \alpha^n G(x,t) \end{aligned}$$

The latter allows one to reduce G to a single variable function $G_0(r)$, $r = \frac{|x|}{\sqrt{t}}$, so that

$$G(r,t) = t^{-n/2} G_0\left(\frac{r}{\sqrt{t}}\right)$$

Substitution in the heat equation yields an ODE for G_0

$$G_0'' + \left(\tfrac{n-1}{r} + \tfrac{r}{2}\right) G_0' + \tfrac{n}{2} G_0 = 0$$

The latter has general solution $e^{-r^2/2}\left(C_1 + C_2 \int\limits_0^r r^{n-1} e^{r^2/2}\right)$. Its second (polynomial) term

$$e^{-r^2/2} \int\limits_0^r \left(r^{n-1} e^{r^2/2}\right) \sim r^{(n-2)/2} + \ldots$$

vanishes at ∞, iff $C_2 = 0$. So G is found to be $Ce^{-r^2/2}$, with constant $C = \frac{1}{(2\pi)^{n/2}}$ after proper normalization.

4. Distributions and Fourier transform

4.1. Distributions

Distributions or *generalized functions* are defined as *linear functionals* on appropriate spaces of *test functions* $\mathcal{S} = \{u(x)\}$. The latter are typically made of continuous or smooth (differentiable) functions on $\mathbb{R}^n$ or domains Ω, vanishing on the boundary, point $\{\infty\}$, or satisfying a suitable boundary condition $B[u]|_\Gamma = 0$. A distribution f , a continuous linear functional on $\mathcal{S}$, is determined by pairing with all test functions

$$f : u \to \langle f|u\rangle$$

All regular functions (or measures)$\{f\}$ become distributions when paired to test-functions $\{u\}$ via integration

$$\langle f|u\rangle = \int f(x)\, u(x)\, dx$$

For this reason pairing of distributions to functions is often denoted by the integral sign, although $\int f(x)\, u(x)\, dx$ may not strictly speaking make sense.

Natural examples of distributions include

- Dirac δ-function $\delta_a = \delta(x-a)$ and its derivatives:

  $$\langle \delta_a|u\rangle = u(a)\,; \quad \langle \partial^m \delta_a|u\rangle = (-\partial)^m\, u(a)$$
  $$\langle \partial^\alpha \delta_a|u\rangle = (-\partial)^{|\alpha|}\, u(a) \text{ for partial derivatives } \partial^\alpha = \tfrac{\partial^{|\alpha|}}{\partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}}$$

  here multi-index $\alpha = (\alpha_1; ...; \alpha_n)$ and its norm $|\alpha| = \alpha_1 + ... + \alpha_n$.

- δ-functions of curves and surfaces Σ

  $$\langle\, \delta_\Sigma|\, u\rangle = \int_\Sigma u dS \text{ - integral with respect to the natural surface area } dS$$

- δ-function of the level set $\Sigma = \{x : \Phi(x) = c\}$,

  $$\delta(\Phi(x) - c) = \frac{1}{|\partial\Phi(x)|}\delta_\Sigma = \int_\Sigma u dS \qquad (4.1)$$

Many operations on test-functions: differentiation, convolution, multiplication, change of variables, Fourier transform, etc., can be transferred to distributions via pairing. We shall mention a few of them

1. **Differentiation:**by definition

$$\langle \partial_j f | u \rangle = - \langle f | \partial_j u \rangle \text{, for any testing } u$$

and similarly for higher derivatives $\{\partial^\alpha f\}$. Indeed, for regular distributions this is nothing but the standard integration by parts formula: $\int \partial_j (f) \; u \, dx = - \int f \, \partial_j (u) \; dx$, for any pair $\{f; u\}$.

2. **Change of variable** formula: if $y = \phi(x)$, then

$$\langle f \circ \phi | u \rangle = \left\langle \frac{1}{|\phi' \circ \phi^{-1}|} f \right| \left. u \circ \phi^{-1} \right\rangle \text{, for all testing } \{u\} \tag{4.2}$$

Here ϕ' denotes the Jacobian matrix of map ϕ, and $|...|$ its determinant. Definition (4.2) is justified for regular distributions by the standard change of variable in integration

$$\int (f \circ \phi) \; u \, dy = \int \frac{1}{|\phi' \circ \phi^{-1}|} f \; (x) \; \left(u \circ \phi^{-1}\right) dx$$

As an example we take an ordinary function $\phi(x)$ with zeroes at $\{x_1; x_2; ...\}$ and $f = \delta(y)$. Then

$$\delta \circ \phi(x) = \sum_{x_j} \frac{1}{|\phi'(x_j)|} \delta(x - x_j)$$

a linear combination of δ's at zeroes $\{x_j\}$. The latter extends to level surfaces of multivariable functions ϕ, where the sum over "zeros of ϕ" becomes the integral over the level surface $\{x : \phi(x) = 0\}$. It justifies our definition of the "level surface " Dirac function (4.1).

3. **Fourier transform** $\mathcal{F} = \mathcal{F}_{x \to \xi} : u \to \hat{u} = \int u(x) \, e^{-ix \cdot \xi} dx$. By definition

$$\langle \mathcal{F}[f] | \, u \rangle = \left\langle f \right| \mathcal{F}^{-1}[u] \left. \right\rangle$$

which for regular functions is justified by the Fourier inversion/Plancherel formula

$$\int f(x) \, \bar{u}(x) \, dx = \int \hat{f}(\xi) \, \hat{u}^*(\xi) \, d\xi \text{, for any } f; u$$

We give a few examples of distributions and their transforms:

Function/distr	$\mathcal{F}$-transform
$\delta(x)$	1
$\delta(x-a)$	$e^{ia\cdot\xi}$
$\partial^{\alpha} f(x)$	$(i\xi)^{\alpha}\hat{f}(\xi)$
$-\Delta f(x)$	$\lvert\xi\rvert^{2}\hat{f}(\xi)$
$\cos ax$	$\frac{1}{2}(\delta(x-a)+\delta(x+a))$

(4.3)

A few examples of surface δ-functions δ_Σ and their transforms $\hat{\delta}_\Sigma$

Surface Σ	$\mathcal{F}$-transform of δ_Σ
linear k -D subspace of $\mathbb{R}^n$	$(2\pi)^{n-k}\,\delta_{\Sigma'}$; Σ' - subspace orthogonal to Σ
sphere $\{x : \lvert x\rvert = a\}$	$C_n \lvert\xi\rvert^{n/2-1} J_{n/2-1}(a\lvert\xi\rvert)$-Bessel
light-cone $\left\{z = c\sqrt{x^2+y^2}\right\}$ in $\mathbb{R}^3$	$\frac{-2\pi\sqrt{1+c^2}}{\left(c\zeta+\sqrt{\xi^2+\eta^2}\right)^2}$

(4.4)

The last example involves of $\hat{\delta}_\Sigma$ involves Heaviside function on the dual cone. Finally, we mention the sum of δ-functions over the lattice[1] $\Gamma \simeq \mathbb{Z}^n$ in $\mathbb{R}^n$,

$$\delta_\Gamma(x) = \sum_{m\in\Gamma} \delta(x-m) \longleftrightarrow \frac{(2\pi)^n}{\operatorname{vol}(\Omega)} \sum_{k\in\Gamma'} \delta(\xi-k) \tag{4.5}$$

is Fourier transformed (by the Poisson summation formula) into the "δ-function" of the dual lattice Γ' times 1"over "volume of the fundamental parallelepiped Ω of Γ".

5. Solution of differential equations.

Now we shall apply the Fourier transform and distributions to construct Green's and Poisson kernels for three basic types of pde's.

[1]Any lattice Γ in $\mathbb{R}$ is made in integral multiples $\{ma : m = 0; \pm1; \pm2; ...\}$ for a fixed length a. Then its dual $\Gamma' = \left\{k\frac{2\pi}{a} : k = 0; \pm1; \pm2; ...\right\}$ has basic length $a' = \frac{2\pi}{a}$. So (4.5) could be stated as

$$\mathcal{F}\left\{\sum_{m=-\infty}^{\infty} \delta(x-m)\right\} = \sum_{m=-\infty}^{\infty} e^{im\xi} = \frac{2\pi}{a} \sum_{k=-\infty}^{\infty} \delta\left(\xi - \frac{2\pi}{a}k\right)$$

In $\mathbb{R}^n$ any lattice Γ is obtained from a basis $\{\mathbf{v}_1; ...; \mathbf{v}_n\}$ by taking all integral linear combinations $\{\mathbf{m} = \sum m_i \mathbf{v}_i; m_i \in \mathbf{Z}$ -integers$\}$. Its dual lattice Γ' is then spanned by the dual basis $\{\mathbf{v}'_1; ...; \mathbf{v}'_n\}$ so that $\mathbf{v}_i \cdot \mathbf{v}'_j = 2\pi\delta_{ij}$- Kronrcker delta.

5.1. Elliptic equations

We shall start with elliptic operators: Laplacian and Helmholtz

$$-\Delta[\mathcal{K}] = \delta; \ \left(-\Delta + m^2\right)[\mathcal{K}] = \delta$$

Fourier transforming both equations gives distribution

$$\hat{\mathcal{K}}(\xi) = \begin{cases} \frac{1}{|\xi|^2} \text{ - Laplace} \\ \frac{1}{|\xi|^2+m^2} \text{ -Helmholtz} \end{cases}$$

The first is radially symmetric and homogeneous of degree two, so its $\mathcal{F}$-transform is also radial and homogeneous of degree $2-n$. Thus we recover the familiar Newton/Coulomb potential

$$\mathcal{K} = \frac{C}{|z|^{n-2}}, \text{ for } n \geq 3 \text{ (case } n=2 \text{ is more subtle)}$$

The "Bessel potential" for the Helmholtz equation is reduced via spherical symmetry (4.4) to the form

$$\mathcal{K}(r) = \frac{\omega_{n-1}}{(2\pi)^n} \int_0^\infty \frac{1}{\rho^2+m^2} (mr\rho)^{-\nu} J_\nu(mr\rho)\, \rho^{n-1} d\rho \text{ with } \nu = \tfrac{n}{2} - 1$$

The latter can easily evaluated for $n = 1; 3$; (or any odd n) to get familiar form

$$\mathcal{K} = \begin{cases} \frac{1}{2m} e^{-mr} \text{ on } \mathbb{R} \\ \frac{1}{2mr} e^{-mr} \text{ on } \mathbb{R}^3 \end{cases}$$

In higher-D cases complex analysis allows to express $\mathcal{K}$ in terms of the modified Bessel function K_ν, as discussed in section 3.2. Namely, of Hemholtz operator on $\mathbb{R}^n$ is given by the Bessel potential

$$\mathcal{K}(r) = r^{-\nu} K_\nu(mr) \sim \frac{e^{-mr}}{r^{\frac{n-1}{2}}}; \text{ for } -\Delta + m^2$$

(exponentially decaying Kelvin function of order $\nu = \frac{n-2}{2}$), and oscillating (incoming/outgoing) MacDonald $H^\pm$-functions

$$\mathcal{K}(r) = r^{-\nu} H_\nu^\pm(mr) \sim \frac{e^{\pm imr}}{r^{\frac{n-1}{2}}}; \text{ for } -\Delta - m^2$$

5.2. Poisson kernel.

The Fourier transform technique could be applied to find the half-space Poisson kernel for Laplaces equation in $\{(y;x) : y > 0; x \in \mathbb{R}^n\}$

$$\begin{matrix} (\partial_{yy} + \Delta)\,\mathcal{P} = 0 \\ \mathcal{P}|_{y=0} = \delta \end{matrix} \implies \mathcal{P} = e^{-y\sqrt{-\Delta}}$$

After F-transform it becomes $\hat{P} = e^{-y|\xi|}$. Then Fourier-inversion yields

$$\mathcal{P}\,(x;y) = \frac{C_n y}{\left(|x|^2 + y^2\right)^{\frac{n+1}{2}}}$$

The same result can be derived by the method of images. The $\mathcal{F}$-transform of the heat equation was already explained. So now we turn to the wave and KG-equations in $\mathbb{R}^n$.

5.3. Hyperbolic equations.

The wave and KG equations possess all the symmetries of Minkowski space $\mathbb{M}^{n+1}$, namely space-time translations as well as 7.1 $\mathbb{SO}\,(1;n)$. The corresponding Green's functions $\mathcal{K} = \mathcal{K}\,(t;x-y)$, could be reduced to a single variable, the hyperbolic radius $\rho = \sqrt{(ct)^2 - |x|^2}$ (section 3.1). They satisfy the Bessel-type equation

$$\mathcal{N}_{\rho\rho} + \frac{n}{r}\mathcal{N}_\rho \pm m^2\mathcal{N} = 0$$

But it is not easy to derive the proper "source condition" at $\{0\}$. Applying the Fourier transform $\mathcal{N}(x;t) \to \mathcal{N}\hat{}\ (\xi;t)$to the wave or KG-equation we get

$$\mathcal{N}\hat{}\ = \begin{cases} \frac{\sin t|\xi|}{|\xi|} & \text{(wave)} \\ \frac{\sin\left(t\sqrt{|\xi|^2+m^2}\right)}{\sqrt{|\xi|^2+m^2}} & \text{(KG)} \end{cases} \tag{5.1}$$

It remains to invert distributions $\mathcal{N}\hat{}$ of (5.1)

$$\mathcal{N}\,(r;t) = c_{n-1}\int_0^\infty \frac{\sin t\xi}{\xi}\,(r\xi)^{-\nu}\,J_\nu\,(r\xi)\,\xi^{n-1}d\xi \text{ (for wave)} \tag{5.2}$$

Once again the calculation is fairly easy in 1-D and 3-D (due to the elementary form of Bessel function $J_{1/2}(r) = \frac{\sin r}{2r}$). In small dimensions $n = 1, 2, 3$ we get the following list of wave-propagators $\mathcal{N}$

$$\mathcal{N}(t;x) = \begin{cases} \frac{1}{2c} H(ct - |x|) & 1D \\ \frac{1}{2\pi\sqrt{(ct)^2 - |x|^2}} & 2D \\ \frac{1}{4\pi ct}\delta(ct - |x|) & 3D \end{cases}$$

all being functions of hyperbolic ρ albeit distributional.

5.4. KG-propagator

The wave operator: $\Box = \partial_t^2 - \Delta$ and the Klein-Gordon: $\Box + m^2 = \partial_t^2 + L$, where $L = -\Delta + m^2$ have fundamental solutions of the type $\mathcal{N} = \frac{\sin t\sqrt{L}}{\sqrt{L}}$; $\mathcal{M} = \cos t\sqrt{L}$. Both integral kernels are functions of $r = |x - \xi|$ and t. The Fourier transform $\mathcal{F}_{x\to k}$ yields a representation of $\mathcal{N}$ in terms of the Bessel function J_ν of order $\nu = \frac{n-2}{2}$

$$\mathcal{N}(r;t) = c_{n-1}\int_0^\infty \frac{\sin t\sqrt{\xi^2+m^2}}{\sqrt{\xi^2+m^2}}(r\xi)^{-\nu} J_\nu(r\xi)\,\xi^{n-1}d\xi \text{ (for KG)} \tag{5.3}$$

In $\mathbb{R}^3$ Bessel $J_{1/2} = \frac{\sin x}{\sqrt{x}}$ is an elementary function, so (5.3) becomes

$$\mathcal{N} = \frac{1}{4\pi^2}\int_0^\infty \frac{\sin t\sqrt{\xi^2+m^2}}{\sqrt{\xi^2+m^2}}\frac{\sin r\xi}{r\xi}\xi^2 d\xi = \frac{1}{r}\int_0^\infty \frac{\cos\left(t\sqrt{\cdots} - r\xi\right) - \cos\left(t\sqrt{\cdots} + r\xi\right)}{2\sqrt{\cdots}}\xi d\xi$$

We take the hyperbolic radius $\rho = \sqrt{t^2 - r^2}$ and replace polar variables $\{r;\xi\}$ by the pair $\{u;v\}$,

$$\xi = \rho\sinh u$$
$$r = \rho\sinh v$$

Then (after another change: $u \to s = \cosh u$) function $\mathcal{N}$ becomes

$$\mathcal{N} = \frac{\sinh v}{\rho}\int_{-\infty}^\infty \cos(m\rho\cosh u)\cosh u\,du = \frac{1}{\rho}\int_1^\infty \cos(m\rho\, s)\frac{ds}{\sqrt{s^2-1}}$$

The latter is recognized as the derivative of J_0 written in the 7.14 form (7.14) (see Appendix 7.3)

$$\mathcal{N}(\rho) = \frac{1}{\rho}\frac{d}{d\rho}\left\{\int_1^\infty \sin(m\rho\, s)\,\frac{ds}{\sqrt{s^2-1}}\right\} = \frac{1}{\rho}\frac{d}{d\rho}\left\{J_0(m\rho)\,H(\rho)\right\}$$

with Heaviside function H. Hence we get the KG-propagator in 3-D

$$\mathcal{N}(\rho) = \frac{1}{\rho}\left[\delta(\rho) - \frac{1}{m}J_1(m\rho)\right] \tag{5.4}$$

in terms of hyperbolic radius $\rho = \sqrt{c^2t^2 - |x-\xi|^2}$. The low-D cases are listed in the table

$$\mathcal{N} = \begin{cases} J_0(m\rho) & 1\text{D} \\ -\frac{C}{\sqrt{\rho}}Y_{1/2}(m\rho) = \sqrt{\frac{2}{m\pi}}\frac{\cos m\rho}{\rho} & 2\text{D} \\ \frac{C}{\rho}\left[\delta(\rho) - \frac{1}{m}J_1(m\rho)\right] & 3\text{D} \end{cases}$$

In higher dimensions $\mathcal{N}$ becomes yet more singular.

6. Huygens principle, energy conservation and causality.

For any hyperbolic equation it obeys the causality and Huygens principle.

Causality Green's function $\mathcal{K}(t;x)$ vanishes outside the light cone $\{|x| \le ct\}$, hence any solution of the initial-value problem vanishes outside the envelope of light-cones based on the initial disturbance (domain of influence).

This result can be shown for fairly general hyperbolic equations, including wave and KG, using

Energy conservation principle Total energy of any solution of a general wave equation $u_{tt} + L[u] = 0$, defined as the sum of its kinetic and potential terms

$$E[u] = \int \tfrac{1}{2}\left(u_t^2 + c^2|\nabla u| + m^2u^2\right)dx$$

is conserved

$$\frac{dE}{dt} = 0 \tag{6.1}$$

Another way to state (6.1) is in terms of the energy-density

$$e\,(x;t) = \tfrac{1}{2}\left(u_t^2 + c^2\,|\nabla u| + m^2 u^2\right) \qquad (6.2)$$

Then (6.1) is equivalent to[2]

$$\partial_t e + \nabla \cdot (u_t \nabla u) = 0 \qquad (6.3)$$

To derive causality we take a backward light cone $\{(x;t) : |x-y| \le c\,(T-t)\,; 0 < t < T\}$ and use integration by parts (Green's identity) to show that for any fixed t the energy of cross-section $B_t = \{|x-y| \le c\,(T-t)\}$

$$E\,(B_t) = \int_{B_t} e\,(x;t)\,dx$$

decreases with t, $E\,(t_1) \le E\,(t_0)$ for all $t_1 > t_0$ (see fig.**??**). Hence any region of zero initial data will produce zero disturbance inside the backward light-cones above it.

For wave equation in odd space-dimensions the causality principle takes on a sharper form of

Huygens principle Green's function of the wave equation $\Box u = 0$in odd space-dimensions $n = 3, 5, ...$ is distributed over the conical surface $\{|x| = ct\}$, rather then solid cone.

This result follows from recurrent formulae for the half-integer Bessel-functions (7.13) applied to Fourier integral (5.2).

Remark 1. *Formula (5.4) shows K to depend on the hyperbolic radius ρ only, and thus gives another demonstration of the Huygens principle (causality) for solutions of the wave and KG equations.*

7. Appendices

7.1. Hyperbolic rotations

Orthogonal group in n-space $\mathbb{SO}\,(n)$ is made of all $n \times n$ matrices $\{U\}$ that preserve the Euclidean inner product: $x \cdot y$. So $Ux \cdot Uy = x \cdot y$ for all vectors $x; y$ in $\mathbb{R}^n$,

[2] Quantity u_t represents the momentum field of u, so (6.3) claims: the rate of change of the energy-density is equal to the momentum flux.

or ${}^{T}U \cdot U = I$. In 2-D those are familiar rotations

$$\left\{ U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : 0 \le \theta \le 2\pi \right\}$$

Similarly, *pseudo-orthogonal* (Lorentz) matrices $\{U\}$ preserve the indefinite inner product on so called *Minkowski* space-time $\mathbb{M}^{n+1} = \{(x_0; x_1; ...x_n)\}$ - important among other in special relativity

$$\langle x|y\rangle = x_0 y_0 - (x_1 y_1 + ... + x_n y_n)\,; \tag{7.1}$$

The corresponding Lorentz group is denoted by $\mathbb{SO}(1, n)$. In $\mathbb{M}^2$ those are hyperbolic rotations

$$\left\{ U = \begin{bmatrix} \cosh\tau & \sinh\tau \\ \sinh\tau & \cosh\tau \end{bmatrix} : -\infty \le \tau \le \infty \right\}$$

Both groups: orthogonal $\mathbb{SO}(n)$ and Lorentz $\mathbb{SO}(1, n)$ preserve radii in $\mathbb{R}^n$ and $\mathbb{M}^{n+1}$

$$\begin{aligned} r &= |x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + ... + x_n^2} \\ \rho &= \sqrt{\langle x|x\rangle} = \sqrt{x_0^2 - (x_1^2 + ... + x_n^2)} \end{aligned}$$

Unlike Euclidean product $x \cdot x$, the Minkowski $\langle x|x\rangle$ is not always positive, so ρ is well defined inside the light cone: $x_0^2 \ge (x_1^2 + ... + x_n^2)$ (fig.**??**).

The analogue of spherical coordinates $(r; \theta, \phi_1 ...)$ on Euclidean $\mathbb{R}^n$are hyperbolic spherical coordinates $(\rho, \tau, \phi_1 ...)$in the light cone. Indeed, the role of unit sphere $r = 1$ is played by a unit hyperboloid $\rho = 1$ (fig.**??**). and the vertical/horizontal Euclidean components

$$\left\{ \begin{array}{c} z = r\cos\theta \\ x = r\sin\theta ... \end{array} ; 0 \le \theta \le \pi \right\}$$

become in the Minkowski space

$$\left\{ \begin{array}{c} z = r\cosh\tau \\ x = r\sinh\tau ... \end{array} ; -\infty < \tau < \infty \right\}$$

An important property of the Laplacian in $\mathbb{R}^n$(exploited in spherical separation) is its rotational symmetry, so for any matrix U in $\mathbb{SO}(n)$ and function f

$$\Delta [f(Ux)] = (\Delta f)(Ux)$$

i.e. rotation of variable x can be interchanged (commutes) with Δ. This readily follows from the general change of variable formula in $\Delta = \nabla \cdot \nabla$: changing $x \to y = Ax$, one gets

$$\Delta_y = \tfrac{1}{J} \nabla_x \cdot J \left(A^t A\right)^{-1} \nabla_x \tag{7.2}$$

where $J = \det(A)$. So orthogonal matrix A ($J = 1$, $A^t A = I$) preserves operator: $\Delta_y = \Delta_x$.

As consequence all "functions of Δ", including its Green's function $\mathcal{K} = (-\Delta)^{-1}$possess the same rotational $\mathbb{SO}(n)$-symmetry. Together with translational symmetry, $\Delta [f(x+a)] = (\Delta f)(x+a)$, for all $a \in \mathbb{R}^n$ it implies that $\mathcal{K}(x,\xi) = \mathcal{K}(r)$ depends on a single variable, the Euclidean distance $r = |x - \xi|$.

Similar arguments apply to the d'Alambertian $\Box = \partial_t^2 - \Delta$ with the Euclidean dot-product and transpose A^t in (7.2) replaced by the Minkowski product and transpose (7.1).

7.2. Derivation of the source-condition

To derive the 3.2 (3.2) for an elliptic operator L one takes any function u on $\mathbb{R}^n$, writes $L[u] = F$, applies Green's kernel $\mathcal{K}(x-y)$ to the equation, and integrates by parts (Green's identity) over the complement Ω_ϵ in $\mathbb{R}^n$ of a small ball $B_\epsilon(x)$ centered at $\{x\}$. On the one hand

$$u(x) = \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \mathcal{K}(x-y)\, L[u]\, dy$$

on the other hand

$$\int_{\Omega_\epsilon} \mathcal{K}(x-y)\, L[u] = \int_{\Omega_\epsilon} L_y[\mathcal{K}]\, u + \oint_{\Gamma} p\left(\partial_n \mathcal{K}\, u - \mathcal{K}\, \partial_n u\right) dS$$

First integral in the r.h.s. vanishes, since $L[\mathcal{K}] = 0$ outside the source. Remembering that $\mathcal{K} = \mathcal{K}(|x-y|)$, we get the second integral

$$\oint_{|x-y|=\epsilon} \ldots = p\omega_{n-1} \epsilon^{n-1} \mathcal{K}'(\epsilon)\, u(x) + \ldots$$

while the third integral

$$\oint_{|x-y|=\epsilon} \ldots \approx \epsilon^n \mathcal{K}(\epsilon)\, p \Delta u(x) \to 0 \text{ as } \epsilon \to 0$$

Thus
$$u\left(x\right)=\lim\left\{p\omega_{n-1}\epsilon^{n-1}\mathcal{K}'\left(\epsilon\right)\right\}u\left(x\right)+\left\{\epsilon^{n}\mathcal{K}\left(\epsilon\right)\right\}p\Delta u\left(x\right)$$
for any function u, which yields the source condition (3.2).

7.3. Bessel functions

7.3.1. Differential equation.

Bessel functions of the *first kind* and order ν solve the differential equation
$$y''+\frac{1}{x}y'+\left(1-\frac{\nu^2}{x^2}\right)y=0 \tag{7.3}$$
while those of the *second kind (modified)* solve
$$y''+\frac{1}{x}y'-\left(1+\frac{\nu^2}{x^2}\right)y=0 \tag{7.4}$$
Formally the transition from (7.3) to (7.4) is by a complex change of variable $x\longrightarrow ix$.

In both case $x=0$ is a singular point, so by the Frobenius method both have a pair of solutions expanded in the series
$$y_{\pm\nu}\left(x\right)=x^{\pm\nu}\sum_{0}^{\infty}a_kx^{2k} \tag{7.5}$$
whose coefficients are computed directly
$$a_k=\frac{\left(-1\right)^k}{2^{2k}k!\Gamma\left(k+\nu+1\right)} \tag{7.6}$$

Solutions (7.5) are linearly independent for non-integral order ν, while the integral case $\nu=n$ is exceptional as two series give the same function: $y_{-n}=\left(-\right)^n y_n$. So the second independent solution is chosen in the form $Y=\log x+$"regular series".

Solutions (7.5) are denoted by $J_{\pm\nu}\left(x\right)$ for (7.3), 1-st kind, and $I_{\pm\nu}\left(x\right)$ for (7.4), 2-nd kind. Any other solution of (7.3-7.4) is a combination of J's or I'ṡ.

Let us remark that $\frac{1}{2}$-order Bessel-J is an elementary function
$$J_{1/2}\left(x\right)=\sqrt{\frac{2}{\pi x}}\sin x$$

as denominator of the k-th coefficient (7.6) becomes $\Gamma\left(\frac{1}{2}\right)(2k+1)!$

Two important combinations of Bessel-J, called the Y- and H-(Hankel or Neumann) are defined as follows

$$\begin{array}{c} Y_\nu = \frac{\cos\pi\nu\, J_\nu - J_{-\nu}}{\sin\pi\nu} \\ H_\nu^{(\pm)}(z) = J_\nu \pm iY_\nu \end{array}$$

7.3.2. Recurrence relations

Bessel functions obey a series of interesting recurrence relations expressed in terms of differential operations

$$A = x\frac{d}{dx}; B = \frac{1}{x}\frac{d}{dx} \tag{7.7}$$

applied to function $y(x)$ or $x^\nu y(x)$. Precisely, for any Bessel y_ν of order ν and $m = 1, 2, ...$

$$\begin{array}{rcl} \left(\frac{1}{x}\frac{d}{dx}\right)^m (x^\nu y_\nu) & = & x^{\nu-m} y_{\nu-m} \\ \left(\frac{1}{x}\frac{d}{dx}\right)^m \left(x^{-\nu} y_\nu\right) & = & x^{-(\nu+m)} y_{\nu+m} \end{array}$$

So operation B lowers order ν of product $x^\nu y_\nu$ by 1, and raises order of $x^{-\nu} y_\nu$.

Relations (7.8) could be verified by direct manipulation of series (7.5) with coefficients (7.6) (for Bessel J_ν). An alternative argument exploits commutation relations of operators A, B (7.7) and Bessel's

$$L_\nu = \partial^2 + \tfrac{1}{x}\partial + \left(1 - \tfrac{\nu^2}{x^2}\right) = BA + \left(1 - \tfrac{\nu^2}{x^2}\right) \tag{7.8}$$

We observe that the commutator

$$[A, B] = A \cdot B - B \cdot A = \left[x\partial, \tfrac{1}{x}\partial\right] = -\tfrac{2}{x}\partial = -2B \tag{7.9}$$

and conjugate Bessel operator (7.8) with multiplications by $x^{\pm\nu}$, i.e. introduce operators

$$\begin{array}{rcl} \hat{L}_\nu & = & x^\nu L_\nu x^{-\nu} = \partial^2 + \frac{1-2\nu}{x}\partial + 1 = B(A - 2\nu) + 1 \\ \check{L}_\nu & = & x^{-\nu} L_\nu x^\nu = \partial^2 + \frac{1+2\nu}{x}\partial + 1 = B(A + 2\nu) + 1 \end{array}$$

Solution of $\hat{L}_\nu[y] = 0$ is $x^\nu y_\nu(x)$ - an analytic (series) part of (7.5), whereas $\check{L}_\nu[y] = 0$ gives $x^{-2\nu} y_\nu(x)$.

Next we observe the following (raising/lowering) relations between operators $\hat{L}_\nu; \check{L}_\nu$and B

$$\begin{aligned} \hat{L}_\nu B &= B\hat{L}_{\nu+1} \\ \check{L}_\nu B &= B\check{L}_{\nu-1} \end{aligned} \tag{7.10}$$

so conjugation $B^{-1}LB$ raises the order of $\hat{L}_\nu$ and lowers that of $\check{L}_\nu$. Clearly, B has the same effect on solutions of $\hat{L}_\nu$ and $\check{L}_\nu$. So $\hat{L}_\nu [y] = 0$ implies that function $z = B[y]$ solves $\hat{L}_{\nu-1}[z] = 0$, and similarly for $\check{L}$. That proves the transformation rules (7.8).

Formulae (7.8) imply some other useful recurrence relations, like

$$\begin{aligned} \left(\partial + \frac{\nu}{x}\right) J_\nu (x) &= J_{\nu-1}(x) \\ \left(\partial - \frac{\nu}{x}\right) J_\nu (x) &= -J_{\nu+1}(x) \end{aligned} \tag{7.11}$$

that give $J_{\nu\pm1}$ in terms of derivative of J_ν, and the two-term recurrence

$$\begin{aligned} J_{\nu-1}(x) + J_{\nu+1}(x) &= \frac{2\nu}{x} J_\nu (x) \\ J_{\nu-1}(x) - J_{\nu+1}(x) &= 2J'_\nu (x) \end{aligned} \tag{7.12}$$

Remembering that half-order Bessels functions are elementary $J_{1/2} = \frac{\sin x}{\sqrt{x}}$; $Y_{1/2} = \frac{\cos x}{\sqrt{x}}$, we get as a consequence of that all half-integral Bessels are elementary. They could be computed either by n-the order differential relation (product of operators (7.11))

$$\begin{aligned} J_{n+1/2}(x) &= \left(\tfrac{n-1/2}{x} - \partial\right) \ldots \left(\tfrac{1/2}{x} - \partial\right) \left[\tfrac{\sin x}{\sqrt{x}}\right] \\ J_{-n+1/2}(x) &= \left(\tfrac{n-1/2}{x} + \partial\right) \ldots \left(\tfrac{1/2}{x} + \partial\right) \left(\tfrac{\sin x}{\sqrt{x}}\right) \end{aligned} \tag{7.13}$$

or the algebraic relation (product of Jacobi matrices $M_\nu = \begin{bmatrix} 0 & 1 \\ -1 & \frac{2}{x}\nu \end{bmatrix}$ from (7.12). We bring a few of them

$$\begin{aligned} J_{1/2}(x) = &\sqrt{\tfrac{2}{\pi x}} \sin x \\ J_{3/2}(x) = &\sqrt{\tfrac{2}{\pi x}} \left(\tfrac{x\cos x - \sin x}{x}\right) \\ J_{5/2}(x) = &\sqrt{\tfrac{2}{\pi x}} \left(\tfrac{\sin x + 3x\cos x - 3x^2 \sin x}{x^2}\right) \\ J_{7/2}(x) = &\sqrt{\tfrac{2}{\pi x}} \left(\tfrac{15\sin x - 15x\cos x - 6x^2\sin x + x^3\cos x}{x^3}\right) \end{aligned}$$

7.3.3. Asymptotics

The choice of linear combinations of Bessel solutions (7.3)-(7.4) is dictated by their asymptotic behavior at ∞. Namely,

$$J_\nu(x) \sim \sqrt{\frac{\pi}{2x}} \cos\left(x - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right)$$

then

$$Y_\nu(x) \sim \sqrt{\frac{\pi}{2x}} \sin\left(x - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right)$$

hence

$$H_\nu^\pm(x) \sim \sqrt{\frac{\pi}{2x}} \exp\left[\pm i\left(x - \tfrac{\pi}{2}\nu - \tfrac{\pi}{4}\right)\right]$$

The modified (2-nd kind) functions I and K are related to J and Y via complex change of variable $x \to ix$, hence they have exponential asymptotics at ∞. Precisely,

$$I_\nu(x) = e^{-i\frac{\pi}{2}\nu} J_\nu(ix) \sim \frac{e^x}{\sqrt{2\pi x}} \left\{1 - \frac{4\nu^2 - 1}{8x} + ...\right\}$$

exponentially increasing, while

$$K_\nu = \frac{\pi}{2}\left(\frac{I_{-\nu} - I_\nu}{\sin \pi\nu}\right) = \frac{\pi i}{2} e^{i\frac{\pi}{2}\nu} H_\nu^+(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}\{1 + ...\}$$

exponentially decreasing

7.3.4. Integral representations

There many integral representations of Bessel functions but most of them fall in two types: *Poisson type* (finite integral) and the *Mehler-Sonine* type (infinite integral). The first type is exemplified by

$$\Gamma\left(\tfrac{1}{2} + \nu\right) J_\nu(x) = \frac{1}{\sqrt{\pi}}\left(\frac{x}{2}\right)^\nu \int_0^\pi e^{ix\cos\theta} \sin^{2\nu}\theta \, d\theta = \left(\frac{x}{2}\right)^\nu \int_{-1}^1 \cos xu \left(1 - u^2\right)^{\nu - 1/2} du$$

The second (Mehler-Sonine) gives

$$\Gamma\left(\tfrac{1}{2} - \nu\right) J_\nu(x) = \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{-\nu} \int_1^\infty \sin\, xt \frac{dt}{(t^2 - 1)^{\nu + 1/2}} \qquad (7.14)$$

and similarly

$$\Gamma\left(\tfrac{1}{2}-\nu\right)\,Y_\nu\,(x) = -\frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{-\nu}\int\limits_1^\infty \cos\,xt\,\frac{dt}{\left(t^2-1\right)^{\nu+1/2}}$$

8. Harmonic oscillator

Differential operator of the form $h = -\partial^2 + x^2$ on $\mathbb{R}$, known as (quantum) harmonic oscillator plays fundamental role in Mathematics and Physics. Its spectrum (eigenvalues/eigenfunctions) can be easily derived from commutation relations for a special pair of first order operators know in Mathematics as *raising/lowering* operators and in Physics as *creation/annihilation* pair

$$\begin{array}{l} a = \partial + x \\ a^\dagger = -\partial + x \end{array}$$

The latter is the formal adjoint of a with respect to the L^2-inner product $\langle u|v\rangle = \int\limits_{-\infty}^{\infty} u\,\bar{v}\,dx$.

Operator h factors into the product of $a; a^\dagger$, namely

$$h = a^\dagger a + 1 = a a^\dagger - 1 \tag{8.1}$$

and the triple $\left\{a; a^\dagger; h\right\}$ is easily shown to obey the following commutation relations

$$\begin{array}{l} \left[a^\dagger; a\right] = -2 \\ \left[h; a\right] = \left[a^\dagger; a\right] a = -2a \\ \left[h; a^\dagger\right] = \left[a; a^\dagger\right] a^\dagger = 2a^\dagger \end{array} \tag{8.2}$$

Here $[a; b] = ab - ba$ denotes the commutator of two operators. In second and third lines we used the product-rule for commutators: $[a; bc] = [a; b]\,c + [a; c]\,b$, for any triple $a; b; c$.

Relations (8.2), particularly the second and the third ones have immediate application to spectrum h. Indeed, if ψ is an eigenfunction $h\,[\psi] = \lambda\psi$ then raising or lowering ψ gives another eigenfunction: $\psi_+ = a^\dagger\,[\psi]$ and $\psi_- = a\,[\psi]$ are also eigenfunctions of h of eigenvalues $\lambda + 2$ and $\lambda - 2$. Indeed,

$$h\,[\psi_+] = h a^\dagger\,[\psi] = \left\{a^\dagger h + \left[h; a^\dagger\right]\right\}[\psi] = (\lambda+2)\,a^\dagger\,[\psi] = (\lambda+2)\,\psi_+$$

and similar one shows

$$h[\psi_-] = ... = (\lambda - 2)\,\psi_-$$

for ψ_-. Here we used the obvious relation $ab = ba + [a;b]$. So operator $a^\dagger$ raises each eigenvalue of h by 2, while a lowers it by 2, whence the terminology.

We proceed to compute the eigenvalues. Take the lowest eigenvalue λ_0 of h and its eigenfunction ψ_0. Such eigen always exists as operator h is positive-definite: $\langle f\,|h|\,f\rangle \geq 0$ for any square-integrable function $f \in L^2$. Clearly, the lowering operator a takes ψ_0 to 0, so it solves an ode

$$a[\psi] = \psi' + x\psi = 0 \implies \psi_0 = e^{-x^2/2} \text{ -the Gaussian}$$

We plug ψ_0 in (8.1) and find the lowest eigenvalue $\lambda_0 = 1$. Next we apply the m-th iterate of the raising operator $a^\dagger$ to ψ_0

$$\psi_m(x) = \left(a^\dagger\right)^m[\psi_0] = (-\partial + x)^m \left[e^{-x^2/2}\right] \tag{8.3}$$

The result is a new eigenfunction of h of eigenvalue $\lambda_m = 1 + 2m$. Thus spectrum of h contains all odd integers with eigenfunctions given by (8.3). To bring $\{\psi_m\}$ to the standard form of the m-th Hermite function we use another commutation relation

$$-\partial + x = e^{x^2/2}(-\partial)\,e^{-x^2/2}$$

whence follows

$$\psi_m(x) = e^{x^2/2}(-\partial)^m \left[e^{-x^2/2}\,e^{-x^2/2}\right] = e^{x^2/2}(-\partial)^m \left[e^{-x^2}\right] \tag{8.4}$$

Clearly, ψ_m is a product of Gaussian $e^{-x^2/2}$ and a Hermite polynomial of degree m

$$\psi_m = e^{-x^2/2} H_m(x)$$

Those can be computed via (8.4)

$$\begin{aligned} H_0 &= 1 \\ H_1 &= 2x \\ H_2 &= 4x^2 - 2 \\ H_3 &= 8x^3 - 12x \end{aligned}$$

Polynomials form a complete set of functions on any finite interval of $\mathbb{R}$ (any f can be approximated by polynomials), hence Gaussian$\times$polynomials approximate all functions on $\mathbb{R}$ that vanish at infinity.. Thus we get a complete *eigenvalue spectrum* of the harmonic oscillator (Hermite operator) h

$$\boxed{\begin{array}{l}\text{eigenvalues} = \{1 + 2m : m = 0; 1; 2; ...\} \\ \text{eigenfunctions} = \{\text{Hermite } \psi_m\}\end{array}}$$

Below we compute and plot the first 7 Hermite polynomials $m = 0, 1, ..., 6$

$$H_m(x) = e^{x^2} \frac{d^m}{dx^m} \left(e^{-x^2}\right)$$

the corresponding (normalized) Hermite functions

$$h_m(x) = e^{-x^2/2} H_m(x) \Bigg/ \left(\int_{-\infty}^{\infty} e^{-x^2} H_m(x)^2 \, dx \right)^{1/2}$$

$H_m(x)$	$h_m(x)$
1	$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$
$-2x$	$\frac{\sqrt{2}}{\sqrt[4]{\pi}} e^{-\frac{1}{2}x^2} x$
$-2 + 4x^2$	$\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2}\sqrt[4]{\pi}} \left(-1 + 2x^2\right)$
$-4x\left(-3 + 2x^2\right)$	$\frac{e^{-\frac{1}{2}x^2}}{\sqrt{3}\sqrt[4]{\pi}} \left(3 - 2x^2\right) x$
$12 - 48x^2 + 16x^4$	$\frac{e^{-\frac{1}{2}x^2}}{2\sqrt{6}\sqrt[4]{\pi}} \left(3 - 12x^2 + 4x^4\right)$
$-8x\left(15 - 20x^2 + 4x^4\right)$	$\frac{e^{-\frac{1}{2}x^2}}{2\sqrt{15}\sqrt[4]{\pi}} \left(15 - 20x^2 + 4x^4\right) x$
$-120 + 720x^2 - 480x^4 + 64x^6$	$\frac{e^{-\frac{1}{2}x^2}}{12\sqrt{5}\sqrt[4]{\pi}} \left(-15 + 90x^2 - 60x^4 + 8x^6\right)$