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Fundamental solutions (Green's f-ns point sources) on $\mathbb{R}$.

Heat

IVP (1) $\begin{cases} u_t - D u_{xx} = F \\ u|_{t=0} = f(x) \end{cases}$

Fund: $G(x, \xi, t)$ – ?

$$\begin{cases} G_t - D G_{xx} = 0, \quad t > 0 \\ G|_{t=0} = \delta(x-\xi) \end{cases}$$

Ans: $G = G(x-\xi, t)$

$$G(x,t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi D t}}$$ – Gaussian

Wave

$\begin{cases} u_{tt} - c^2 u_{xx} = F \\ u|_{t=0} = f(x); \quad u_t|_0 = g(x) \end{cases}$

$K(x, \xi, t)$ – ?

$$\begin{cases} K_{tt} - c^2 K_{xx} = 0; \quad t > 0 \\ K|_{t=0} = 0; \quad K_t|_0 = \delta(x-\xi) \end{cases}$$

$K = K(x-\xi, t)$ – convolut

$$K(x,t) = \frac{1}{2c} H\left((ct)^2 - x^2\right)$$

Thm: Any IVP (1) with initial f, g & source F is given by integral

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi, t) f(\xi) + \int_0^t \int_{-\infty}^{\infty} G(x-\xi, t-\tau) F(\xi, \tau)$$

IC

source

$$u(x,t) = \int_{-\infty}^{\infty} K(x-\xi, t) g(\xi) + \int_{-\infty}^{\infty} \frac{\partial}{\partial t} K(x-\xi, t) f(\xi) + \int_0^t \int_{-\infty}^{\infty} K(x-\xi, t-\tau) F(\xi, \tau)$$

Check for Heat: $L[\ldots]$ designates diff. op-n L applied to f-n

$$(\partial_t - D\partial_x^2)[u] = \int_{-\infty}^{\infty} \underbrace{(\partial_t - D\partial_x^2)[G(x-\xi,t)]}_{\delta(x-\xi)\,\delta(t)}\, f(\xi) + \qquad\Big|\qquad f(x)\,\delta(t)$$

$$+\ \partial_t\left[\int_0^t \int G(..)F(..)\right] - D^2\partial_x^2\left[\int_0^t \int \ldots\right]$$

$$\underbrace{\int_{-\infty}^{\infty}(G_{t-\tau} * F_\tau)\Big|_{\tau=t}} + \int_0^t \int \underbrace{(\partial_t - D\partial_x^2)[G(..)]}\, F(..)$$

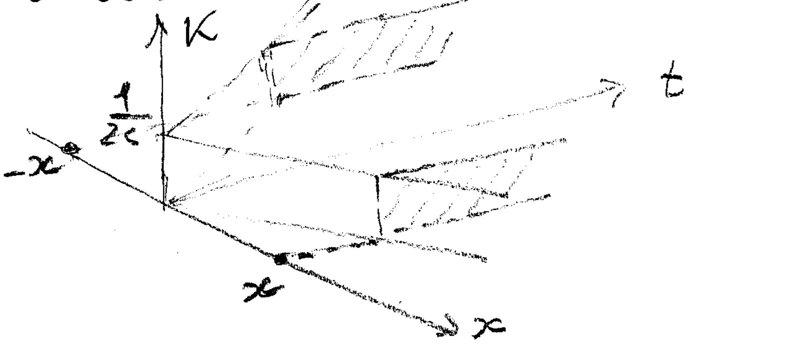
$$\int_{-\infty}^{\infty}\delta(x-\xi)F(\xi,t) + \iint\ 0\cdot F(..) \qquad\Big|\qquad F(x,t)$$

Balance: $(\partial_t - D\partial_x^2)[u] = F(x,t) + \underbrace{f(x)\,\delta(t)}_{IC} \Leftrightarrow \begin{cases} DE \\ u\big|_{t=0} = f \end{cases}$

Q.E.D.

Problem: Do similar derivation for the wave eq-n. Note for K

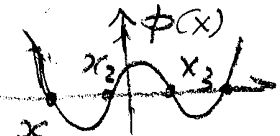
$$\partial_t K = \frac{1}{2}\left[\delta(x-ct) + \delta(x+ct)\right]$$



Use i) $\partial_x\Big[\underbrace{\ \ 0 \ \big|\ 1\ \ }_{H(x)}\Big] = \delta(x)$

(ii) $\partial_t\left[\frac{1}{2c}H\left((ct)^2 - x^2\right)\right] = \frac{1}{2c}\,\underbrace{\delta\left((ct)^2 - x^2\right)}\,\partial_t\left[(ct)^2 - x^2\right] = \ldots$

(iii) For any change of variable $\phi(x)$:



$\delta(\phi(x)) = \sum_j \frac{1}{\phi'(x_j)}\,\delta(x - x_j)$ – sum over roots