Fourier transform

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Fourier harmonics

$$\begin{aligned}&\{\cos kx, \sin kx : k \ge 0\} - \text{ real}\\ &\{e^{ikx} : -\infty < k < \infty\} - \text{ complex}\end{aligned} \qquad (1.1)$$

k – wave number/vector.

Fourier expansion

Any f(x) can be expanded into a F - series

$$\begin{aligned}&f(x) \sim \sum_{k=0,1,\ldots} (a_k \cos kx + b_k \sin kx)\\ &f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx}\end{aligned} \quad \text{- for } 2\pi \text{ -periodic}$$

or F - integral

$$f(x) \sim \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \text{ - on } \mathbf{R} \text{ or } \mathbf{R}^n \qquad (1.2)$$

with F coefficients $\{a_k, b_k\}$ - real, or $\{\hat{f}_k\}$ - complex.

Orthogonality relation of F modes

The inner product on (complex) functions is given by

$$\langle f \mid g \rangle = \int f(x) \bar{g}(x) dx$$

Then system (1.1) obeys orthogonality relations

$$\langle e^{ikx} \mid e^{imx} \rangle = 2\pi\delta(k-m) \qquad (1.3)$$

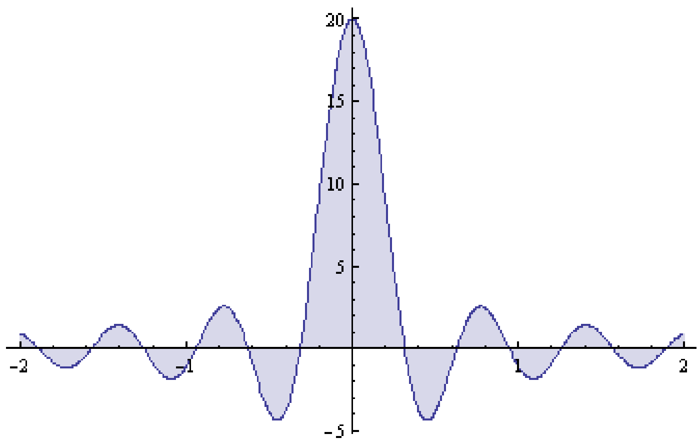
Kronecker delta for integers for 2π - periodic, and Dirac delta on R.

To check compute

$$\int_{-\pi}^{\pi} e^{ipx} dx = \begin{cases} 2\pi; \, p = 0 \\ 0; \, p \neq 0 \end{cases} \text{ on } 2\pi \text{-periodic (p – integer)}$$

On R take finite limits $\int_{-N}^{N} \ldots$ and let $N \to \infty$

$$\int_{-N}^{N} e^{ipx} dx = \frac{2\sin Np}{p} \to 2\pi\delta(p)$$



General orthogonal expansion

If $\{X_k\}$ - orthogonal system of vectors in $\mathbf{R}^n; \mathbf{C}^n$, then any vector Y is expanded into a series

$$Y = \sum_k a_k X_k \tag{1.4}$$

With coefficients

$$a_k = \frac{Y \bullet X_k}{|X_k|^2} \tag{1.5}$$

And

$$|Y|^2 = \sum_k a_k^2 |X_k|^2 \tag{1.6}$$

Applied to F modes we get for any f(x)

(i) $\hat{f}(k) = \frac{1}{2\pi}\int f(x)e^{-ikx}$ - F coefficient or F transform

(ii) $f(x) = \int \hat{f}(k)e^{ikx}dk$ - inverse F transform

(iii) Plancherel formula: for any square-integrable f(x)

$$\|f\|^2 = \int |f(x)|^2\,dx = 2\pi\int |\hat{f}(k)|^2\,dk = \|\hat{f}\|^2$$

Table of Fourier transforms

$e^{-a\,\mathrm{Abs}[k]}$	$\frac{a}{\pi(a^2+x^2)}$
$\frac{1}{a^2+k^2}$	$\frac{e^{a x}\,\mathrm{HeavisideTheta}[-x]+e^{-a x}\,\mathrm{HeavisideTheta}[x]}{2 a}$
$e^{-a k^2}$	$\frac{e^{-\frac{x^2}{4 a}}}{2\sqrt{a}\,\sqrt{\pi}}$
$\mathrm{Sech}[a\,k]$	$-\frac{\mathrm{Sech}\left[\frac{\pi x}{2 a}\right]}{2 a}$
$\mathrm{Tanh}[k]$	$\frac{1}{2}\,i\,\mathrm{Csch}\left[\frac{\pi x}{2}\right]$
$\mathrm{UnitStep}[1-k^2]$	$\frac{\mathrm{Sin}[x]}{\pi x}$

Fourier transform method

Direct $\mathcal{F}$

$$f(x) \longrightarrow \hat{f}(k) = \int f(x) e^{-ik\cdot x} dx$$

Inversion in $\mathbb{R}^n$

$$f(x) = \underbrace{\frac{1}{(2\pi)^n}}_{\text{"norming const"}} \int \hat{f}(k) e^{ik\cdot x}$$

Plancherel f-la: $\|f\|^2 = \int |f(x)|^2 dx = \frac{1}{(2\pi)^n} \int |\hat{f}(k)|^2$

Basic properties: shift, differentiation, convolution

(i) $f_a = f(x-a) \overset{\mathcal{F}}{\rightleftarrows} \hat{f}_a(k) = e^{-ik\cdot a} \hat{f}(k)$

(ii) $\partial_m f \rightleftarrows ik_m \hat{f}$; $\nabla f \to ik\hat{f}$; $\nabla^2 f \to -k^2 \hat{f}$

(iii) $f * g = \int f(x-\xi) g(\xi) d\xi \rightleftarrows \hat{f}(k)\hat{g}(k)$

Any c.c. diff. operator $L = \sum a_m \partial^m \longmapsto \sigma_L(k) = \underbrace{\sum a_m (ik)^m}_{\text{polynomial}}$, $m = (m_1, m_2 \ldots)$

Fourier inversion involves complex analysis (residue calculus)and other techniques.

Application to DE's on $\mathbb{R}$; $\mathbb{R}^n$

DE	solution	Green's
K-G.: $(-\partial_x^2 + m^2)u = F$	$\hat{u} = \frac{\hat{F}}{k^2+m^2} \Rightarrow$ $\boxed{u = K * F}$	$K = \mathcal{F}^{-1}\left(\frac{1}{k^2+m^2}\right) = \frac{e^{-m\lvert x\rvert}}{2m}$
$\begin{cases}(\partial_t - \alpha\Delta)u = F \\ u\vert_0 = f\end{cases}$ Heat:	$\begin{cases}(\partial_t + \alpha k^2)\hat{u} = \hat{F} \\ \hat{u}\vert_0 = \hat{f}\end{cases} \Rightarrow \boxed{u = G * f + \int_0^t \ldots}$	$G = \mathcal{F}^{-1}(e^{-t\alpha k^2}) = \frac{1}{(4\pi\alpha t)^{n/2}} e^{-\lvert x\rvert^2/4\alpha t}$ Gaussian
$\begin{cases}u_{yy} + u_{xx} = 0;\ y > 0 \\ u\vert_{y=0} = f(x)\end{cases}$ Harmonic:	$\begin{cases}\hat{u}_{yy} - k^2\hat{u} = 0 \\ \hat{u}\vert_0 = \hat{f}\end{cases} \Rightarrow \hat{u} = e^{-\lvert k\rvert y}\hat{f}$ $\Rightarrow \boxed{u = P * f}$	Poisson $P = \mathcal{F}^{-1}[e^{-y\lvert k\rvert}] = \frac{1}{\pi(x^2+y^2)}$

1D wave		
$\begin{cases}(\partial_t^2 - c\partial_x^2)u = F \\ u\|_0 = f;\ u_t\|_0 = g\end{cases}$	$\begin{cases}\hat{u}_{tt} + c^2k^2\hat{u} = \hat{F} \\ \ldots\end{cases}$ $\hat{u} = \int_0^t \frac{\sin ck(t-\tau)}{ck}\hat{F}(\tau)\,d\tau + \ldots$	$u = \int_0^t \underbrace{K_{t-\tau} * F(\tau)}_{\text{space convol.}}$ $K = \mathcal{F}^{-1}\left[\frac{\sin ckt}{ck}\right] \overset{?}{=} \frac{1}{2c}H(ct-\|x\|)$ -step. Check: $\partial_t K = \mathcal{F}^{-1}(\cos ckt) = \frac{1}{2}[\delta(x-ct)+\delta(x+ct)]$
2D K-G $(-\Delta+m^2)u = F$	$\hat{u} = \frac{\hat{F}(k)}{k^2+m^2} \Rightarrow u = K * F$	$K = \mathcal{F}^{-1}_{2D}\left(\frac{1}{k^2+m^2}\right) = K(r)$

$$K(r) = \frac{1}{(2\pi)^2}\int_0^\infty \boxed{\int_0^{2\pi} \frac{e^{ikr\cos\theta}}{k^2+m^2}\,d\theta}\; k\,dk$$

$$\int_0^\infty J_0(kr)\frac{k}{k^2+m^2}\,dk$$

$$\Rightarrow \boxed{K(r) = K_0(mr)}$$ – Kelven (modified Bessel)