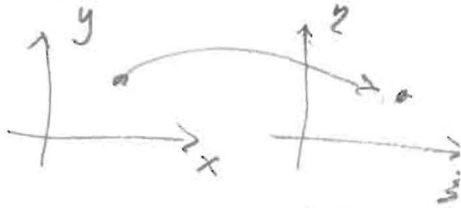
Change of variables in PDE | DG

Characteristic coordinates



Change: $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{pmatrix}$

2nd PDE: $\boxed{L = a\partial_x^2 + 2b\partial_{xy}^2 + c\partial_y^2 \longleftrightarrow A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}}$

Ask: $\tilde{L} = \tilde{a}\,\partial_\xi^2 + 2\tilde{b}\,\partial_{\xi\eta}^2 + \tilde{c}\,\partial_\eta^2$? ? ?

Chain rule $\begin{cases} \partial_x = \xi_x \partial_\xi + \eta_x \partial_\eta \\ \partial_y = \xi_y \partial_\xi + \eta_y \partial_\eta \end{cases} \Rightarrow \underbrace{\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}}_{\text{grad}} = \underbrace{\begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix}}_{\text{Jacob.}} \underbrace{\begin{pmatrix} \partial_\xi \\ \partial_\eta \end{pmatrix}}_{\text{grad}}$

Ans: $\tilde{L} = \underbrace{(a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)}_{\tilde{a} = \nabla\xi\cdot A\cdot\nabla\xi}\partial_\xi^2 + 2\underbrace{\left[a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y\right]}_{\tilde{b} = \nabla\xi\cdot A\cdot\nabla\eta}\partial_{\xi\eta}^2$

$+ \underbrace{\tilde{c}}_{(\nabla\eta\cdot A\cdot\nabla\eta)}\partial_\eta^2 + \text{l.o.t.}$

Ex: 1) Linear change in Laplace $\Delta = \partial_x^2 + \partial_y^2$

$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \to \tilde{\Delta} = (\alpha^2+\beta^2)\partial_\xi^2 + 2(\alpha\gamma + \beta\delta)\partial_{\xi\eta}^2 + (\gamma^2+\delta^2)\partial_\eta^2$

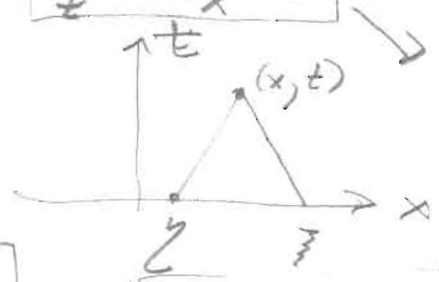
Charact. coord. for $L = a\partial_x^2 + \dots$

Def: $\boxed{\nabla\xi \cdot A \cdot \nabla\xi = 0} \Rightarrow \begin{matrix}\tilde{a} = 0 \\ \tilde{c} = 0\end{matrix} \Rightarrow \tilde{L} = 2\tilde{b}\,\partial^2_{\xi\eta} + \dots$

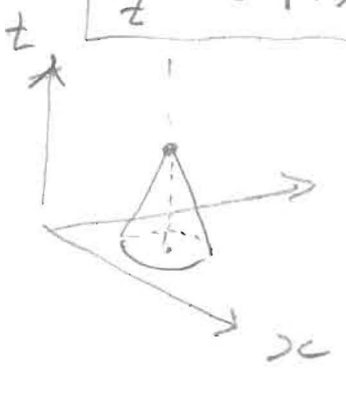
1° Elliptic L $\Rightarrow$ no real charact.

2° Hyperbolic: $L = \partial_t^2 - c^2\partial_x^2$ or $\partial_t^2 - c^2\nabla^2$

2D $\boxed{\xi_t^2 - c^2\xi_x^2 = 0} \Rightarrow \boxed{\xi_t \pm c\,\xi_x = 0}$ $\nearrow \xi = x + ct$, $\searrow \eta = x - ct$



nD $\boxed{\xi_t^2 - c^2|\nabla\xi|^2 = 0} \Rightarrow \boxed{\xi_t = c|\nabla\xi| = c\sqrt{\xi_x^2 + \xi_y^2}}$ NL



charact. cone

Solution: $u_{\xi\eta} = 0 \Rightarrow u = f(\xi) + g(\eta)$