M445: Heat equation with sources

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I. On Fourier and Newton's cooling laws

The Newton's law claims the temperature rate to be proportional to the difference:

$$\frac{d}{dt}T = -\alpha\,(T - T_0) \tag{1}$$

The Fourier law postulates the heat-flux to be proportional to the temperature gradient:

$$\begin{aligned} \frac{d}{dt}Q &= -\int_{\Sigma} \kappa \nabla T \cdot N dS; \\ Q &= cT \end{aligned} \tag{2}$$

with coefficients c=heat capacity; κ=heat conductivity. Two descriptions deal with different time scales: fast for the Fourier and slow for the Newton.

A physical model could be a fluid undergoing turbulent mixing as it cools down, e.g. buoyancy-driven convection in a pool with a freezing surface. Call temperatures T_0 (air) and T_b (initial water).

We consider a circulation pattern that randomly replaces surface parcels at a constant rate (so called *renewal model*). The fluid patches that come to the surface could have the ambient water temperature $T > T_0$, and thus capable of releasing heat. Or they've already participated in the heat-exchange on the previous time-step, which brought their temperature down to $T \approx T_0$, hence rendered them incapable of further cooling.

Let $\phi(t)$ denotes a fraction of the surface exposed (and capable) of the heat exchange. A constant replacement rate makes $\phi(t)$ an exponential function, $\phi(t) = \frac{1}{\tau}e^{-t/\tau}$, where τ is the slow time-scale.

We take the standard erf -solution of the half-space problem and its heat-flux

$$\begin{aligned} T(z,t) &= (T_0 - T_b)\,\mathrm{erf}\left(\frac{z}{\sqrt{\kappa t}}\right) + T_b \\ Q(0,t) &= -k\,(\partial_z T)|_{z=0} = -\frac{k}{\sqrt{\pi \kappa t}}\,(T_0 - T_b) \end{aligned}$$

Here κ denotes the heat-diffusivity, and k -heat conductivity.

Averaging out the fast time scales we get

$$\begin{aligned} T(z) &= \int_0^\infty \phi(t)\, T(z,t)\, dt = (T_0 - T_b)\, e^{-z/\sqrt{\kappa\tau}} \\ Q(0) &= \int_0^\infty \phi(t)\, Q(0,t)\, dt = -\frac{k}{\sqrt{\kappa\tau}} (T_0 - T_b) = -\frac{\kappa c_p \rho}{l_\theta} (T_0 - T_b) \end{aligned} \tag{3}$$

where l_θ is the length scale of the average temperature profile, c_p-specific heat at constant pressure and ρ -density.

The second line of (3) is essentially the Newton's law.

Mathematically, the erf-solution (e.g. for a cooling bar $[-a, a]$)

$$T(x,t) = \operatorname{erf}\left(\frac{x+a}{\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-a}{\sqrt{t}}\right) \approx \frac{2a}{\sqrt{t}} + O\left(t^{-3/2}\right)$$

has a polynomial fall-off in t (the same would hold for the space-average temperature over $[-a, a]$). However, averaging over the slow (Newton) time scale τ, i.e. taking time-convolution of T with $e^{-t/\tau}$, we get

$$\int_0^t \frac{e^{-(t-s)/\tau}}{\sqrt{s}} ds \approx e^{-t/\tau}\left(c_0 + \frac{c_1}{\sqrt{t}} + \ldots\right)$$

i.e. Newton's exponential fall-off.

II. Heat equation with delta-sources

We write a typical heat-diffusion problem using symbolic operator notation

$$\begin{aligned} u_t + L[u] &= F \\ u|_{t=0} &= f \end{aligned} \tag{4}$$

Here L could be an ordinary differential operator $-\partial p \partial + q$ on $[0, l]$ with suitable boundary conditions at $\{0; l\}$, or more general elliptic pde $L = -\nabla \cdot p \nabla + q$, on region $D \subset \mathbb{R}^n$ with boundary Γ, and boundary condition $B[u] = (a + b\partial_n)\, u|_\Gamma$.

The formal (ODE-type) solution of (4) is given by the *operator-exponential*

$$u = e^{-tL}[f] + \int_0^t e^{-(t-s)L}[F(s)]\, ds \tag{5}$$

analogous to the matrix-exponential.

Such operator-exponential represents a fundamental solution of problem (4). One could show that operator e^{-tL} acting on functions $\{f(x)\}$ is given by an integral kernel $G(x,\xi,t)$, called *Green's function* of the problem,

$$G[f] = \int_D G(x,\xi,...)\, f(\xi)\, d\xi \tag{6}$$

We are interested in the delta-source $F = h(t)\,\delta(x - x_0)$. If $G(x,\xi,t)$denotes the Green's function of $L - B$, then solution (5)

$$u(x,t) = \int_0^t G(x, x_0, t - s)\, h(s)\, ds \tag{7}$$

We evaluate (7) for the standard Gaussian $G = \frac{1}{(4\pi\alpha t)^{n/2}} e^{-x^2/4\alpha t}$ i.e. $L = -\alpha\Delta$ on $\mathbb{R}^n$

$$u(x,t) = \frac{1}{(4\pi\alpha)^{n/2}} \int_0^t \frac{e^{-x^2/4\alpha s}}{s^{n/2}} h(t-s)\, ds \tag{8}$$

and treat 2 cases.

A. Constant source $h = \text{Const}$

Here (8) yields after the change $s \to z = \frac{x^2}{4\alpha s}$

$$u = \frac{|x|^{2-n}}{4\alpha} \int_{x^2/4\alpha t}^{\infty} z^{n/2-2} e^{-z} dz = \frac{|x|^{2-n}}{4\alpha} \Gamma\left(\tfrac{n}{2} - 1, \frac{x^2}{4\alpha t}\right)$$

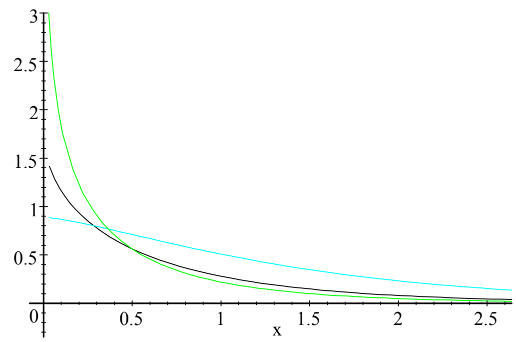
expressed in terms of incomplete Euler gamma function

$$\Gamma(\nu, p) = \int_p^{\infty} e^{-z} z^{\nu - 1} dz$$

of order $\nu = \frac{n}{2} - 1$, depending on space-dimension. Dimensions $n = 1, 2, 3$ can be expanded for small x as

$$\begin{aligned}
\Gamma\left(-\tfrac{1}{2}, p\right) &= \frac{2}{\sqrt{p}} - 2\sqrt{\pi} + 2\sqrt{p} - \frac{1}{3}p^{3/2} + \frac{1}{15}p^{5/2} + O\left(p^{7/2}\right) \\
\Gamma(0, p) &= \left(-\gamma + \ln\frac{1}{p}\right) + p - \frac{1}{4}p^2 + \frac{1}{18}p^3 + O\left(p^4\right) \\
\Gamma\left(\tfrac{1}{2}, p\right) &= \sqrt{\pi} - 2\sqrt{p} + \frac{2}{3}p^{3/2} - \frac{1}{5}p^{5/2} + \frac{1}{21}p^{7/2} + O\left(p^4\right)
\end{aligned}$$

with Euler constant $\gamma = .577$. We plot all 3 gammas



Incomplete gamma-functions $\Gamma(\nu, p)$ for $\nu = -\frac{1}{2}; 0; \frac{1}{2}$

and write the corresponding solutions u expanded in small $p = \frac{x^2}{4\alpha t}$

dim	u
1	$\sqrt{\frac{t}{\alpha}} - \frac{\sqrt{\pi}}{2}\frac{\lvert x\rvert}{\alpha} + \frac{1}{4}\frac{x^2}{\alpha^{3/2}\sqrt{t}} - \frac{1}{96}\frac{x^4}{\alpha^{5/2}t^{3/2}} + \ldots$
2	$\frac{1}{4\alpha}\left(-.577 + \ln\frac{4\alpha t}{x^2} + \frac{1}{4}\frac{x^2}{\alpha t} - \frac{1}{64}\frac{x^4}{\alpha^2 t^2} + \ldots\right)$
3	$\frac{1}{4\alpha}\left(\frac{\sqrt{\pi}}{\lvert x\rvert} - \frac{1}{\sqrt{\alpha t}} + \frac{\lvert x\rvert^2}{12(\alpha t)^{3/2}} - \frac{\lvert x\rvert^4}{160(\alpha t)^{5/2}} + \ldots\right)$

Notice that in 1D solution has an asymptotic limit as $t \to \infty$

$$u(x,t) \simeq \sqrt{\frac{t}{\alpha}} - \frac{\sqrt{\pi}}{2}\frac{|x|}{\alpha}$$

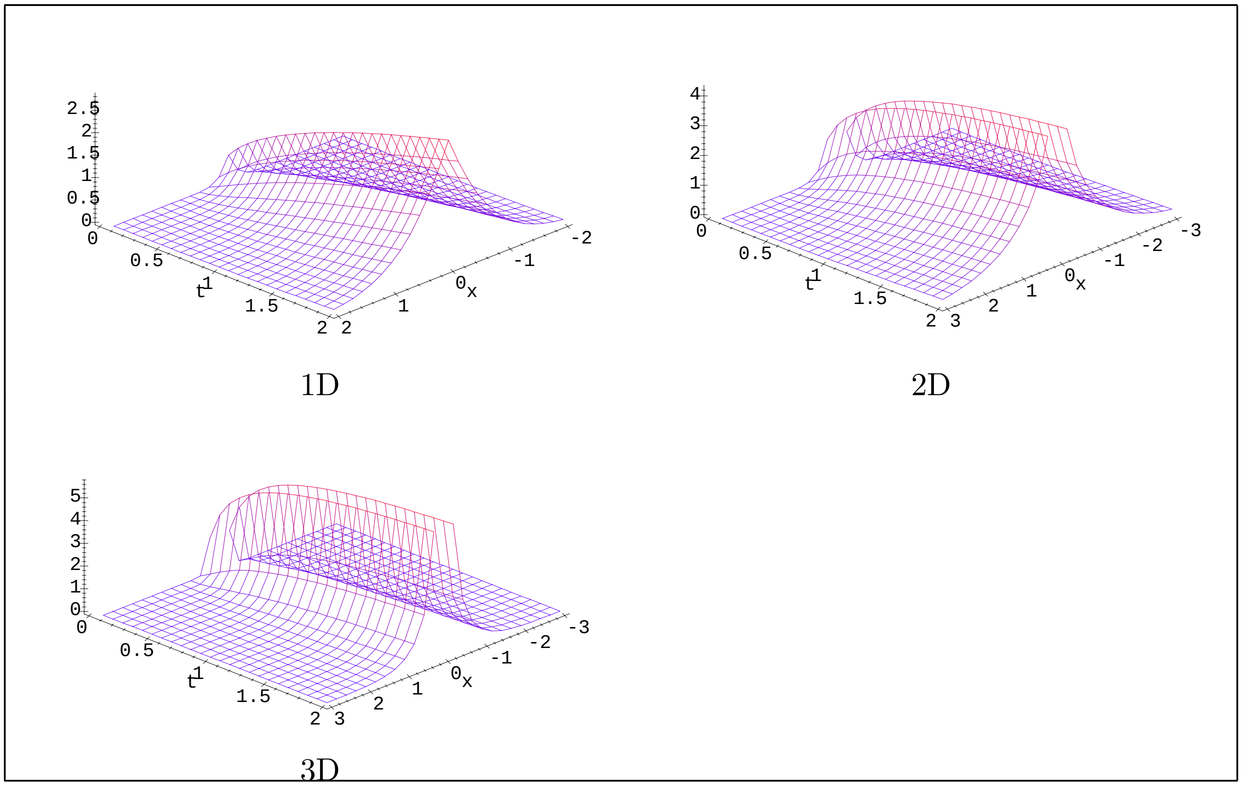
whereas 3D-one converges to a potential-type equilibrium

$$u(x,t) \to \frac{\sqrt{\pi}}{4\alpha|x|}$$

The exact solutions for $n = 1, 2, 3$

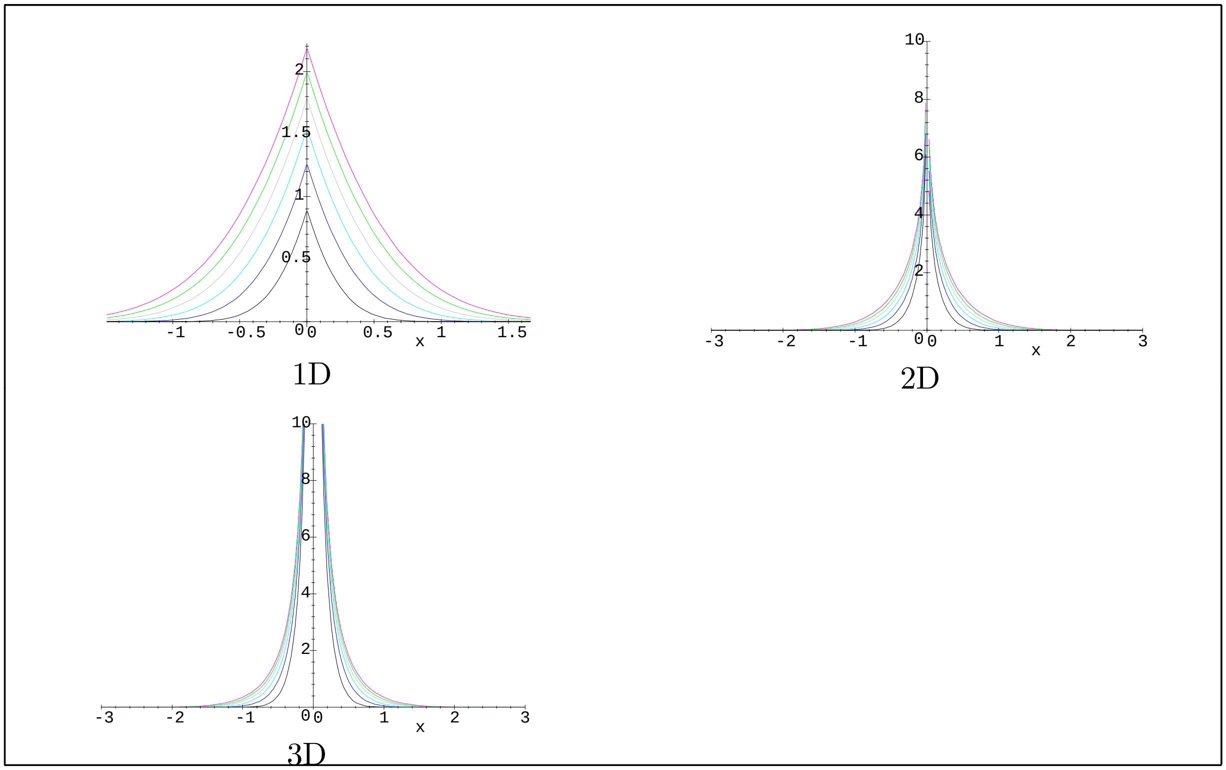
$$\begin{aligned} u &= \frac{|x|}{4\alpha}\Gamma\left(-\tfrac{1}{2}, \frac{x^2}{4\alpha t}\right) \\ u &= \frac{1}{4\alpha}\Gamma\left(0, \frac{x^2}{4\alpha t}\right) \\ u &= \frac{1}{4\alpha|x|}\Gamma\left(\tfrac{1}{2}, \frac{x^2}{4\alpha t}\right) \end{aligned}$$

are plotted below as radial profiles $u\,(r,t)$ in 3D-space-time view



Temperature profile for 1D, 2D and 3D steady point sources

We also show their time snapshots



Time slices of temperature profiles in 1, 2 and 3D

A typical pattern shows accumulation of heat near the source and its spread outward. The rate of accumulation depends on space dimension and steepens with the increase of n.

B. Time periodic source: $h = \cos \omega t$

Here solution

$$u = \int_0^t \cos \omega (t-s) \frac{e^{-x^2/4\alpha s}}{(4\pi\alpha s)^{n/2}} ds$$

The integral has no closed form expression in known (elementary or special) functions. But its large-time asymptotics could be reduced to Fourier transforms of function: $f(t) = \frac{e^{-x^2/4\alpha t}}{(4\pi\alpha t)^{n/2}}$. Namely,

$$u \approx \cos \omega t \left(\underbrace{\int_0^\infty \cos \omega s \, f(s) \, ds}_{\hat{f}_c(\omega)} \right) - \sin \omega t \left(\underbrace{\int_0^\infty \sin \omega s \, f(s) \, ds}_{\hat{f}_s(\omega)} \right)$$

The complete (half-line Fourier) transform of f in expressed through the modified Bessel (Kelvin) function $K_{n/2-1}$

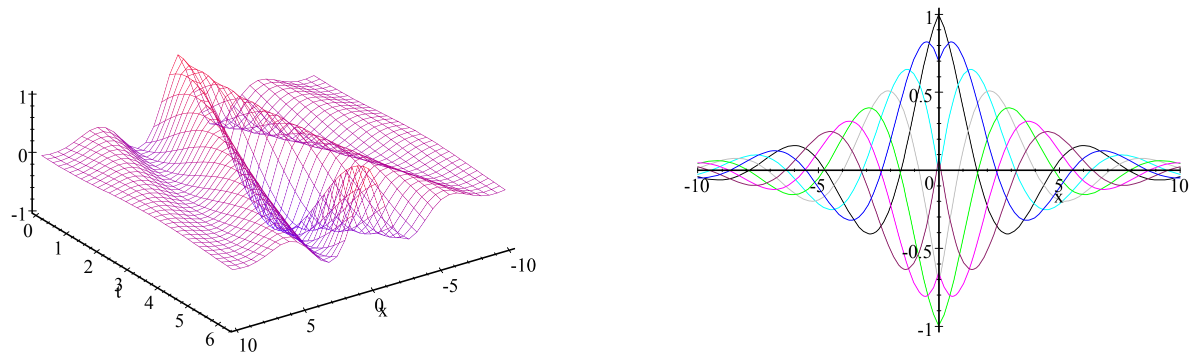
$$\hat{f}\left(\omega\right)=\int_{0}^{\infty}e^{i\omega t}\frac{e^{-x^2/t}}{t^{n/2}}=2e^{i\pi\left(\frac{n-2}{8}\right)}\left(\frac{\sqrt{\omega}}{|x|}\right)^{n/2-1}K_{n/2-1}\left(2\sqrt{-i\omega}\,|x|\right) \tag{9}$$

In special cases, e.g. 1D $K_{-1/2}$ is an elementary function, so integral (9) is simplified to

$$\hat{f}=\frac{1+i}{\sqrt{2\omega}}e^{-\sqrt{2\omega}|x|}\left(\cos\sqrt{2\omega}\,|x|+i\sin\sqrt{2\omega}\,|x|\right)$$

We have thus shown that the asymptotic pattern consists of exponentially attenuated propagating heat-waves

$$u\approx e^{-\sqrt{2\omega}|x|}\cos\left(\sqrt{2\omega}\,|x|\pm\omega t\right)$$



Modulated heat-wave and its time snapshots

Let us remark that the relation between the wave number $k\approx\sqrt{\omega}$ and frequency ω is consistent with the heat-diffusion dispersion law: $i\omega=k^2$.

III. Equilibria for heat-diffusion problems

We use operator formalism (5) for a typical heat-diffusion problem (4) to write its formal solution in terms of operator exponentials, analogous to the matrix-exponential. All functions u,f,F could be expanded in terms of eigenfunctions $\{\psi_k\}$ of operator L, (rather eigenmodes of the *boundary value problem* $L;B$)

$$\begin{aligned} L\left[\psi_k\right] &= \lambda_k\psi_k \\ B\psi_k|_{\Gamma} &= 0 \end{aligned} \tag{10}$$

In particular, Green's function is expanded as

$$G\left(x,\xi,t\right)=\sum_k e^{-t\lambda_k}\frac{\psi_k\left(x\right)\bar{\psi}_k\left(\xi\right)}{\left\|\psi_k\right\|^2}$$

and solution (**??**) becomes

$$u(x,t) = \sum_k \left(\hat{f}_k e^{-t\lambda_k} + \int_0^t e^{-(t-s)\lambda_k} \left[\hat{F}_k(s) \right] ds \right) \psi_k(x). \tag{11}$$

Here $\left\{ \hat{f}_k \right\}; \left\{ \hat{F}_k(t) \right\}$ denote generalized Fourier coefficients of f and F

$$\hat{f}_k = \frac{\langle f(x) | \psi_k \rangle}{\|\psi_k\|^2}$$

in the sense of L^2 (square-mean) inner product.

A simple equilibrium solution v of problem (4) with a stationary (time-independent) source F is given by

$$L[v] = F \Rightarrow v = L^{-1}[F] \tag{12}$$

By analogy with exponential e^{-tL} operator L^{-1} could be represented by an integral kernel (Green's function)

$$K(x,\xi) = \sum_k \frac{1}{\lambda_k} \frac{\psi_k(x) \bar{\psi}_k(\xi)}{\|\psi_k\|^2}$$

expanded in eigenmodes of L. Hence

$$v(x) = \sum_k \frac{\hat{F}_k}{\lambda_k} \psi_k(x)$$

The latter is easily shown to be a limit of solution (**??**) as $t \to \infty$, provided all eigenvalues $\{\lambda_k\}$ of L are positive. Indeed, convolution integral (**??**) becomes

$$u = \frac{I - e^{-tL}}{L}[F] + e^{-tL}[f] \to L^{-1}[F] = v, \text{ as } t \to \infty$$

A. Periodic equilibria

More interesting case arises for a periodic source $F(x,t)$. One asks the same two questions as above

1. whether periodic solutions v exist for (4)
2. whether they are stable, in the sense that any $u(x,t) \to v$ as $t \to \infty$

Both are easily answered using the above operator (ODE)-formalism.

We first consider a single frequency case $F = F(x) e^{i\omega t}$ in the complex form,

$$\begin{cases} u_t + L[u] = F e^{i\omega t} \\ u|_0 = f \end{cases} \tag{13}$$

Formal solution of IVP (13)

$$\begin{aligned} u &= \frac{e^{i\omega t} - e^{-Lt}}{i\omega + L}[F] + e^{-Lt}[f] \\ &= e^{i\omega t}\left(\frac{1}{i\omega + L}\right)[F] + e^{-Lt}\left[f - \left(\frac{1}{i\omega + L}\right)F\right] \end{aligned} \tag{14}$$

is decomposed into the periodic component $v(x)\,e^{i\omega t}$, where equilibrium v satisfies

$$(i\omega + L)\,v = F \Rightarrow \boxed{v = (i\omega + L)^{-1}F} \tag{15}$$

and negative exponential $e^{-Lt}[...]$. As above operators $(i\omega + L)^{-1}; e^{-Lt}$ are given by (complex-valued) Green's functions $K(x,\xi;i\omega)$; $G(x,\xi;t)$, or else could be expanded in eigenmodes

$$v(x) = \sum_k \frac{\hat{F}_k}{i\omega + \lambda_k}\psi_k(x) \tag{16}$$

From complex form (16) one could easily get the real periodic solution

$$\begin{cases} u_t + L[u] = F\cos\omega t \\ u|_0 = f \end{cases} \tag{17}$$

by taking the real and imaginary parts of (14)

$$\operatorname{Re}\left(\frac{e^{i\omega t}}{i\omega + L}\right) = \frac{L}{L^2 + \omega^2}\cos\omega t + \frac{\omega}{L^2 + \omega^2}\sin\omega t$$

This yields the IVP-solution (17) written as

$$u = \left(\cos\omega t\frac{L}{L^2 + \omega^2} + \sin\omega t\frac{\omega}{L^2 + \omega^2}\right)F + e^{-Lt}\left(f - \frac{L}{L^2 + \omega^2}F\right)$$

in the operator-form, or an equivalent series expansion

$$\begin{aligned} u = \sum_k &\left\{\frac{\lambda_k\cos\omega t + \omega\sin\omega t}{\lambda_k^2 + \omega^2}\hat{F}_k \right. \\ &\left. + e^{-\lambda_k t}\left(\hat{f}_k - \frac{\lambda_k}{\lambda_k^2 + \omega^2}\hat{F}_k\right)\right\}\psi_k(x) \end{aligned} \tag{18}$$

The latter clearly demonstrates that $u(x,t)$ converges to a periodic equilibrium $v(x,t) = \operatorname{Re}(v(x)\,e^{i\omega t})$, provided all eigenvalues λ_k are positive, so exponential terms drop in (14)-(18).

B. Multiple frequency case

Here F is represented by a time-Fourier series

$$F = \sum_m e^{i\omega_m t} F_m(x)$$

In the periodic case all frequencies are multiples of a single (lowest) one $\omega_m = m\omega$ and the period of F is $T = \frac{2\pi}{\omega}$. More generally, $\{\omega_m\}$ are arbitrary real numbers, the so called *frequency spectrum*[1] of F.

We seek partial (periodic) solution v in the form

$$v = \sum_m e^{i\omega_m t} v_m(x) \tag{19}$$

with undetermined Fourier coefficients $\{v_m\}$. The substitution in (4) determines each one of them via (15)

$$v_m = (i\omega_m + L)^{-1} F_m$$

So v is expanded in the time Fourier series (19) with the same period (or quasiperiods) as F.

An interesting example of multiple frequencies arises for a periodically moving point-source

$$F = \delta(x - a\cos\omega t) \tag{20}$$

We consider it on a symmetric interval $[-l, l]$ with amplitude of oscillation $a < l$. Generalized time-periodic function (20) has a frequency Fourier expansion $F = \sum_m e^{im\omega t} F_m(x)$ with coefficients

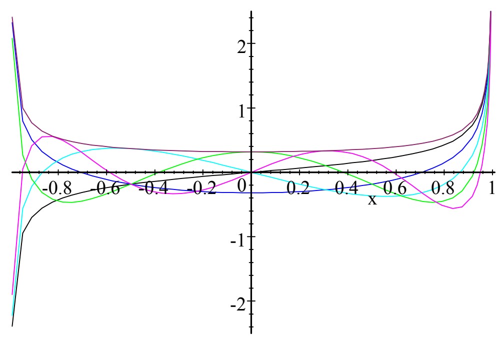
$$F_m(x) = \frac{\cos\left(m\cos^{-1}\left(\frac{x}{a}\right)\right)}{\pi\sqrt{a^2 - x^2}} = \frac{T_m\left(\frac{x}{a}\right)}{\pi\sqrt{a^2 - x^2}}$$

whose numerators are made of the classical Tchebyshev polynomials of the first kind. We plot a few of them

[1] Function F is called *quasi-periodic* if its spectrum is made of linear combinations of a finite (basic) set $\{\omega_1; ...; \omega_p\}$

$$\omega_m = \sum_{k=1}^{p} n_k \omega_k$$

with integer coefficients n_k. Otherwise, it is called almost periodic.



As a consequence we get the periodic equilibrium for the moving-source problem, expanded in the double series

$$v(x,t) = \frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{\lambda_k}\left(\frac{\lambda_k\cos m\omega t + m\omega\sin m\omega t}{\lambda_k^2+(m\omega)^2}\right) \times \frac{\left\langle T_m\left(\frac{x}{a}\right)/\sqrt{a^2-x^2}\right|\psi_k\rangle}{\|\psi_k\|^2}\psi_k(x) \tag{21}$$

assuming all eigenvalues of L positive.

Problems:

1. Specify expansion (21) for the Dirichlet and Neumann problem on $[-l,l]$, $L = -\partial^2$.
2. Compute the first 5 frequency modes $m = 0, ..., 5$.
3. Plot approximate periodic equilibrium (21) by truncating both series. Use Mathematica!