Math224: Final exam. May 4, 2007

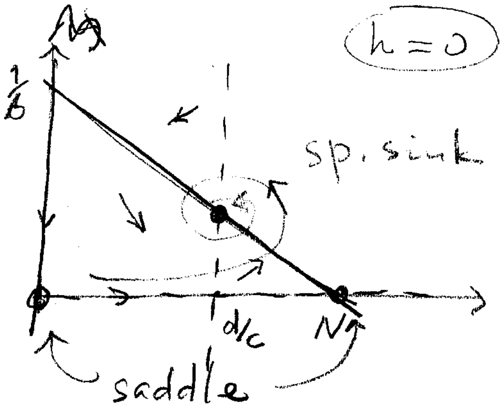
Print Name: ______________________________

1. **Set up DE/DS models**, classify them (linear, nonlinear, separable, Hamiltonian etc.), indicate solution method
   a. Volterra-Lotka predator-prey model with logistic prey x and predator y, when prey is hunted at a rate h (**Extra** credit for phase-plane)

$$\begin{cases} \dot{x} = a\left(1-\frac{x}{N} - by\right)x - h \\ \dot{y} = (cx-d)y \end{cases}$$

i) NL 2nd DS

ii) Numeric



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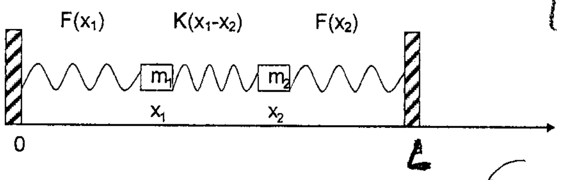
   b. A coupled oscillator system (below) with forces: $F(x_i) = ax_i$, and coupling $K(x_1 - x_2)$ Set it as a second order (Newton) system for position variables: $\{x_1, x_2\}$, then convert to a first order system for position-velocity variables: $\{x_1, x_2, v_1, v_2\}$.

$$\begin{cases} m\ddot{x}_1 = -ax_1 + K(x_1 - x_2) \\ m\ddot{x}_2 = a(L - x_2) - K(x_1 - x_2) \end{cases}$$



Linear (or NL depending on K) 4th order DS

$$\begin{cases} \dot{x}_1 = v_1 \\ \dot{x}_2 = v_2 \\ \dot{v}_1 = \frac{1}{m_1}\left[-ax_1 + K(x_1 - x_2)\right] \\ \dot{v}_2 = \frac{1}{m_2}\left[a(L - x_2) - K(x_1 - x_2)\right] \end{cases}$$

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Problem	Score
1(40)	
2(40)	
3(35)	
4(30)	
5(30)	
6(35)	
Total	

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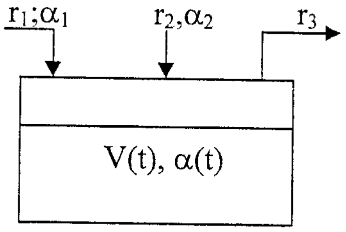
c. Mixing system with 2 inflow channels: rates $r_1;r_2$ [m³/s] and concentrations $\alpha_1;\alpha_2$ [mg/m³], and outflow rate r_3. Write DE and IVP for the amount of chemical $q(t)$, given initial volume V₀ m³ and initial concentration 0.

$$\boxed{V(t) = V_0 + (r_1+r_2-r_3)t} \; ; \quad \alpha(t) = \frac{q(t)}{V(t)}$$



$$\begin{cases} \dot{q} = r_1\alpha_1 + r_2\alpha_2 - r_3\alpha \\ q(0) = 0 \end{cases}$$

$$\begin{cases} \dot{q} + \dfrac{r_3}{V(t)} q = r_1\alpha_1 + r_2\alpha_2 \\ q(0) = 0 \end{cases}$$

$$\dot{\alpha} + \frac{r_1+r_2}{V}\alpha = \frac{r_1\alpha_1 + r_2\alpha_2}{V}$$

Linear 1st DE; multipliers

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d. Write Energy ~~conservation~~ law for pendulum, and Duffing oscillator ($f = x - x^3$). Explain the effect of friction on energy $E(t)$ (its change in time) (**Extra**: phase-plots in 2 cases with or w/o friction)

Pendulum:

$$\begin{cases} \dot{\theta} = v \\ \dot{v} = -\frac{g}{\ell}\sin\theta - \alpha v \end{cases}$$

Total energy:

$$\alpha = 0 \quad \boxed{E = \frac{v^2}{2} - \frac{g}{\ell}\cos\theta - \text{const}}$$

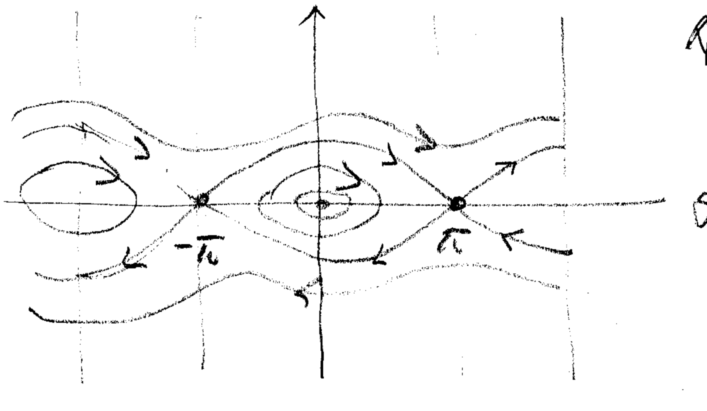
✓

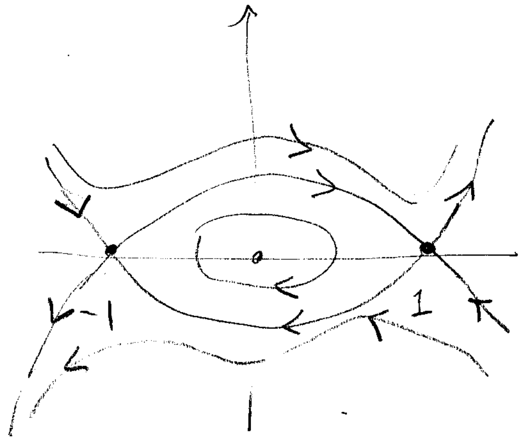
Duffing

Friction $\alpha > 0$ will dissipate Energy: $\frac{dE}{dt} \le 0$

$$\begin{cases} \dot{x} = v \\ \dot{v} = f = x - x^3 - \alpha v \end{cases}$$

$$\boxed{E = \frac{v^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}} - \text{const}$$





2. **Linear oscillator:** $\ddot{x} + d\dot{x} + \frac{9}{4}x = f(t)$.

a. Write characteristic polynomial and find its roots for 3 values of damping below. Find critical value d_c, explain what happens to solutions as d crosses from under-damped to over-damped range. For constant force $f=1$ compute general solution in each case (Hint: use equilibrium x_p and homogeneous $x_h = c_1x_1(t) + \ldots$).

$x_p(t) = 4/9$ - equil.

d	Characteristic polynomial $p(\lambda)$	Characteristic roots $\lambda_{1,2}$	General solution
0	$\lambda^2 + 9/4$	$\pm i\,3/2$	$4/9 + \overbrace{c_1 \cos \frac{3}{2}t + c_2 \sin 3/2\, t}^{x_h}$
1	$\lambda^2 + \lambda + 9/4$	$-\frac{1}{2} \pm i\sqrt{2}$	$4/9 + e^{-t/2}(c_1 \cos\sqrt{2}t + c_2 \sin\sqrt{2}t)$
4	$\lambda^2 + 4\lambda + 9/4$	$-2 \pm \sqrt{7}/2$	$4/9 + c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

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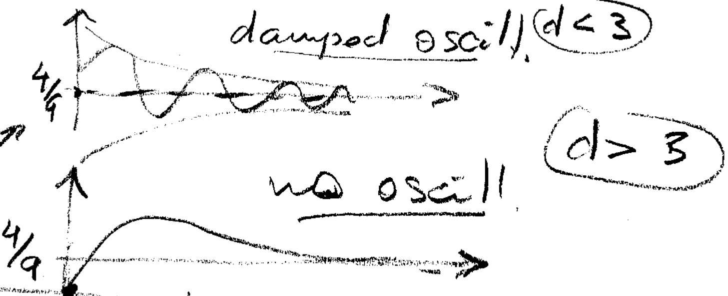
$$\boxed{d_{crit} = 2\sqrt{mk} = 3}$$

Transition from



b. Take periodically forced oscillator: $\ddot{x} + \frac{9}{4}x = f(t) = 1 + .5\cos\omega t$. Write its general solution and IVP-solution: $x(0) = 4/9; \dot{x}(0) = 0$ (Hint: use equilibrium for constant part $f_0 = 1$ and complex exponential for trigonometric part, $\cos\omega t = \mathrm{Re}(e^{i\omega t})$).

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$$x_p(t) = \frac{4}{9} + .5\,\mathrm{Re}\left[\frac{e^{i\omega t}}{9/4 - \omega^2}\right] = \frac{4}{9} + \frac{.5}{9/4 - \omega^2}\cos\omega t$$

$\omega \neq 3/2$

$$x_h = c_1 \cos\frac{3}{2}t + c_2 \sin\frac{3}{2}t$$

$$\text{IVP:} \Rightarrow \begin{cases} \frac{.5}{9/4 - \omega^2} + c_1 = 0 \\ 3/2\, c_2 = 0 \end{cases} \Rightarrow \begin{array}{l} c_1 = -\frac{.5}{9/4 - \omega^2} \\ c_2 = 0 \end{array}$$

$$\rightarrow x(t) = \frac{4}{9} + .5\,\frac{(\cos\omega t - \cos 3/2\, t)}{9/4 - \omega^2}$$

3. Linear DE

a. Solve IVP: $y'-\frac{3}{t}y=1; y(1)=y_0$, using multipliers. Sketch solution for several values: $y_0>1/3;=1/3;<1/3$. **Extra**: Can one take initial value $y(0)=\ldots$, at t= 0 for this DE? Explain.

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Multiplier: $\mu(t)=e^{-\int 3/t\,dt}=e^{-3\ln t}=\boxed{t^{-3}}$

IVP at $t_0=1$: $y(t)=\frac{1}{\mu(1)}\left[y_0+\underbrace{\int_1^t \mu(s)ds}_{-\frac{1}{2}(t^{-2}-1)}\right] \Rightarrow$

$$\boxed{y(t)=t^3\left[\left(y_0+\frac{1}{2}\right)-\frac{1}{2}t^{-2}\right]}=\left(y_0+\frac{1}{2}\right)t^3-t/2$$

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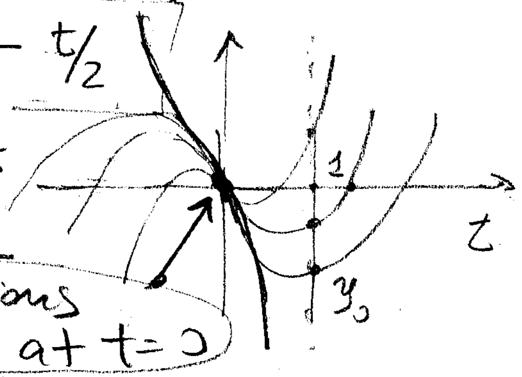
$\boxed{t_0=0}$ is singular p-t for DE



All solutions tangent at t=0

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b. Explain the Laplace transform method, and apply it to IVP: $\boxed{y'+3y=\cos 2t; y(0)=1}$ by (Hint: $\frac{s}{(s+a)(s^2+b^2)}=\frac{1}{a^2+b^2}\left(\frac{b-as}{s^2+b^2}-\frac{a}{s+a}\right)$. Sketch solutions.

$\mathcal{L}$

1^0 $(s+3)Y=1+\frac{s}{s^2+4} \Rightarrow$ $a=3;\ b=2$

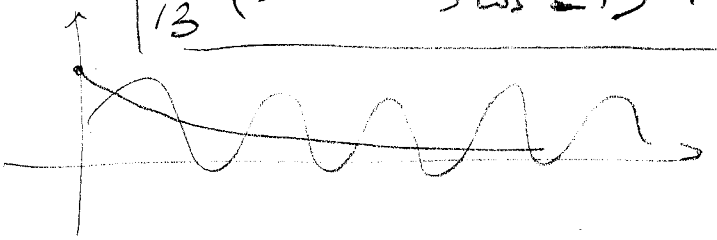
2^0 $Y(s)=\frac{1}{s+3}+\frac{s}{(s+3)(s^2+4)}=\frac{1}{s+3}+\frac{1}{13}\left(\frac{2-3s}{s^2+4}-\frac{3}{s+3}\right)$

$$\boxed{y(t)=\underbrace{e^{-3t}}_{y_h}+\frac{1}{13}\underbrace{\left[\sin 2t-3\cos 2t-3e^{-3t}\right]}_{y_p}}$$

$$\boxed{\frac{1}{13}(\sin 2t-3\cos 2t)+\frac{10}{13}e^{-3t}}$$

1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
e^{at}	$\frac{1}{-a+s}$
Cos[b t]	$\frac{s}{b^2+s^2}$
Sin[b t]	$\frac{b}{b^2+s^2}$
UnitStep[-a + t]	$\frac{e^{-as}}{s}$

Shift: $f(t-a)u_a(t)$	$e^{as}F(s)$
Multiplication: $e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-\frac{d}{ds}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-f_0s-f_1$



c. Explain the *modulation* and *resonance* phenomena. Use solution of part (b) to find resonant frequency ω, and sketch the corresponding solution curves $x(t)$, as ω approaches the natural frequency and ω = "natural frequency".

Solution of (b): $x(t) - \frac{4}{9} = .5 \left(\frac{\cos\omega t - \cos\frac{3}{2}t}{9/4 - \omega^2} \right) =$

$$\boxed{x = \frac{4}{9} + .5 \frac{2\sin\left(\frac{\omega - 3/2}{2}\right)t \quad \sin\left(\frac{\omega + 3/2}{2}\right)t}{(3/2 - \omega) \quad (3/2 + \omega)}}$$

$\omega \approx 3/2$

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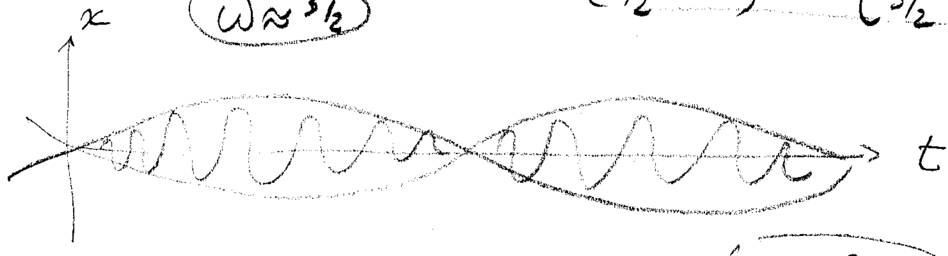
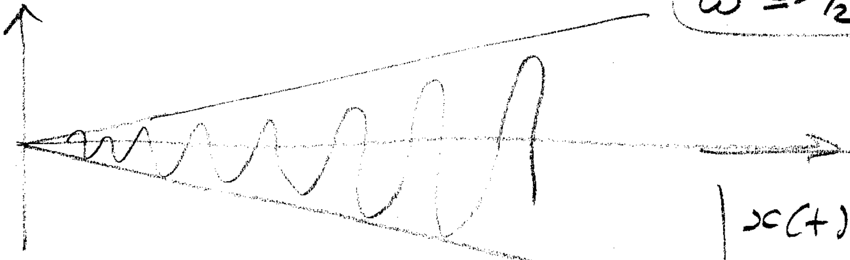


modulation

$\omega = 3/2$ resonance



$$\boxed{x(t) = \frac{4}{9} + .5t \frac{\sin\frac{3}{2}t}{3}}$$

d. Take damping d=1 and force *f*=sin*ωt*. Find the response solution $x_p(t)$, plot its amplitude as function of forcing frequency ω. **Extra**: i) Find the maximal response frequency and its amplitude; ii) Explain whether a damped forced oscillator can resonance as in part (c).

$\ddot{x} + \dot{x} + \frac{9}{4}x = \sin\omega t \Rightarrow$ char: $p(\lambda) = \lambda^2 + \lambda + 9/4$

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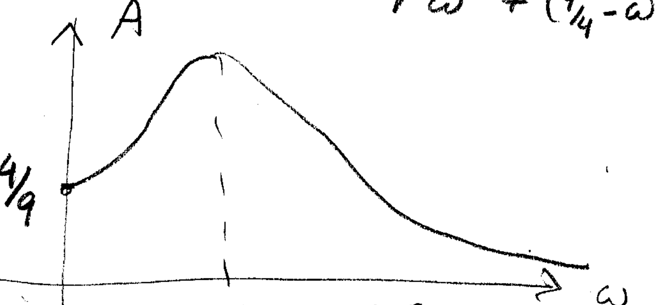
$$x(t) = \mathrm{Im}\left[\frac{e^{i\omega t}}{p(i\omega)}\right] = \mathrm{Im}\left[\frac{e^{i\omega t}}{(9/4 - \omega^2) + i\omega}\right] = A\sin(\omega t - \phi)$$

Amplit: $A = \frac{1}{\sqrt{\omega^2 + (9/4 - \omega^2)^2}}$; Ph. shift: $\phi = \tan^{-1}\left(\frac{\omega}{9/4 - \omega^2}\right)$



$\omega_c = \sqrt{9/4 - 1/2} = \frac{\sqrt{7}}{2}$

No resonance for $d > 0$ as any forcing frequency ω has <u>finite</u> responce amplitude $A(\omega) < \infty$!

$\phi_c = \tan^{-1}(-\sqrt{22})$

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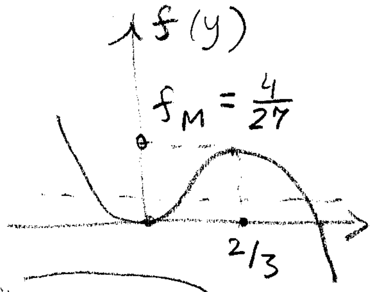
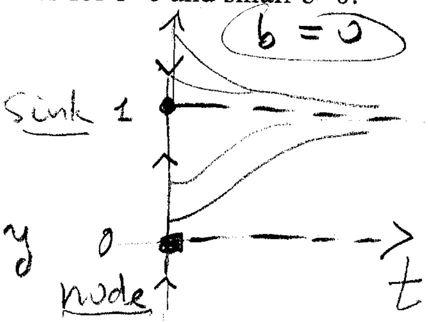
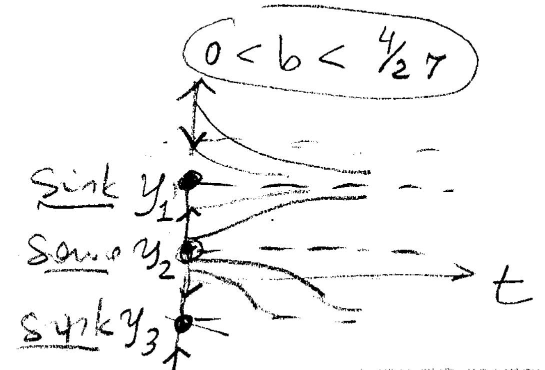
RHS: $\boxed{f(y)} \Rightarrow \boxed{f' = 2y - 3y^2 = 0}$

4. 1D Bifurcations.

a. Modified logistic model: $y' = \boxed{y^2(1-y)} - b$ is harvested at a constant rate b. Sketch equilibria, phase-line and typical solution curves for b=0 and small b>0.







Equil $y_3 < 0 < y_2 < y_1 < 1$

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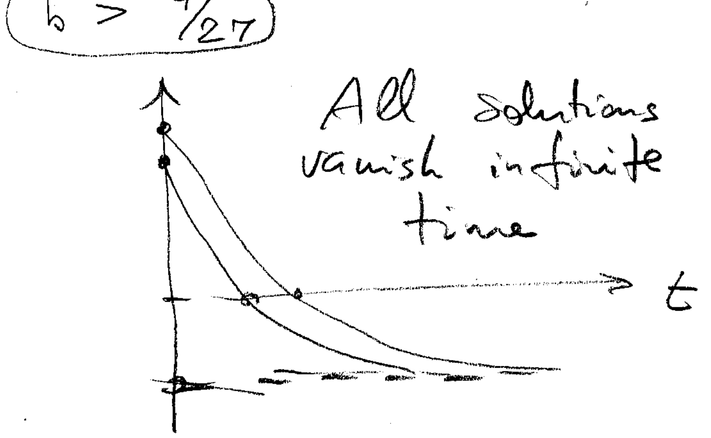
$b > 4/27$



b. Find bifurcation value b, sketch bifurcation diagram and explain which b would allow a sustainable survival level of $y(t)$?

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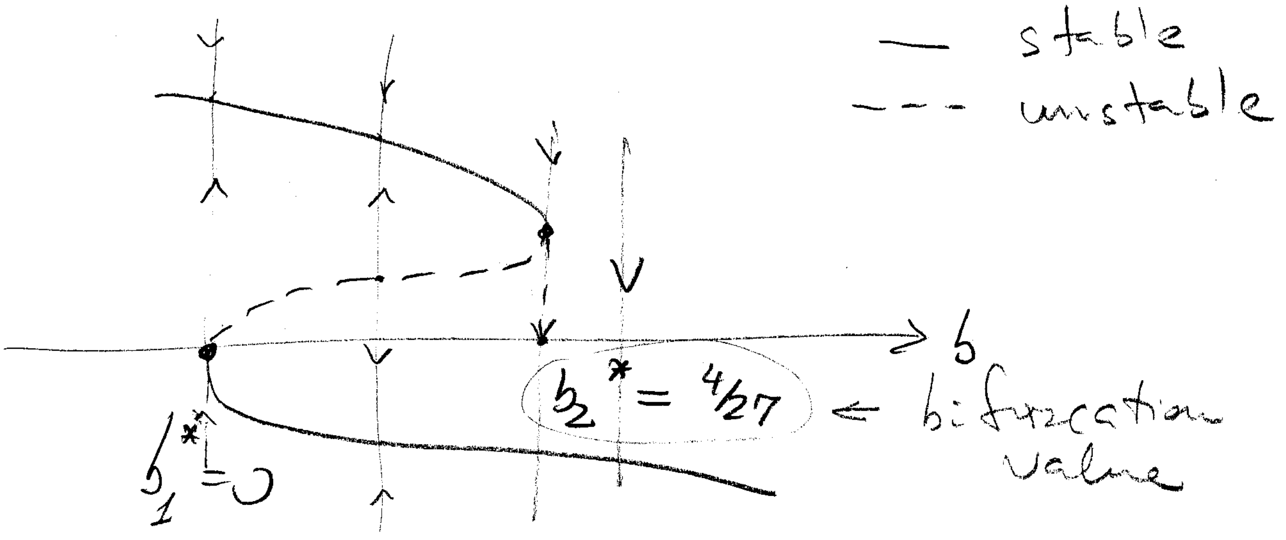

i) For all $0 \le b < \frac{4}{27}$ population is sustained provide $\boxed{y > y_2}$ ← const.

ii) If $b > \frac{4}{27}$, all initial y_0 will die in finite time.

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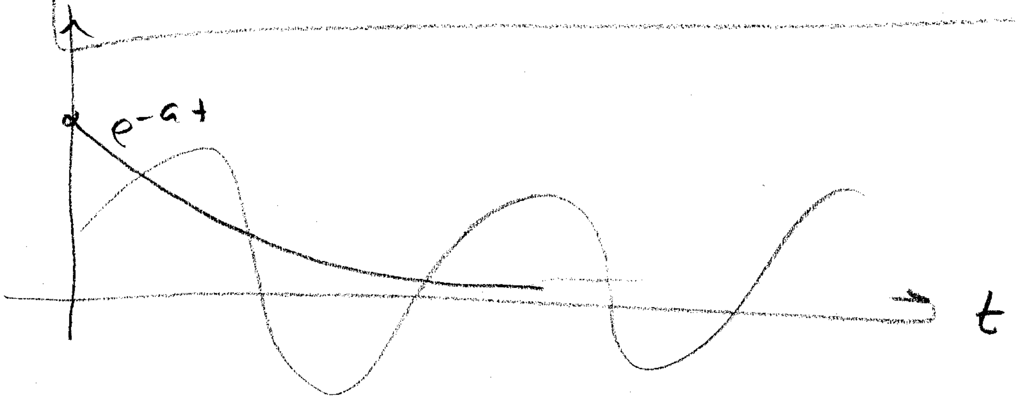
c. Solve IVP: $\begin{array}{l} y''+4y=e^{-at}; \\ y(0)=1; y'(0)=0 \end{array}$ by the Laplace transform method (Hint: $\frac{1}{(s+a)(s^2+4)}=\frac{1}{4+a^2}\left(\frac{1}{s+a}+\frac{a-s}{s^2+4}\right)$. Sketch solutions and explain their behavior at large t.

1° $(s^2+4)\underline{Y} = s + \frac{1}{s+a}$

2° $\underline{Y}(s) = \frac{s}{s^2+4} + \frac{1}{(s+a)(s^2+4)} = \frac{s}{s^2+4} + \frac{1}{4+a^2}\left[\frac{1}{s+a} + \frac{a-s}{s^2+4}\right]$

$$\downarrow$$

3° $$\boxed{y(t) = \underbrace{\cos 2t}_{y_h} + \underbrace{\frac{1}{4+a^2}\left[e^{-at} + \frac{a}{2}\sin 2t - \cos 2t\right]}_{y_p}}$$



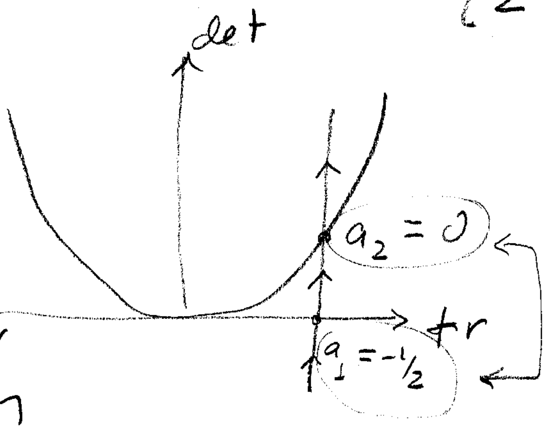
$a < 0$ - cooper.

$a > 0$ - y-competes

5. **Linear systems and bifurcations.** Competition-cooperation model $\begin{array}{l} x' = x - ay \\ y' = 2x + y \end{array}$.

   a. Explain when two species compete or cooperate (in terms of a). Find the bifurcation values of parameter a, and describe the type of bifurcations. Sketch the bifurcation (trace-determinant) diagram

Matrix: $A = \begin{bmatrix} 1 & -a \\ 2 & 1 \end{bmatrix}$; $tr = 2$; $\boxed{det = 1 + 2a}$



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$a < -1/2$	$-\frac{1}{2} < a < 0$	$0 < a$
saddle	source	sp. source

Bifurcations:

$1°$ $det = 0 \Rightarrow a_1 = -1/2$

$2°$ $det = tr^2/4 \Rightarrow a_2 = 0$

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   b. Verify your conclusions in part (c) by taking 2 special values: $a=\pm 2$. Find eigendata, sketch phase plots and write general solution in both cases (Extra: Show that for a=2 species y is driven to extinction in finite time for any initial values (x_0,y_0), in the first quadrant)

$\boxed{a=2}$

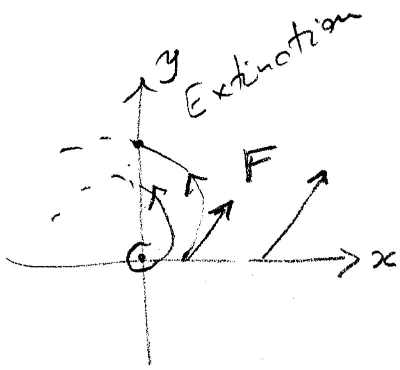
$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

$p = \lambda^2 - 2\lambda + 5$

$\lambda = 1 \pm 2i$

$X = \begin{pmatrix} 1 \\ \mp i \end{pmatrix}$

sp. source



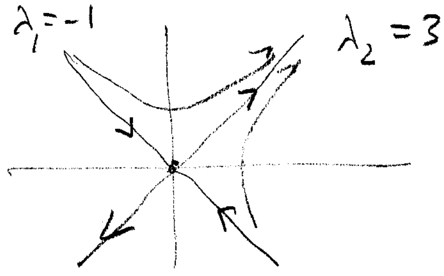
$\boxed{a=-2}$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$p = \lambda^2 - 2\lambda - 3$

$\lambda_{1,2} = -1, 3$

-1	3
$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



6. Nonlinear systems

a. Take modified predator-prey system: $\begin{cases} \dot{x} = x(1 - x/5 - .3y) \\ \dot{y} = y(x-d) \end{cases}$. Explain the meaning of variables *x*,*y* and coefficients *5*,*.3*,*d*. Take two values: (i) d=1 and (ii) d=6; sketch in each case phase plane: null-clines and equilibria (determine their stability-types). Confirm your conclusion by linearized analysis (Jacobians) at equilibria equilibrium in each case. Find the bifurcation values of parameter d. Can two species coexist in case (ii)?

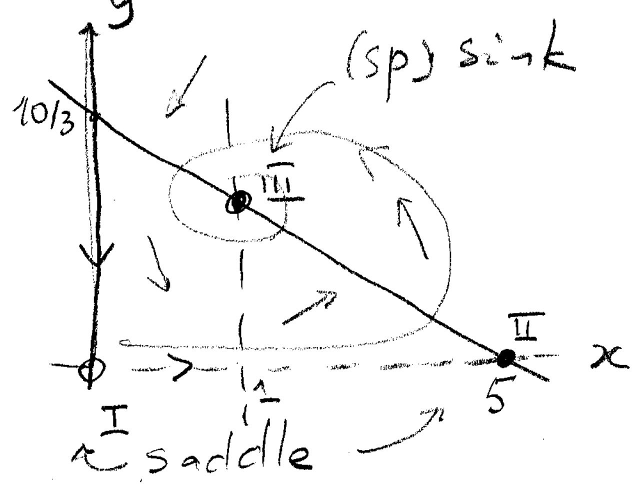
$\left[\begin{array}{l} x - \text{prey} \\ y - \text{predator} \end{array}\right.$

5 = Carrying Capac. of x

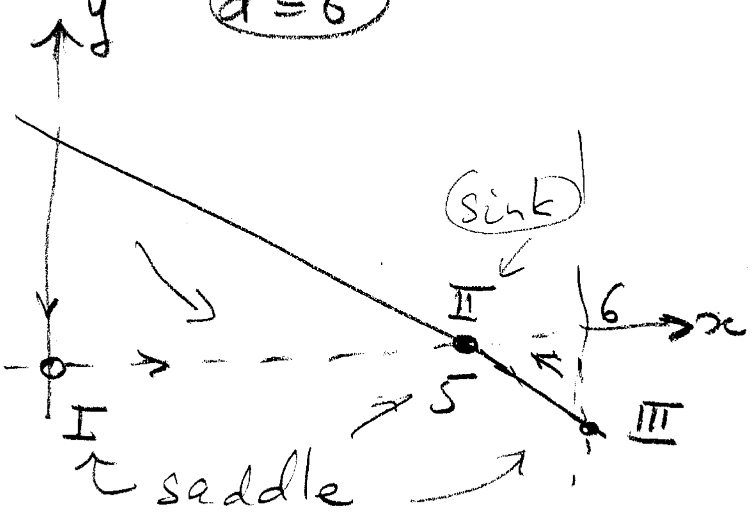
•3 = killing rate / pred × prey

d = pred. attrition (death) rate

$\boxed{d=1}$



$\boxed{d=6}$



As d passes bifurcation value $\boxed{5 = cc.}$ coexist. equil. III moves to non-phys. domain and pair II + III exchange their type (stable/unstable)

Jacobian: $A = \begin{bmatrix} 1 - \frac{2}{5}x - .3y & -.3x \\ y & x - d \end{bmatrix}$

I = (0,0)	II = (5,0)	III = (d, (1-d/5)/.3)
$\begin{bmatrix} 1 & 0 \\ 0 & -d \end{bmatrix}$	$\begin{bmatrix} -1 & -1.5 \\ 0 & 5-d \end{bmatrix}$	$\begin{bmatrix} -d/5 & -.3d \\ (1-d/5)/.3 & 0 \end{bmatrix}$
saddle	saddle → sink	sp sink → sink → saddle

b. Determine which of the following vector fields is Hamiltonian, gradient, dissipative. Find the appropriate function (Hamiltonian, potential, Lyapounov) in each case:

$$\mathbf{F}=\left(y-2,x-y^2\right);\mathbf{G}=\left(v,-\sin x-.5v\right);\mathbf{H}=\left(y,x^2-x\right)$$

Sketch phase portrait: null clines, equilibria, separatrices for any one of 3 (your choice). Verify stability types by the analysis of Jacobian matrices at equilibria.

Potential f-n

Ⓕ $\nabla\cdot F=-2y\neq 0 \rightarrow$ non-Ham.

$f_y-g_x=1-1=0 \rightarrow \boxed{\text{Gradient}}$

$$\boxed{S=-2x+xy-y^3/3}$$

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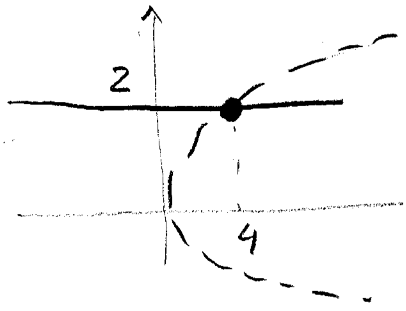

Equil: (4, 2)

$$A=\begin{bmatrix}0 & 1\\ 1 & -2y\end{bmatrix}$$

$$\underbrace{A=\begin{bmatrix}0 & 1\\ 1 & -4\end{bmatrix}}_{\text{saddle}}$$

Ⓖ - damped pendulum $\Rightarrow$ $\boxed{\text{Dissipative}}$ w. Lyapunov f-n $L(x,v)=\frac{v^2}{2}-\cos x \leftarrow$ Energy

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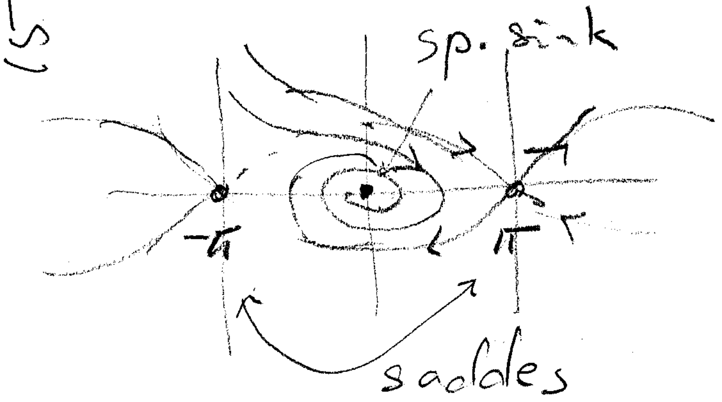

$$A=\begin{bmatrix}0 & 1\\ -\cos & -.5\end{bmatrix}$$

(0,0)	(±π, 0)
$\begin{bmatrix}0 & 1\\ -1 & -.5\end{bmatrix}$	$\begin{bmatrix}0 & 1\\ 1 & -.5\end{bmatrix}$
sp. sink	saddle

Ⓗ $\nabla\cdot H=0 \Rightarrow \boxed{\text{Hamilton}}$ $\begin{cases}h_x=y\\ h_y=-x^2+x\end{cases} \Rightarrow \boxed{h=\frac{y^2}{2}+\frac{x^2}{2}-\frac{x^3}{3}}$

NL oscillator

U - potential

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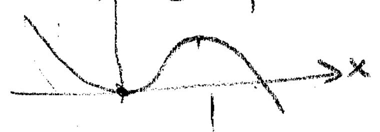
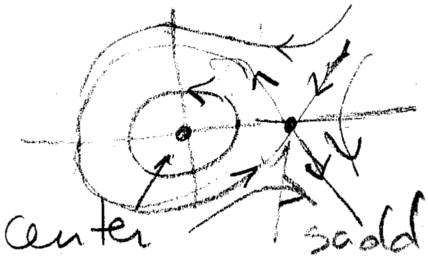


$$A=\begin{bmatrix}0 & 1\\ 2x-1 & 0\end{bmatrix}$$

(0,0)	(1,0)
$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
center	saddle



Enjoy your summer!