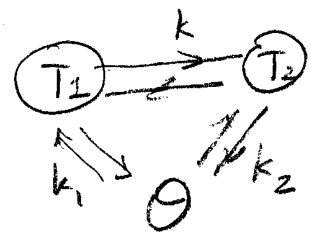
Math. 224: Test I. Feb. 22, 2007

Print Name: ______________________________

1. Set up differential equation models, describe them (DE/DS, first or higher order, linear/nonlinear, autonomous, separable, etc.) and indicate solution method(s) for each one (analytic, numeric).
   a. Two bodies exchange heat with the medium and among themselves according to Newton's law. The exchange coefficients are k (among the bodies) and $k_1;k_2$ (between bodies and the medium). The medium temperature $\theta°$.

$$\begin{cases} \dot{T}_1 = -k_1(T_1-\theta) - k(T_1-T_2) \\ \dot{T}_2 = -k_2(T_2-\theta) - k(T_2-T_1) \end{cases}$$



6 Linear 1st order DS

   b. Volterra-Lotka competition: 2 species x,y grow logistically with growth rates $a_1;a_2$ and carrying capacities $N_1;N_2$ with product-type competition terms with coefficients $b_1;b_2$. Besides, the first species is hunted at a constant rate h.

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$$\begin{cases} \dot{x} = a_1\left(1-\frac{x}{N_1}\right)x - b_1xy - h \\ \dot{y} = a_2\left(1-\frac{y}{N_2}\right)y - b_2xy \end{cases}$$

Nonlinear DS, numeric equilibria

Problem	Score
1(24)	
2(20)	
3(20)	
4(28)	
5(28)	
Total	

c. Oscillator (mass-spring) system of mass m, friction coefficient α, spring constant k, suspended vertically in the external gravity force mg Write it as a 2nd order equation, and convert to a differential system. Extra: find equilibrium position x_0

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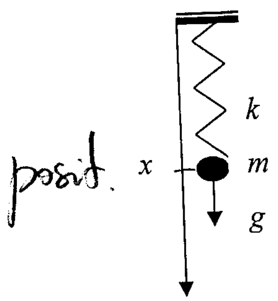
$$m\ddot{x} = \underbrace{-kx + mg}_{f} - \alpha\dot{x}$$

2nd order DE

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{k}{m}x + g - \alpha v \end{cases}$$

1st –"– DS



Equil: $\begin{cases} f(x) = 0 \\ v = 0 \end{cases} \Rightarrow$ $x_0 = \frac{m}{k} g$, $v_0 = 0$ +2

Linear DE/DS, analytic

d. Financing loan model with interest rate α, and payment rate p. Sketch solution curves. Extra: explain the minimal payment rate, find total payment and the overpay factor.

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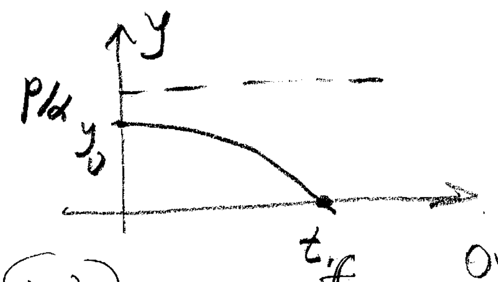
$y(t)$ – dept at time t

Linear DE IVP $\begin{cases} y' = \alpha y - p \\ y(0) = y_0 \end{cases} \Rightarrow$ $y(t) = \frac{p}{\alpha} + \left(y_0 - \frac{p}{\alpha}\right)e^{\alpha t}$



Total pay $= \int_0^{t_f} p\,dt = p t_f$

overpay $A = \frac{\text{Tot. pay}}{y_0} > 1$

(+3)

2. For logistic model $y' = r(1 - y/N)y - b$, explain the meaning of variables and coefficients.

$y(t)$ – populat. r = max per cap. growth rate

3 N = carrying capac. b = harvest

a. Write general solution of DE: $y' = .2(1 - y/10)y$, and IVP-solution for y(0)=3. Plot phase line, equilibria (with marked stable/unstable types) and typical solution curves.

$$y(t) = \frac{y_0 N}{y_0 + (N - y_0)e^{-rt}} = \frac{30}{3 + 7e^{-.2t}}$$

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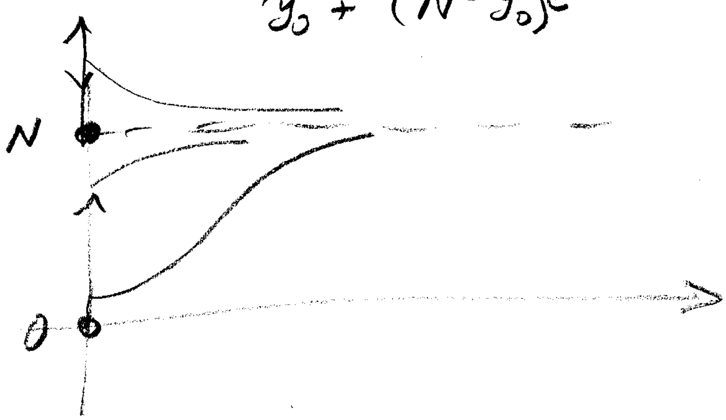


10 b. Write linearized equations (LDE) at 2 equilibria of the logistic model (a) with added harvesting function *b(t)*. Take specific $b(t) = 5e^{-2t}$, and solve the resulting LDE to find approximate solution $y(t) \approx$ ... (Hint: use undetermined-coefficient method for $u' + mu = be^{-ct}$).

Equil.	slope $f' = r(1 - \frac{2y}{N})$	Lin. DE for $u = y - y_0$
1) 0	$r = .2$	$u' = ru - b(t)$
2) $N = 10$	$r = -.2$	$u' = -ru - b(t)$

Solution: $\boxed{u' - ru = -be^{-ct}} \Rightarrow u(t) = \underbrace{Ae^{-ct}}_{u_p} + \underbrace{(y_0 - A)e^{rt}}_{u_h}$?

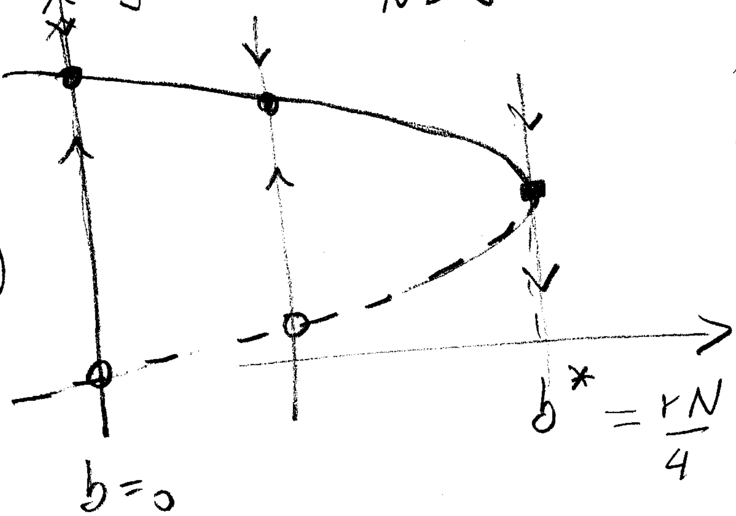
To find A, substitute u_p $\Rightarrow$ $\boxed{A = \frac{b}{r+c}}$

$\Rightarrow$ 1) $y(t) \approx u(t) = \frac{5}{2.2}e^{-2t} + \left(y_0 - \frac{5}{2.2}\right)e^{.2t}$

2) $y(t) \approx 10 + u(t) = -5/1.8\, e^{-2t} + (y_0 + 5/1.8)\, e^{-.2t}$

3. Discuss **bifurcation problem** for logistic equation 2(a) harvested at a constant rate b. Find critical (bifurcation) value(s) b^* and sketch the bifurcation diagram (with marked equilibria types).

$y' = r(1-\frac{y}{N})y - b \Rightarrow$ Bifurc. value $\boxed{b^* = \frac{rN}{4}}$



• – stable —

o – unstable - - -

saddle-node

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$b^* = \frac{rN}{4} = \frac{.2\cdot 10}{4} = .5$

a. Take 3 values: $b_1 = .3; b_2 = b^*; b_3 = 1$ and sketch phase-lines with equilibria and typical solutions $y(t)$. Is b=.3 a sustainable harvest level (explain)?

$b_1 = .3 < b^*$ $\quad$ $b_2 = b^* = .5$ $\quad$ $b_3 = 1$

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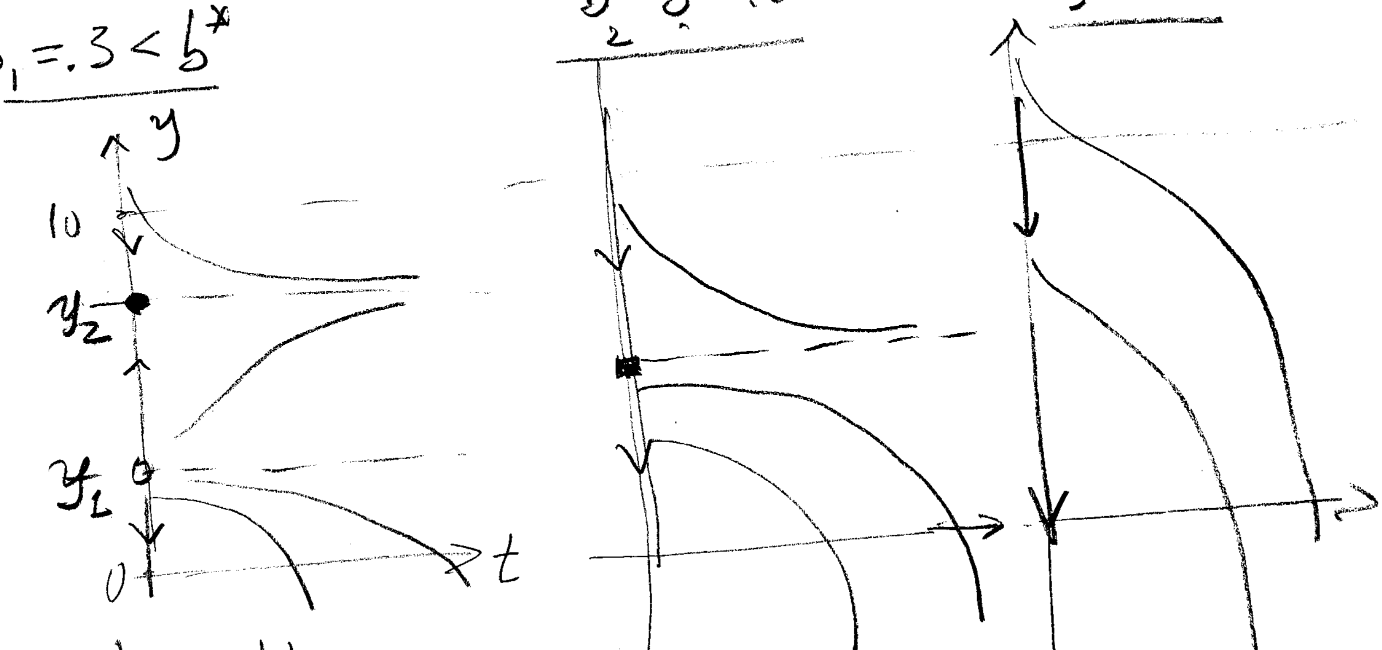

sustainable

above y_1

4. Solution methods

a. Solve IVP: $y' = 3y^{2/3}; y(0) = 1$. Explain its relation to a spherical droplet. Plot solutions. Explain non-uniqueness of solutions with initial value $y(t_0) = 0$.

$r(t)$ – radius; $A(t) = 4\pi r^2$ – area

$y = V(t) = \frac{4}{3}\pi r^3$ – volume

$$\frac{dV}{dt} = \alpha A \Rightarrow \frac{dV}{dt} = \alpha\left(\frac{3}{4\pi}V\right)^{2/3} = \alpha_1 V^{2/3}$$

$$\int_{y_0}^{y} \frac{dy}{y^{2/3}} = 3t \Rightarrow y(t) = \left(t + y_0^{1/3}\right)^3$$

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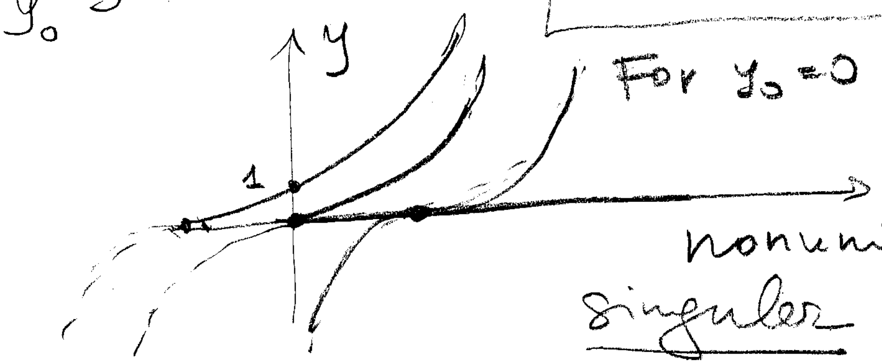

For $y_0 = 0 \Rightarrow y(t) = \begin{cases} 0; \\ t^3 \end{cases}$ –

nonunique as $y = 0$ is singular $f_y = \frac{2}{3}y^{-1/3}$ - singul. at $y = 0$!

b. Write general solutions of DEs: (i) $y' + ty = a$; (ii) $y' + ay = b\cos 2t$

i) multiplier: $\mu = e^{\int t\,dt} = e^{t^2/2}$

$$y(t) = e^{-t^2/2}\left[c + a\int_0^t e^{s^2/2}\,ds\right]$$

Erfi(t)

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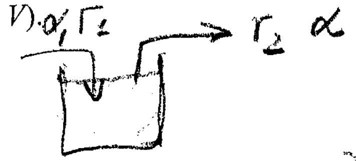
ii) Undet. coeff: $y_p = A\cos 2t + B\sin 2t$? ?

$$\Rightarrow A = \frac{ab}{4+a^2}; \quad B = \frac{2b}{4+a^2}$$

$$y = b\,\frac{a\cos 2t + 2\sin 2t}{4+a^2} + c_1 e^{-at}$$

c. Mixing problem with incoming rate $r_1 = .8m^3/s$ and concentration $\alpha_1 = 3gr/m^3$; outgoing rate $r_2 = 1.3m^3/s$; initial volume $V_0 = 10m^3$, and amount of chemical $q_0 = 0$. Write DE/IVP for function $q(t)$, solve it and sketch solutions. Explain what happens to $q(t)$ and $\alpha(t)$ as volume approaches 0 (Hint: handout formulae $q = \alpha_1 V + CV^{-\beta}; \beta = \frac{r_2}{r_1 - r_2}$, for volume-function V).



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$$\begin{cases} q' + \frac{r_2}{V} q = r_1 \alpha_1 \\ q(0) = 0 \end{cases}$$

$$V = V_0 + (r_1 - r_2)t$$

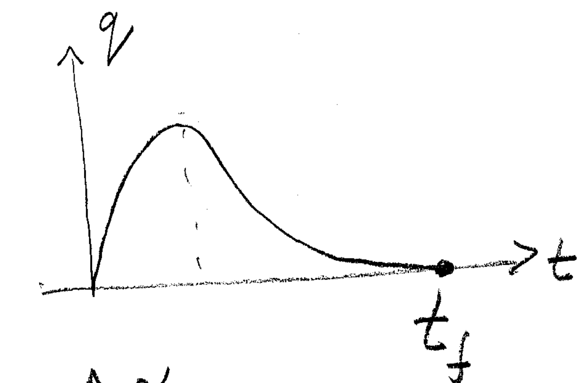
$$\beta = \frac{r_2}{r_1 - r_2} = \frac{1.3}{-.5} = -2.6$$

$$\Rightarrow \quad q(t) = \alpha_1 V + C V^{-\beta}$$

Const $\quad C = V_0^{\beta}(q_0 - \alpha_1 V_0) = 10^{-2.6}(-30) = -3 \cdot 10^{-1.6}$

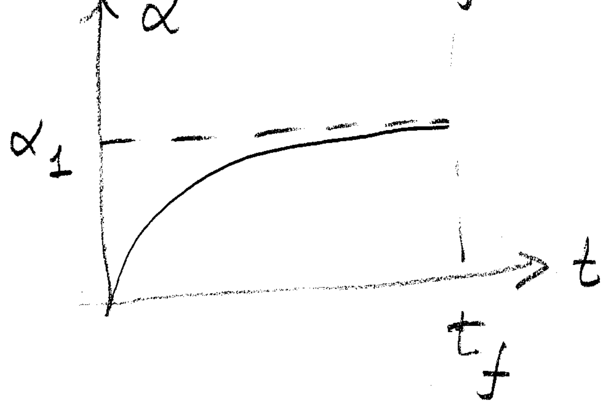
$$V(t) = 10 - .5t$$

Solution blows up at $t_f = 20$



$$\alpha(t) = \frac{q(t)}{V(t)} = \alpha_1 + C V^{-\beta - 1}$$

$$\alpha(t) = 3 - 3\left(\frac{V}{10}\right)^{1.6}$$



5. **Dynamical systems:** phase-plane, equilibria

a. Identify each plot below with a suitable dynamical system (vector-field) F from the following list, and describe the corresponding physical model (predator-prey, pendulum, damped oscillator, competing species):

1) $\begin{pmatrix} v \\ -3x-.1v \end{pmatrix}$ 2) $\begin{pmatrix} .4x-.01xy \\ .05xy-.3y \end{pmatrix}$ 3) $\begin{pmatrix} x(1-x-y) \\ y(.6-x-.36y) \end{pmatrix}$ 4) $\begin{pmatrix} v \\ -\sin x \end{pmatrix}$

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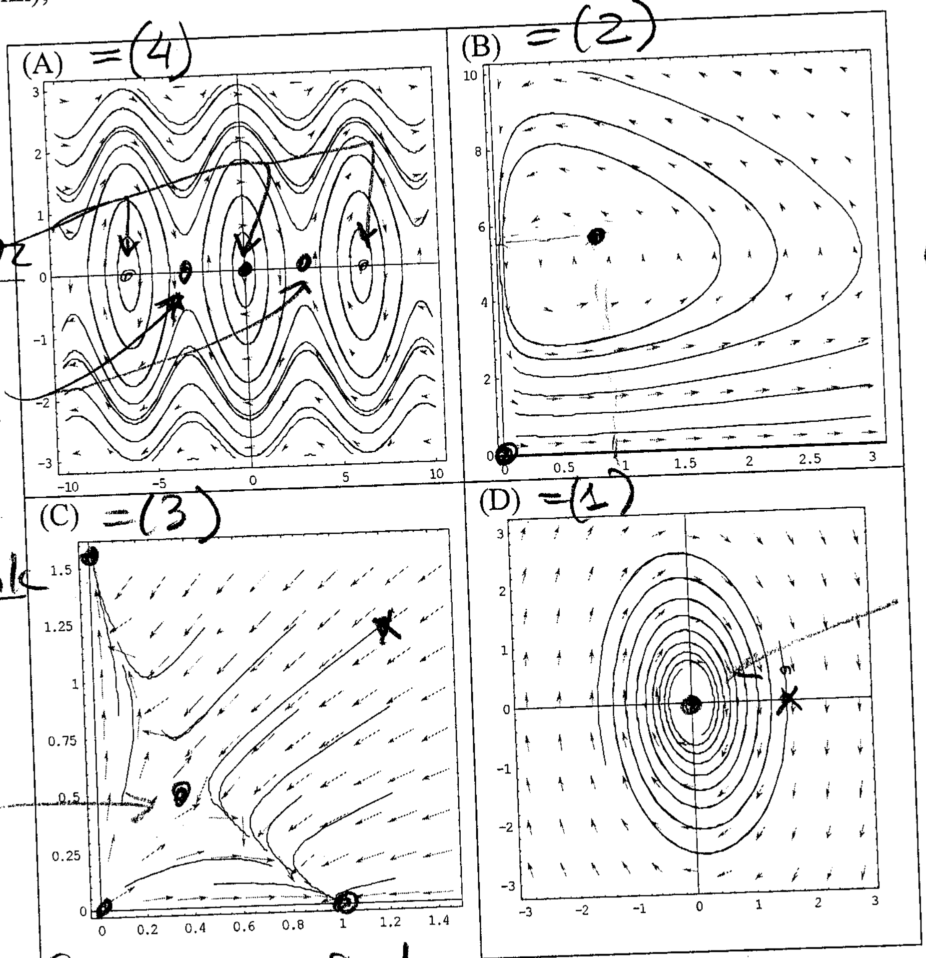
(D) damped osc. (B) pred. prey (C) comp. spec. (A) pendul.

b. Locate equilibria on each plot and describe their type (center, saddle, source, sink, spiral source/sink), based on vector field and trajectories.



equil: (0,0) - center; $(\pm\pi, 0)$ - saddle

equil: (0,0) - saddle; (1, 5.8) - center

Sink; Saddle; Source; Sink

(0,0) spiral sink

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c. Compute equilibria of system (B) from its algebraic equations

$$\begin{cases} x(.4 - .01y) = 0 \\ y(.05x - .3) = 0 \end{cases} \Rightarrow$$

x	y
(0, 0)	
(6, 40)	

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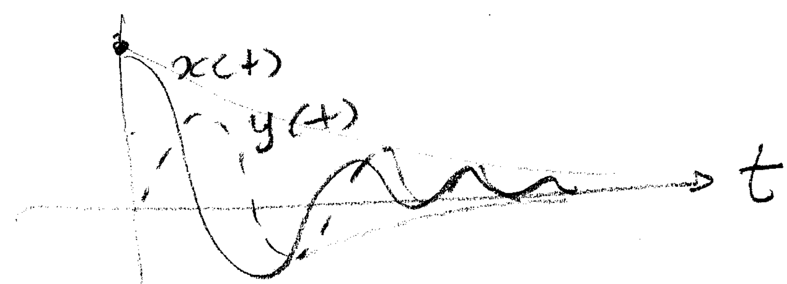
d. Pick a particular trajectory shown on plots (D) and (C) sketch its solution-curves: x(t) (solid),y(t) (dashed). Describe what happens to solutions at large time t

(D)



(C)

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