Math. 224. Test 1

FALL 2002. OCT. 3

David Gurarie   Name ____________________

1. **Modeling.** Set up differential equation model of the following systems. Classify them as autonomous/non-autonomous DE/DS, first or higher order, linear/nonlinear, separable. Indicate solution method(s) for each one (either analytic or numeric), but **do not solve!**

   (a) DE model for two bodies exchanging heat with the medium and among themselves according to the Newton's law. The exchange coefficients are k (among the bodies) and k_1, k_2 (between a body and the medium). The medium has temperature $\theta = 60°K$.

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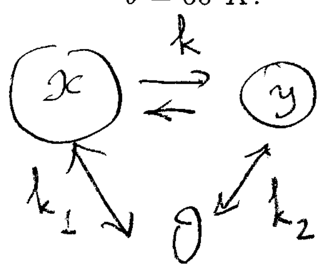
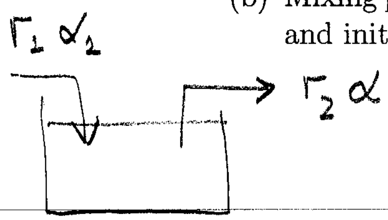


x, y – temperatures

$$\begin{cases} \dot{x} = -k\,(x-y) - k_1\,(x-\theta) \\ \dot{y} = -k\,(y-x) - k_2\,(y-\theta) \end{cases}$$

Autonomous linear DS

   (b) Mixing problem with incoming rate $r_1 = 3$, concentration α_1, outgoing rate $r_2 = 1$, and initial volume $V_0 = 20$



Volume: $V(t) = V_0 + (3-1)t = 20 + 2t$

Amount: $q' + \frac{r_2}{V}\,q = r_1\alpha_1$; $\boxed{q' + \frac{1}{20+2t}\,q = 3\alpha_1}$

8 Linear DE solved by multipliers:

$$\mu(t) = e^{\int \frac{r_2}{V}\,dt} = V^{r_2/r_1 - r_2}$$

$$\boxed{\mu(t) = (20+2t)^{1/2}}$$

Problem	Score
1(25)	
2(25)	
3(30)	
4(20)	
Total	

Math. 224. Test 1

FALL 2001. OCT. 4

David Gurarie

Name ______________________________

1. **Modeling.** Write differential equations or systems for any 4 of the following models. Identify them as DE/DS, autonomous/non-autonomous, first or higher order, linear/nonlinear, separable. Indicate solution methods for each one (analytic, numeric etc.), but **do not solve!**

   (a) A cooling body in the room temperature $\theta = 45$, initial temperature $T_0 = 100$, and the cooling rate α. Extra: explain how to find α from two temperature measurements $T_1 > T_2$ taken 3 hrs apart.

$$\begin{cases} T' = -\alpha(T-\theta); \ \theta = 45 \\ T(0) = 100 \end{cases}$$

Lin. DE; sep.

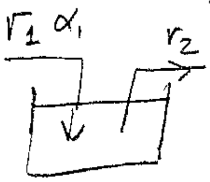
$$\alpha = \frac{1}{3} \ln\left(\frac{T_1 - \theta}{T_2 - \theta}\right)$$

   (b) The logistic growth model with the growth rate α, carrying capacity $N = 20$, initial population y_0, and harvesting rate $a(t)$. Which a allow analytic solutions?

$$\begin{cases} y' = \alpha y(1 - y/20) - a(t); \\ y(0) = y_0 \end{cases}$$

Non Lin DE; sep. for a - const

   (c) Mixing problem with incoming rate $r_1 = 1.4$, concentration α_1, and outgoing rate $r_2 = .7$.



$$V = V_0 + (r_1 - r_2)t = V_0 + .7t$$

Lin. DE; multipl.

$$\boxed{q' + \frac{r_2}{V} q = r_1 \alpha_1}$$

$$q' + \frac{.7}{V_0 + .7t} q = 1.4\alpha_1$$

Problem	Score
1(25)	
2(25)	
3(25)	
4(25)	
Total	

2. **Rocket science**. As rocket burns fuel its mass $m(t)$ decreases according to the burning strategy (the handout considered constant burning rate). Hence the Newton equation in constant gravity: $\boxed{\frac{d}{dt}(mv) = -U\dot{m} - gm}$ where U is the nozzle velocity, $\dot{m}$ - the burning rate

Mass

(i) $m = m_0(1-\alpha t)$; α - burning rate as a fraction of m_0

$\dot{m} = -m_0\alpha \implies$

$$\boxed{\frac{d}{dt}\left[(1-\alpha t)v\right] = \alpha U - g(1-\alpha t)}$$

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(a) Write specific rocket equations for two burning strategies (i) constant burning rate α relative to the total initial mass m_0; (ii) burning fuel at a rate proportional to the available fuel (Hint: mass $m = m_C + m_F(t)$ - "cargo plus fuel", compute m_F).

(ii) Mass $m(t) = m_c + m_F e^{-\alpha t}$; m_F - initial fuel mass

$\dot{m} = -\alpha m_F e^{-\alpha t}$; α - relative burning rate

$$\boxed{\frac{d}{dt}\left[(m_c + m_F e^{-\alpha t})v\right] = U\alpha m_F e^{-\alpha t} - g(m_C + m_F e^{-\alpha t})}$$

or $\frac{d}{dt}\left[(b+e^{-\alpha t})v\right] = U\alpha e^{-\alpha t} - g(b+e^{-\alpha t})$ $\quad b = \frac{m_C}{m_F}$

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(b) Describe the type of equation, and solution method. Write solution $v(t)$ in the first case (**extra credit** for second case);

Multiplier $\mu = m(t)$ or direct. integration of r.h.s

(i) $(1-\alpha t)v = (\alpha U - g)t + g\alpha t^2/2 \implies$

$$\boxed{v(t) = \frac{(\alpha U - g)t + g\alpha t^2/2}{1-\alpha t}}$$

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(ii) $(b+e^{-\alpha t})v = (U\alpha - g)\frac{1-e^{-\alpha t}}{\alpha} - bgt$

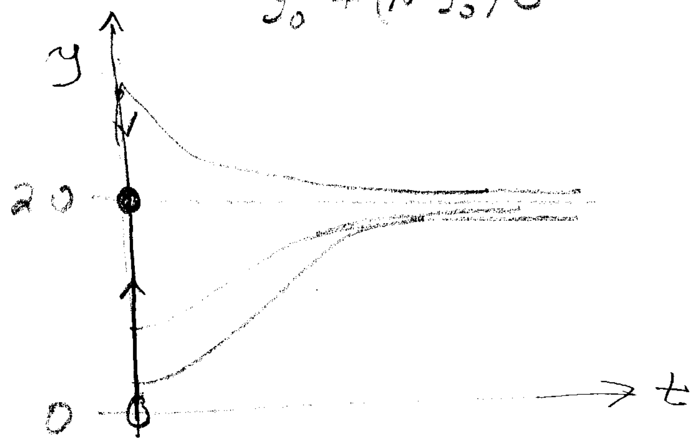
$$\boxed{v(t) = \frac{(U - g/\alpha)(1-e^{-\alpha t}) - bgt}{b + e^{-\alpha t}}}$$

GS: $y(t) = \frac{N}{1+Ce^{-\alpha t}}$

3. **Logistic equation with harvest**

(a) Write general solution of the logistic model: $y' = y(1 - y/20)$. Find IVP-solution $y(0) = 3$, plot phase line, equilibria and typical solution curves.

(IVP) $y(t) = \frac{N y_0}{y_0 + (N - y_0) e^{-\alpha t}} = \frac{60}{3 + 17e^{-t}}$;



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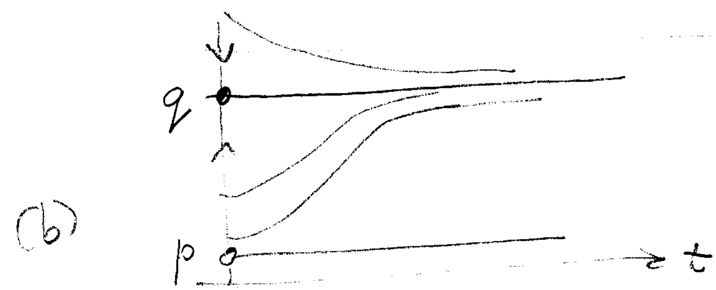

(b) Same problem for logistic population (a) harvested at a constant rate $b = 2$ (Hint: find two equilibria $p < q$ of $f(y) - b = 0$; use solution formula: $y' = \alpha(y-p)(q-y) \Rightarrow y = \frac{q + pCe^{-\alpha(q-p)t}}{1 + Ce^{-\alpha(q-p)t}}$)

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$y' = y(1 - y/20) - 2 = \frac{1}{20}(q-y)(y-p)$;

Roots: $f(y) - b = -\frac{1}{20}(y^2 - 20y + 40) = 0 \Rightarrow \begin{cases} p = 10 - \sqrt{60} \approx 2.25 \\ q = 10 + \sqrt{60} \approx 17.75 \end{cases}$

Solution: $\alpha = \frac{1}{20} = .05$; $p, q = - -$; $\alpha(q-p) = .08$

$$\boxed{y(t) = \frac{17.8 + C\, 2.2\, e^{-.08t}}{1 + Ce^{-.08t}}}$$ GS

Solve for IVP: $y_0 = \frac{q + Cp}{1 + C} \Rightarrow C = \frac{q - 1}{y_0 - p} = \frac{16.75}{.75}$

$\boxed{C \approx 22.3}$

IVP $y(t) = \frac{17.8 + 49e^{-.08t}}{1 + 22.3e^{-.08t}}$

(c) Consider periodic harvesting $b(t) = b\cos(t)$, with small b. Find linearized equations near equilibria, and solve them (Hint: particular solution of DE $u' + mu = b\cos t$, is $u_p = b\frac{m\cos t + \sin t}{1+m^2}$).

Linearized: $m = f'(0) = 1$; $m = f'(20) = -1$

$u = y(t) - y_0$

$\boxed{0}$ $u' = u + b\cos t \Rightarrow u(t) = b\,\frac{-\cos t + \sin t}{2} + ce^{t}$

$\boxed{20}$ $u' = -u + b\cos t \Rightarrow u(t) = b\left(\frac{\cos t + \sin t}{2}\right) + Ce^{-t}$

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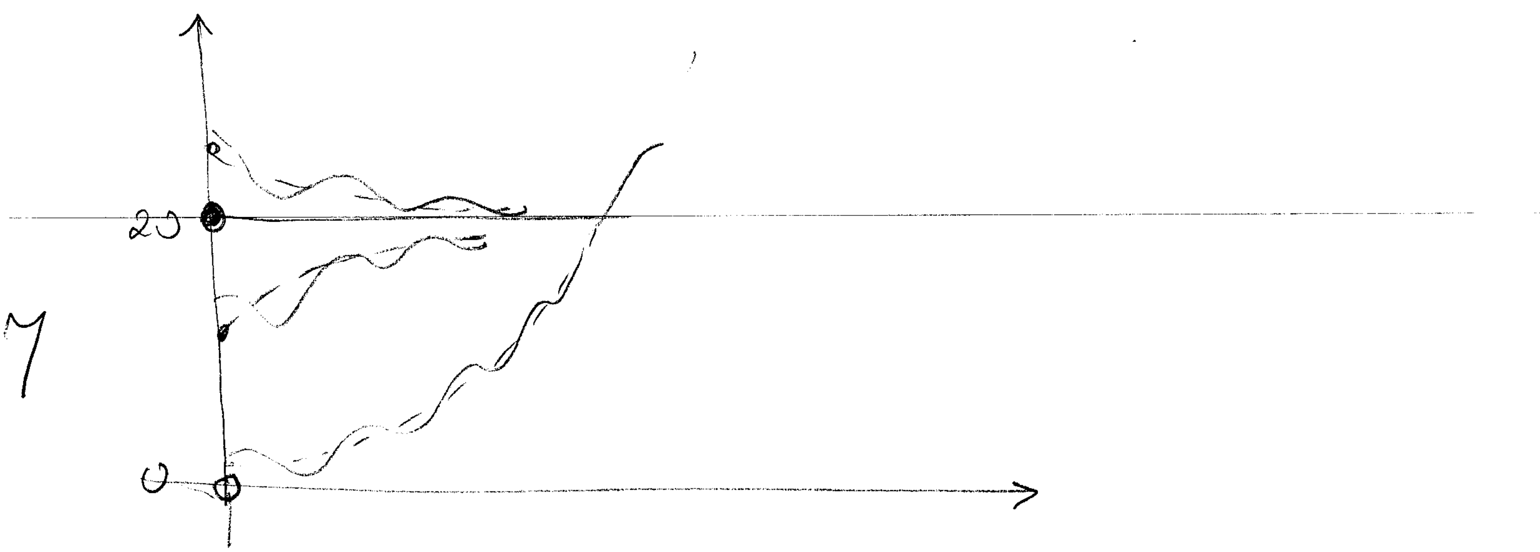
(d) Plot linearized solutions and discuss the validity of such approximations for the nonlinear system.



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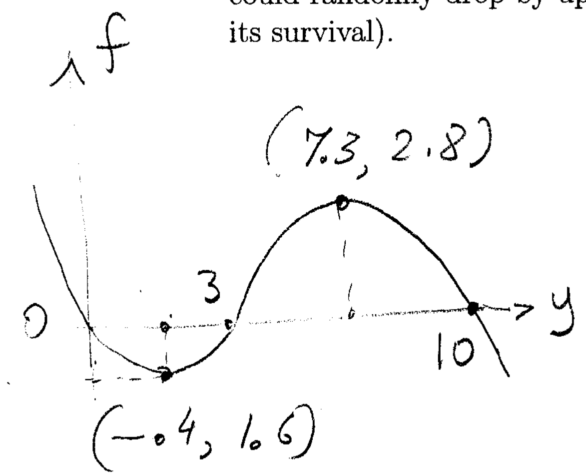
* For stable equil. $y=20$, lin. solutions approx. exact $y(t)$
* For unstable $y=0$, diverge from $y(t)$

20

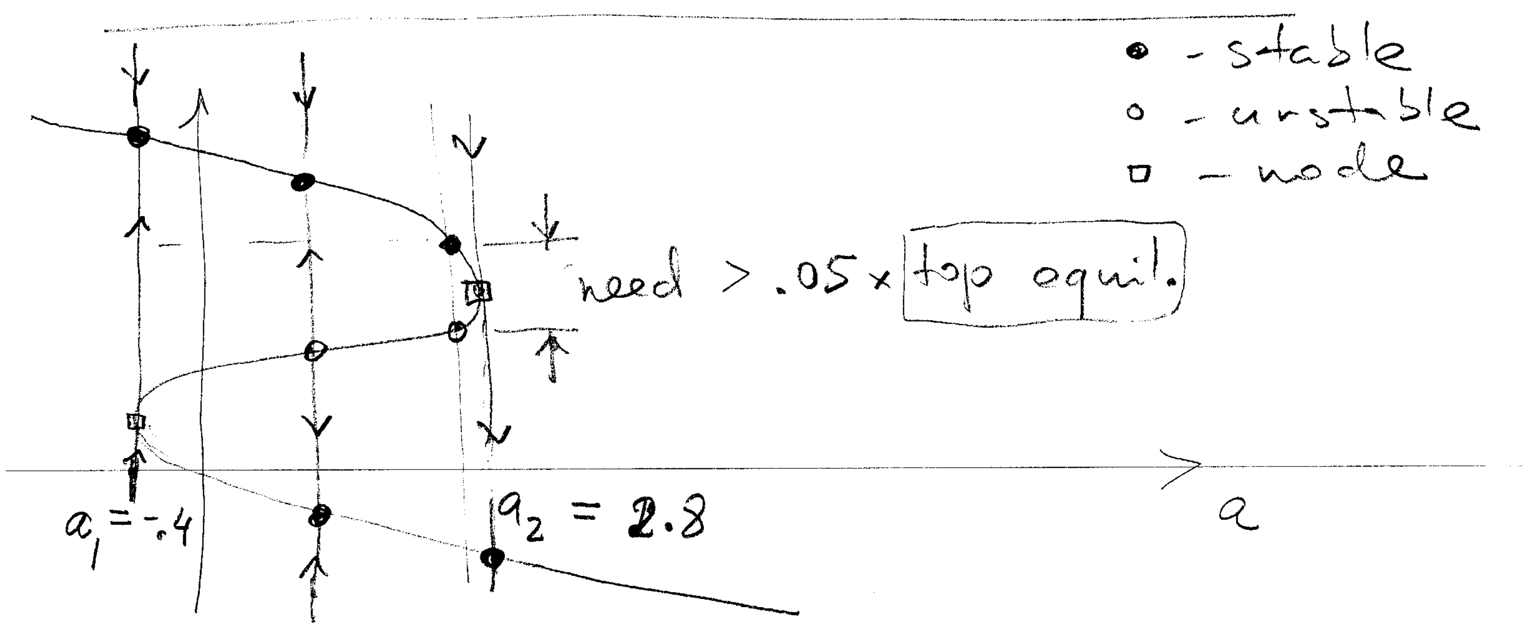
4. **Equilibria and bifurcations.**

(a) Consider cubic logistic equation $y' = y\left(\frac{y}{3} - 1\right)\left(1 - \frac{y}{10}\right)$ with constant harvest rate a. Describe the effect of a on equilibria and sustainability, find value a that ensures survival. Sketch the bifurcation diagram.(Extra: given that population could randomly drop by up to 5% due to illness, what harvest rate would ensure its survival).



Bifurc. = critical values f

$$f' = -1 + \frac{13}{15}y - \frac{y^2}{10} = 0$$



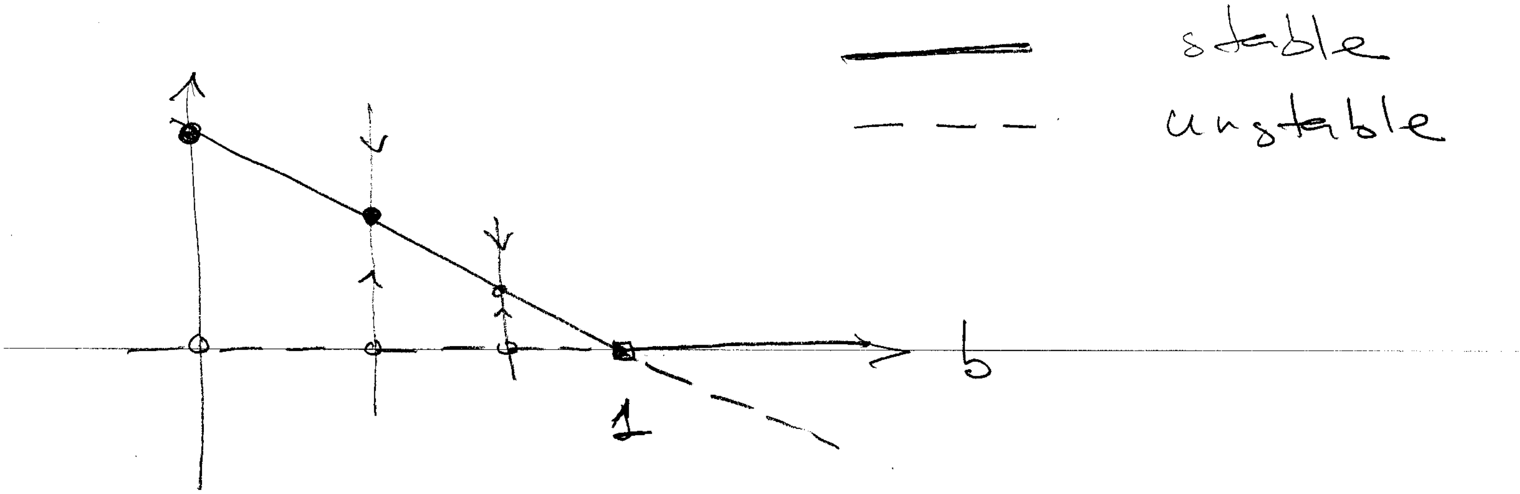
(b) Take standard (quadratic) logistic model $f = y\left(1 - \frac{y}{10}\right)$ harvested as a fraction of population by. Find bifurcation value b^* and sketch bifurcation diagram. Plot phase-lines and solution curves in 3 cases: (i) $b_1 < b^*$, (ii) $b_2 = b^*$ and (iii) $b_3 > b^*$.

20 $$y' = y(1 - y/10) - by = y\left[(1-b) - \frac{y}{10}\right] = (1-b)\,y\left(1 - \underbrace{\frac{y}{(1-b)10}}\right)$$

Change carrying $N(b)$

Bifurcations: $\begin{cases} f = 0 \\ f_y = 0 \end{cases} \Rightarrow$

$$\begin{cases} (1-b)\,y\left(1 - \frac{y}{N(b)}\right) = 0 \\ (1-b) - \frac{2}{N}\,y = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} y = 0 \\ b_{cr} = 1 \end{array}}$$



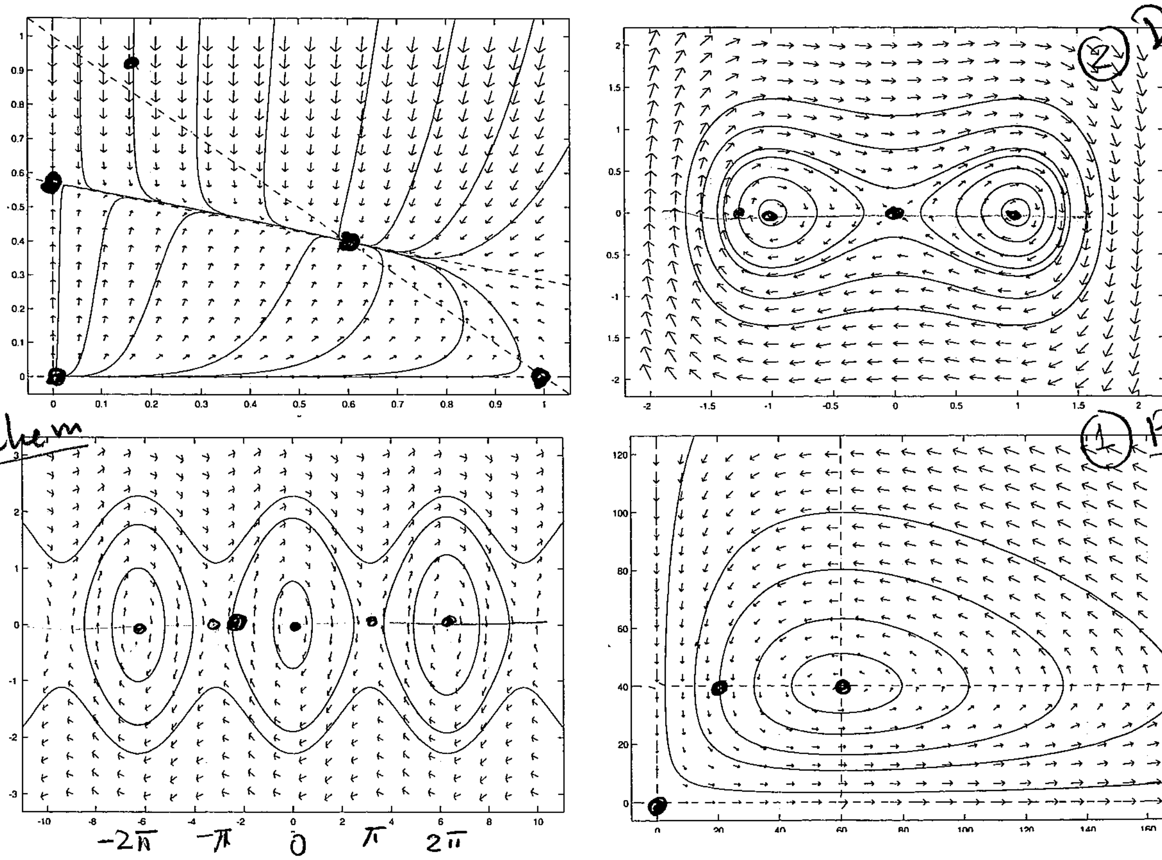
4. **Dynamical systems; phase-plots; equilibria**

(a) Identify each plot with a suitable dynamical system (vector-field) (f, g) from the following list, and describe its physical model (predator-prey; oscillator; pendulum; competing species; Duffing; etc.).

1) $\begin{pmatrix} .4x - .01xy \\ .005xy - .3y \end{pmatrix}$	2) $\begin{pmatrix} y \\ x - x^3 \end{pmatrix}$	3) $\begin{pmatrix} x(1 - x - y) \\ y(4 - 2y - 7x) \end{pmatrix}$	4) $\begin{pmatrix} y \\ -\sin x \end{pmatrix}$

(3) comp. species

(2) Duff



(4) pendulum

(1) Pred pre

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(b) Locate equilibria on each plot, and compute them for any two of four systems.

(1) pred-prey

$(0,0)$; $\left(\frac{.3}{.005}, \frac{.4}{.01}\right) = (60,40)$

(2) Duffing

$(0,0)$ $(\pm 1, 0)$

(3) Comp. spec.

$(0,0)$

$(1,0)$

$(0,2)$

$(. \; .)$

(4) pendulum: $\{(\pi k, 0) : k = 0, \pm 1, \ldots\}$

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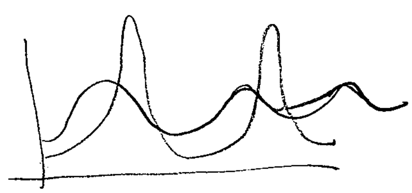
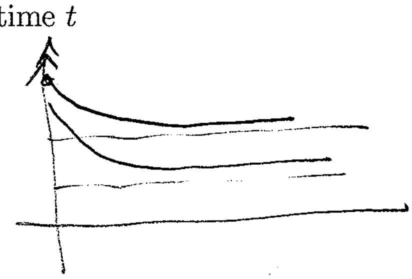
(c) Pick a particular (marked) trajectory for any two of the phase-plots (your choice) and sketch its solution-curves $x(t), y(t)$ (solid/ dashed). Describe in words what happens to solutions at large time t

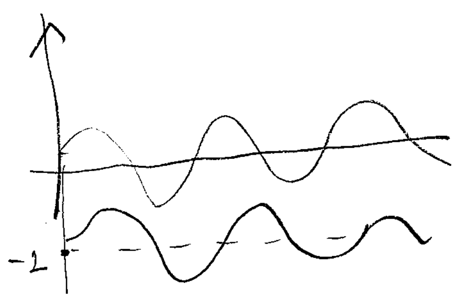
(1)



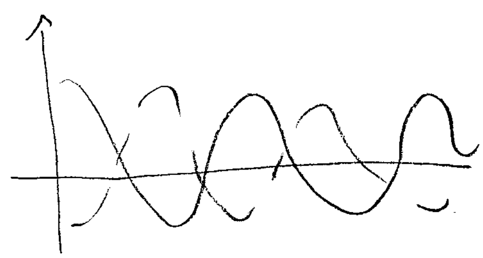
(3)



(2)



(4)



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