Math. 224. Review problems 2

SPRING 02

David Gurarie Name ______________________________

Topics

- **Linear systems**: oscillators, sliding chain, competition-cooperation, migration, buyer-seller markets; boundary value problems (equilibrium heat equation)
- Fundamental pair, general solution, IVP-solution, BVP-solutions
- Phase-plane, equilibria
- **Eigenvalue method** (real, complex, repeated)
- Applications (damped oscillators, migration et al.)
- Bifurcations: trace-det plane
- **Linear equations**: Method of characteristic polynomial for 2nd (and higher) order DE's
- **Forced oscillations**: characteristic potential and undetermined coefficients. **Periodic forcing**: modulation and resonance, damping (responce amplitude and phase)
- **Nonlinear systems:** phase plane, equilibria, linearizarion (Jacobian matrix)
- Hamiltonian and conservative systems: energy conservation, equilibria, trajectories
- **Specific models**: (i) nonlinear oscillators and pendulum; (ii) predator-prey; (iii) competition
- **Bifurcations**: (i) effect of friction, (ii) analysis of competition

Sample problems

1. **Modeling**

   (a) **Competition-cooperation:** show that competing/cooperating species $\{x, y\}$ with the natural growth rates a, b, and interaction coefficients $\{b, c\}$ obey a differential system with matrix $A = \begin{bmatrix} a & \pm b \\ \pm c & d \end{bmatrix}$. (i) Explain the meaning of $\pm$ sign in terms of competition-cooperation (ii) Show that competition/ cooperation matrix A has real eigenvalues. (iii) What happens to eigenvalues of A when b has positive sign and c negative ?

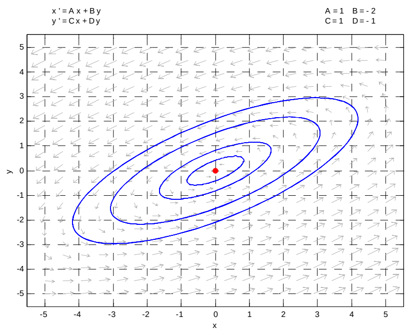
   (b) Write DE, DS models for sliding chain

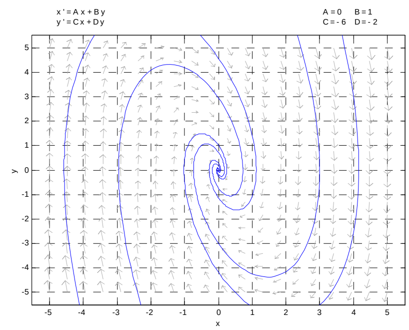
(c) **Predator-prey:** write linear differential model for a predator-prey model, and discuss its eigenvalues and behavior of solutions.

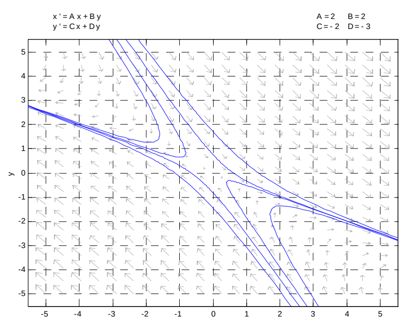
(d) **Migration model**: write a migration model for two states, with annual exodus rates p and q. Discuss its eigenvalues and solutions at large time t.

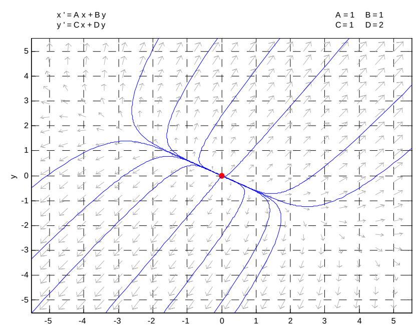
2. **Linear systems: phase-plots and equilibria**

(a) Identify each plot with a linear system given by matrix A with the following eigenvalues $\{\lambda_1, \lambda_2\}$: (a) $\{-1 \pm 2i\}$; (b) $\{-1, -3\}$; (c) $\{-1, 2\}$; (d) $\{\pm i\}$; (e)$\{1, 3\}$; (f) $\{1 \pm 2i\}$. Explain your choice.









(b) Sketch solution curves for each one.

(c) Locate equilibria on each plot and describe their type. Find eigendirections (whenever appropriate)

3. Linear algebra and differential systems:

(a) Convert DE $y'' + ay' + 4y = f(t)$ into a differential system: $X' = AX + \vec{b}$. Write $X; A; \vec{b}$

(b) Convert (3-rd order) DE $y''' + ay'' + 4y = f(t)$ into a differential system: $X' = AX + \vec{b}$. Write $X; A; \vec{b}$.

(c) A 2×2 real matrix A, has complex eigenvalues: $\lambda = -1 + 3i$, and eigenvector: $(1, 2 - 3i)$. Find a fundamental pair and general solution of ODS: $Y' = AY$. Solve IVP: $Y(0) = (a, 1)$

(d) A 3×3 matrix A has eigenvalues: $\lambda_1 = -1; \lambda_2 = -2; \lambda_3 = 0$ and the corresponding eigenvectors

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}; X_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

1. Write the characteristic polynomial of A.
2. Write the general solution of ODS $X' = AX$
3. Solve the initial value problem: $X(0) = (1, 0, 0)$. Hint

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 & -2 \\ 1 & -1 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

4. Solve the eigenvalue problem for matrices

(a) $A = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}; C = \begin{bmatrix} -1 & 4 \\ -1 & 0 \end{bmatrix}$

(b) Write general solutions of the homogeneous differential systems:

$$(i)\ X' = AX \text{ and } (ii)\ X' = BX$$

and plot their phase-portraits indicating eigen-directions.

5. **Linear systems and bifurcations**

(a) Consider a one-parameter family of matrices (linear systems): $A = \begin{bmatrix} a & a-1 \\ 2 & 1 \end{bmatrix}$. Sketch stability diagram in the pq-plane ($p = -tr(A); q = \det(A)$) and find bifurcation values of a

(b) Solve eigenvalue problem and sketch phase-plots of the system in different intervals of a and at the bifurcation values of a (Hint: there are 4 bifurcation points and 5 intervals)

6. **Linear DE with sources**: $y' + ay = f(t); y(0) = 1$

(a) Write general solution and IVP solution for $f(t) = Fe^{-\alpha t}$, sketch them and describe their asymptotics as $t \to \infty$. Sketch phase-plots

(b) Do the same problem for periodic forcing: $f(t) = F\cos\omega t$.

7. **Forced oscillations**: $y'' + dy' + \frac{9}{4}y = f(t);\ y(0) = 1; y'(0) = 0$

(a) Write general solution and IVP solution for $f(t) = Fe^{-\alpha t}$, in two cases: $d = 0$, and $d = 2$. Sketch solutions (in both cases) and describe their asymptotics as $t \to \infty$. Sketch phase-plots

(b) Do the same problem for periodic forcing: $f(t) = F\cos\omega t$.

(c) Take undamped case, write the IVP-solution for $y(0) = y'(0) = 0$, for periodic forcing $f = \cos \omega t$ with arbitrary ω. Explain the modulation and resonance phenomena (determine resonant ω), and plot the corresponding solutions.

(d) For damping coefficient $d = 2$, find and plot the response amplitude as function of ω, identify the peak response amplitude, and discuss its behavior as friction coefficient $d \to 0$.

8. **Oscillators and bifurcations.**

(a) Write a linear oscillator model of mass $m = 2$, spring constant $k = 5$ and damping coefficient d, as second order DE and differential system (DS)

(b) Write the general solution (DE and DS), and the IVP-solution: $x(0) = 1; \dot{x}(0) = 0$. Find the critical damping d_{crit}. Consider cases:

$$d = 0; d < d_{crit}; d = d_{crit}; d > d_{crit};$$

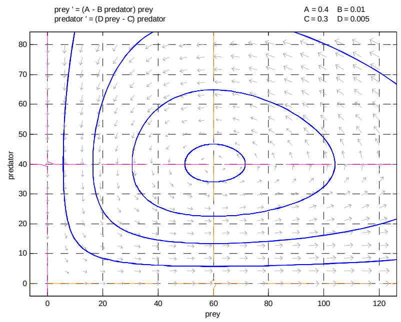
(c) Discuss the bifurcation of the system as parameter d varies from -10 to 10 (some values could be unphysical). Specifically, find the bifurcation (critical) values of d . Sketch the phase-plots and solution-plots in each case (use initial values $x(0) = 1; \dot{x}(0) = 0$ for solution plots).

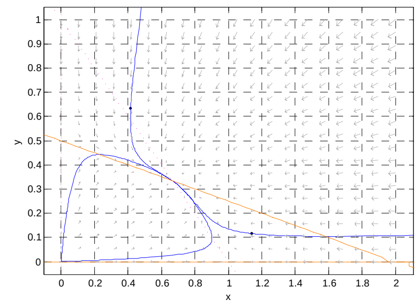
(d) Describe the oscillator motion in each case: no damping, underdamped, critically damped, overdamped. Describe the effect of damping on frequency and amplitude of oscillations.

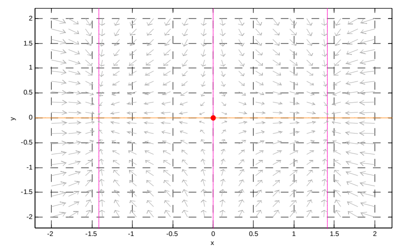
9. **Nonlinear systems and phase-plots**

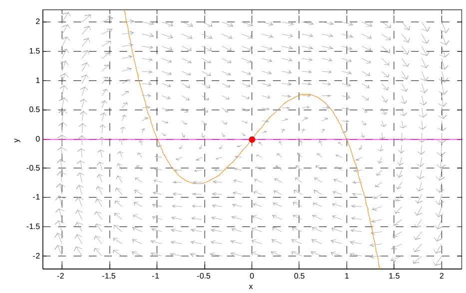
(a) Identify each plot with a suitable dynamical system (vector-field) $F(Y)$ from the following list, and give the physical model for each one. Explain the meaning of variables and coefficients

(1)	(2)	(3)	(4)
$\begin{pmatrix} 2x - x^3 \\ -y \end{pmatrix}$	$\begin{pmatrix} y \\ -y/2 + x - x^3 \end{pmatrix}$	$\begin{pmatrix} x - xy/3 \\ xy - 2y \end{pmatrix}$	$\begin{pmatrix} x - \frac{x^2}{3} - xy \\ y - \frac{xy}{2} - \frac{y^2}{2} \end{pmatrix}$









(b) Locate equilibria on each plot and describe their type, based on plots and part (a)

(c) Describe each plotted system as Hamiltonian, dissipative, conservative, gradient, based on plots.

(d) determine which of systems (1-4) is Hamiltonian. Find the Hamiltonian, potential, Lyapounov functions (whichever is appropriate) for vector fields (1)-(4)

10. **Linearization.** Take any systems (1)-(4) of the previous problem

(a) Find all equilibria and write the linearized system near each equilibrium

(b) Compute the eigendata, determine the type and stability of equilibria. Sketch phase plots near equilibria.

(c) Take any two equilibria of different types and write the general solution of the linearized system for each one

(d) Verify your qualitative conclusion in parts (a-c) of the previous problem by the linearized analysis.

(e) Sketch a few solution-plots $\{x(t), y(t)\}$ that start near your equilibria (part (c)). Compare them to solutions of the nonlinear system

(f) Discuss the asymptotics of (nonlinear) solutions as $t \to \infty$, and interpret your results qualitatively in terms of competition/cooperation, survival etc. Discuss the role and meaning of closed trajectories and separatices.