Math. 224: Review

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Fall 01

1 Sample problems

1. Find equilibria, determine stability, draw phase-line of logistic-type model: $y' = P(y)$-polynomial of y, e.g. $P = y(1 - y/N) \ldots.$ Write analytic solution of the logistic equation: $y' = ay(1 - y/N)$.

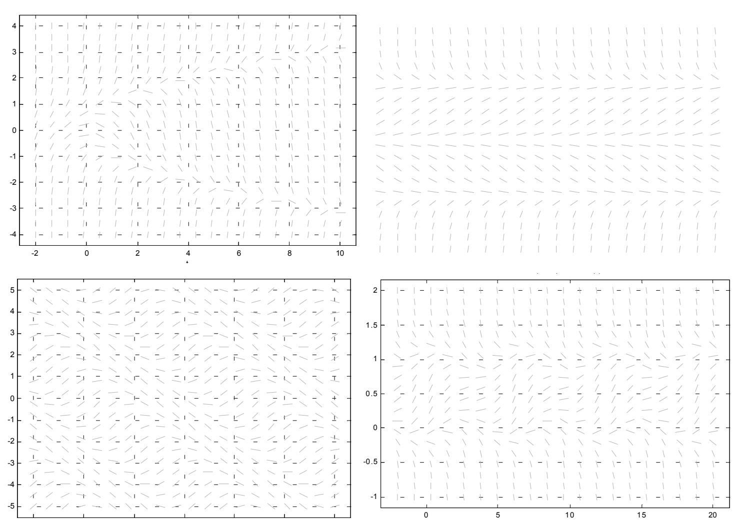
2. Determine bifurcation values $\{\alpha\}$ and the type of bifurcation for ODE: $y' = P(y, \alpha)$, e.g. $P = y(1 - y/N) + \alpha$.

3. Identify the following slope-fields with 1-st order DE's: (a) $y' = y(1 - y^2)$ (b) $y' = \cos(y + t)$ (c) $y' = y^2 - t$ (d) $y' = y(1 - y) + a\cos t$ (e) $y' = -y + \cos t$.

   (a) Identify the autonomous equation, find its equilibria and sketch phase-line.

   (b) Sketch solution-curves for all equations.

   (c) Determine which of the five could be solved analytically, and describe the method. Give the general solution



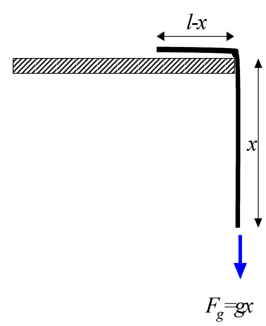
4. Set DE models for the following systems. Describe them as DE/DS; 1-st (or higher) order; linear/nonlinear; separable, autonomous, conservative, etc. Outline solution method for each.

   (a) Growth with migration: two populations (x, y) in regions A, B grow logistically and migrate between two regions, so that fraction a of x-population migrates to B, while fraction b of y-populations migrates to A.

   (b) Logistic model with harvesting (discuss solution methods for constant and periodic harvesting)

   (c) Competing/cooperating species with logistic growth rates for each one. Explain signs $\pm$ of interaction coefficients.

   (d) Sliding chain with friction coefficient proportional to the horizontal (table) portion of the chain. Write it as a 2-nd order DE and the differential system. Indicate solution method with or without friction.



   (e) Food chain system made of trees, moose, wolves, where p percent of moose are hunted annually

   (f) Tank of 100 gal half-filled with lead-contaminated solution of 5 mg/gal is flushed with the fresh water at the rate 3 gal/sec and the mixture is taken out at the rate 1 gal/sec. Find how long it would take to fill the tank and what would be the terminal value of lead concentration

   (g) Logistic population growth model with the rate of increase $\alpha = .5$, the threshold capacity $N = 20$, the harvesting rate $r = .5$, the initial population 5. Find equilibria and plot solutions that start above or below equilibria

   (h) Determination of the time of death. A body was found on the ice at 20°F temperature. The body temperature has dropped in 3 hours from 70° to 55°. Determine the time of death (take the normal body temperature to be $100°F$).

5. Discuss stability and bifurcations for a family of linear systems: $X' = \begin{bmatrix} \alpha & 1 \\ -4 & 0 \end{bmatrix} X$

6. Find equilibria and plot phase-plots for the competing species model

$$\begin{cases} x' = x\,(1 - x/a - y/4) \\ y' = .5y\,(1 - x/3 - y/a) \end{cases}$$

for $a = 1$ and $a = 5$. Discuss bifurcations of the model as a changes from 1 to 5.

7. Determine whether a given system is Hamiltonian, conservative, dissipative, gradient. Find its Hamiltonian /potential function, or conserved integral. Sketch phase-plots.

   (a) $F = \left(ax - y, x^2 + by\right)$, $(y, -\sin x - .2y)$, $(2x - 3y, -3x + y)$; $\left(-2xy + y^2, x^2 + y^2\right)$

   (b) Write equation of motion for nonlinear Duffing oscillator, $U\,(x) = x^4 - x^2$. Find its equilibria and determine their types.

   (c) Discuss the effect of friction, external force on energy of the oscillator.

8. **Linear oscillator:** $2y'' + \alpha y' + 8y = f\,(t)$

   (a) Take $\alpha = 0$ and find the undetermined coefficient solution for two choices of $f = e^{-t}$, and $\cos \omega t$.

   (b) Solve the above two problems by the Laplace transform. Take initial data: $y\,(0) = 0; y'\,(0) = 1$, and $\omega = 2$. Compare solutions with part (b). (Partial fractions:

$$\frac{1}{(s+1)\,(s^2+4)} = \frac{1}{5\,(s+1)} - \frac{1}{5}\frac{s-1}{s^2+4}$$
$$\frac{s}{(s^2+\omega^2)\,(s^2+4)} = \frac{1}{-4+\omega^2}\left\{\frac{s}{s^2+4} - \frac{s}{s^2+\omega^2}\right\}$$

   (c) Plot solution curves of parts (c). Explain what happens to solutions as $t \to \infty$

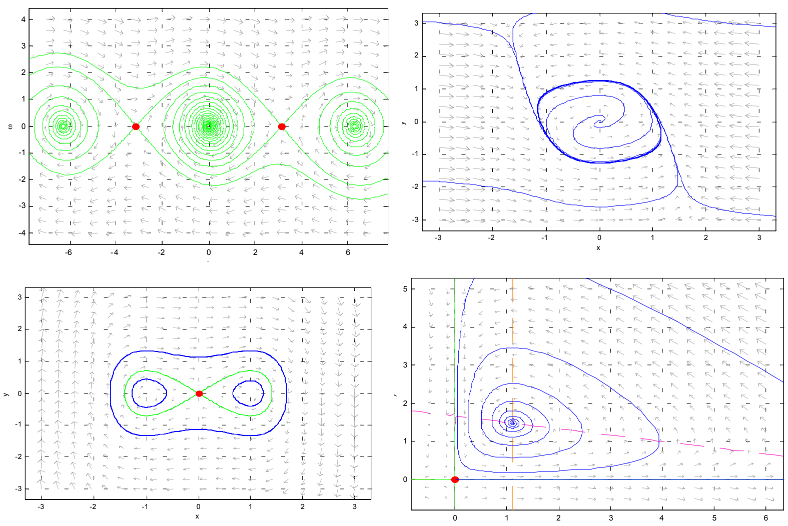
   (d) What happens to solution of $f = \cos \omega t$ as forcing frequency $\omega \to 2$? Sketch hypothetical plots for $\omega = 1.8$ and $\omega = 2$.Explain the modulation and resonance phenomena.

   (e) Take $\alpha = 4$ and write solution of the oscillator problem with zero initial data and arbitrary force f as a convolution-integral; find fundamental solution K

   (f) Explain the role of parameter α on solutions of oscillator problem (underdamped, critical damped, overdamped), find α_{critical}. Plot typical solutions in each case.

   (g) Plot the maximal response amplitude as a function of parameter α

9. Identify the follow phase-plots with system and write DE models (a) predator-prey with logistic growth; (b) Van der Pole oscillator: $F = \left(-y + ax - x^3, x\right)$ (c) nonlinear spring with force: $\pm\left(x - x^3\right)$ (d) damped pendulum; (e) competing species



   (a) Discuss special solutions in all 4 cases: equilibria (types), stable/unstable orbits, periodic solutions, limit cycles

   (b) Sketch solution curves for trajectories shown on the plots

10. Write the food chain model for trees, moose and wolves with growth rates $\alpha_1, \alpha_2, \alpha_3$, carrying capacities $N_1; N_2; N_3$ and all interaction coefficients equal 1. Assume all $\alpha = 1$, and all $N = 1$, while moose are hunted at a arte proportional to their population with coefficient γ. Find the coexistence equilibrium and discuss its stability as γ changes.

11. **Laplace transform.**

   (a) Produce solution of part c, or d of problem 3 (your choice) by the Laplace transform method

   (b) Write solution of $\begin{array}{c} y'' + 2y' + 4y = f(t) \\ y(0) = 0;\ y'(0) = 0 \end{array}$ for arbitrary force f as convolution-integral. Write and plot the fundamental solution $K(t)$

12. **First order DE and bifurcations.**

   (a) Solve initial value problem for the linear decay model with periodic source: $y' + 2y = 3\sin t$; $y(0) = 0$. Sketch solution and describe its behavior of solution at large t.

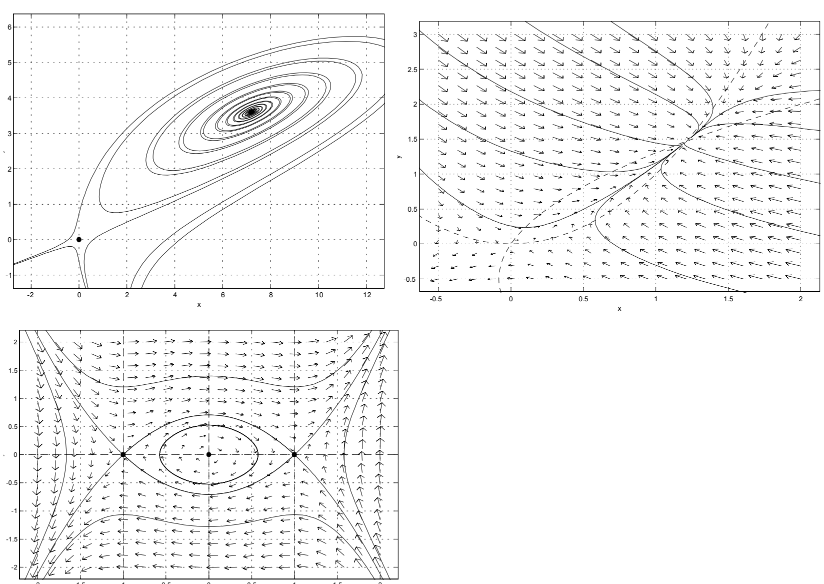
(b) Solve equation $y' = \sqrt{a + \cos y}$ by separation (leave your answer in the integral form, do not attempt to evaluate!). **Extra**: what relation this equation and its solution has to the pendulum motion.

(c) Solve logistic equation: $y' = .2y\,(1 - y/4)$ with arbitrary initial value y_0. Sketch solutions for different y_0, above and below its equilibria

(d) Discuss the effect of constant harvesting b in part (c): find the bifurcation value(s) b. Sketch solution curves at the bifurcation value. Which values b allow species to survive ?

13. **Dynamical systems**

(a) Determine which of the following plots is Hamiltonian, dissipative, gradient. Explain



(b) Find value(s) a that make vector field $F = \left(ax - y^2, x - 2y\right)$ Hamiltonian, and find its Hamiltonian function. **Extra**: does any of the above plots resemble phase-plot of F for some value a (not necessarily "Hamiltonian" one)?

(c) Find all equilibria of field F (with arbitrary a) and write linearized system for each one.

(d) Show that one of equilibria is always saddle (for positive a), whereas the other one could change its type.

(e) Find the bifurcation values of parameter $0 \le a \le 5$. Sketch the trace-determinant diagram (for the second equilibrium). Explain what happens to equilibria at bifurcation values.

(f) (Extra) Sketch typical phase plane of the system at each intermediate value of a (between bifurcations).

14. **Linear competition-cooperation** $\begin{cases} \dot{x} = x + 2y \\ \dot{y} = -x + y \end{cases}$

    (a) Explain which of two species cooperates and which competes with the other

    (b) Find the general solution, and IVP-solution for $Y(0) = (0, 1)$

    (c) Sketch phase plot, and solution-plot for IVP.

    (d) Show that y-species is reduced to zero in finite time for any positive initial values (x_0, y_0). Find this time for IVP of part (b)

15. Homework problems: ch. 1-6 and handouts.