224 Review

D. GURARIE

I. Models; systems; processes (DE; DS)

⊛ Growth - decay; cooling (heat exchange); mixing; free fall; CR - circuits

⊛ Mechanical / Physical:

i) oscillators (linear/non-lin; pendulum; Duffing...)
ii) Newtonian gravity (projectiles; rocket...)
iii) Lorenz (convective motion)

⊛ Population / Biology

i) Logistic (constrained growth)
ii) Migration
iii) competition/cooperation (linear, non-lin)
iv) Predation (Volterra-Lotka; satiation)
v) SIR - infection

⊛ Chemical kinetics (reactions)

⊛ Geometric / static: fastest slope; heavy chain; minimal surface

III. Methods (analytic)

Order	Linear		Non-linear		
1st	$\begin{cases} y' + ay = f \\ y(0) = \ldots \end{cases}$	Multiplier	$y' = f(y)$	separation	$y' = f(y,t) \ldots$?
2nd	$y'' + ay' + by = f$, $y(0) = \ldots$	Charac. polyn, Undet. coeff, Laplace	$y'' = f(y)$	Conserved integral $\Rightarrow$ 1st order	$y'' = f(y, y') \ldots$?
3rd etc	$y''' + ay'' + \ldots$, $y(0) = \ldots$	Same as above!	$y'''' = f(y, y', y'') \ldots$?		
DS	$Y' = AY + B$	Eigenvalue, Matrix Exp	$Y' = F(Y)$	$\to$ Conserved integr. $\to$ Special solutions $\to$ Equilibria	

Linear DE & DS

1st multip!

$$\mu(t) = e^{\int a\,dt}$$

$$y = \frac{1}{\mu(t)}\left[C + \int_0^t (\mu f)\,dt\right]$$

$(a = \text{const})$

$$\underbrace{Ce^{-at}}_{y_h} + \underbrace{\int_0^t e^{-a(t-s)} f(s)\,ds}_{y_p}$$

2nd Charact. polyn.

$$p(\lambda) = \lambda^2 + a\lambda + b \Rightarrow (e^{\lambda_1 t}\ e^{\lambda_2 t})$$

$$y = \underbrace{c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}_{y_h} + \underbrace{y_p}_{\text{Responce}}$$

y_p:

- $\dfrac{e^{\alpha t}}{p(\alpha)}$; $f = e^{\alpha t}$
- $Re\left[\dfrac{e^{i\beta t}}{p(i\beta)}\right]$; $f = \cos\beta t$
- $\mathcal{L}^{-1}\left[\dfrac{F(s)}{p(s)}\right]$ - Inverse Laplace
- $y_p = \int_0^t K(t-\tau) f(\tau)$ - convol.

DS

$$A: \begin{Bmatrix} \lambda_1; \lambda_2 \ldots \\ X_1; X_2 \ldots \end{Bmatrix} \Rightarrow e^{\lambda_1 t} X_1 \ldots$$

$$Y(t) = c e^{\lambda_1 t} X_1 + \ldots + Y_p$$

$$Y_p = \int_0^t \underbrace{e^{A(t-\tau)}}_{\text{Matrix exp}} F(\tau)\,d\tau$$

IV Qualitative methods

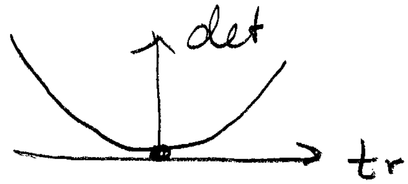
* Phase-space, null-clines
* Equilibria, linearization, stability

1) $\boxed{F(Y)=\vec{0}} \to \begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases} \to \boxed{Y_1 = \binom{x_1}{y_1};\ Y_2 = (\cdot);\ \dots}$

2) $A = F' = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Big|_{Y_1 \dots} = A_1;\ \ A_2;\ \dots$ – matrices of linearized DS

Linear approx: $Y(t) \approx Y_1 + V(t)$, where $\boxed{V' = A_1 V}$ linear DS

3) Stability-types:



* Bifurcations:

DE: $\boxed{y' = f(y,b)}$ DS: $\boxed{Y' = F(Y,b)}$

$\Downarrow$

$\begin{cases} f(y,b)=0 \to \{y_1(b);\ y_2(b)\dots\}; \\ f_y(y,b)=0 \end{cases}$

slope at equilibria

$m(b) = f_y(y_1(b), b)$

$\begin{cases} F(Y,b)=\vec{0} \to \{Y_1(b); \dots \\ \underbrace{A_1(b) = F'|_{Y_1(b)}}_{\text{Bifurcations of } A_i(b)}; \dots \end{cases}$

V. Special systems: Hamiltonian, gradient

Conservative Test: $\nabla \cdot F = 0$; $h = \int f\,dy - g\,dx$

Equilibria, separatrices, ...